11 Modelling periodic phenomena: trigonometric functions

Skills check

1 $a^2 + b^2 = c^2$

$5^2 + b^2 = 13^2$

$b^2 = 169 - 25 = 144$

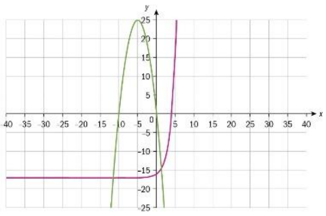
$PR = 12\,\text{m}$

$\hat{P} = \sin^{-1}\dfrac{5}{13} = 22.6^o.$

$\hat{Q} = \sin^{-1}\dfrac{12}{13} = 67.4^o\ .$

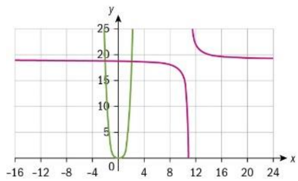
2 $h = 73\tan(43) = 68.1$

3 a



Solution for $x \le 0$ is -11.5

b



Solution for $t > 0$ is 2.08

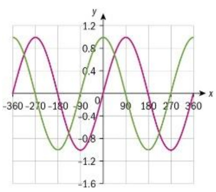
Exercise 11A

1 a $\theta = \sin^{-1}\dfrac{30}{70} = 25.4$. So the values of θ are $25.4,\ 180 - 25.4,\ 180 + 25.4,\ 360 - 25.4.$

All possible values are $25.4^o,\ 155^o,\ 205^o,\ 335^o.$

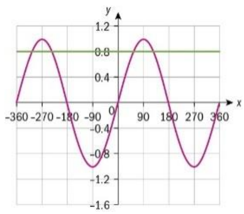
b $\theta = \sin^{-1}\dfrac{20}{70} = 16.6$. The values are: $90 - 16.6 = 73.4^o$ and $270 + 16.6 = 287^o.$

2 a i



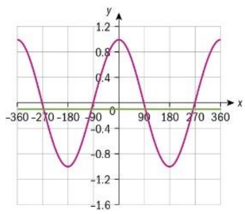
Solutions are $\theta_1 = 45^o,\ 225^o.$

ii



Solutions are $\theta_2 = 53.1^o, 126.9^o.$

iii



Solutions are $\theta_3 = 95.7^o, 264.3^o.$

b i $\theta_1 = 45, \quad (x,y) = (-70\sin(45), 70\cos(45)) = (-49.5, 49.5)$

$\theta_1 = 225 \quad (x,y) = (-70\sin(225), 70(\cos(225)) = (49.5, -49.5)$

ii $\theta_2 = 53.1 \quad (x,y) = (-70\sin(53.1), 70\cos(53.1)) = (-56.0, 42.0)$

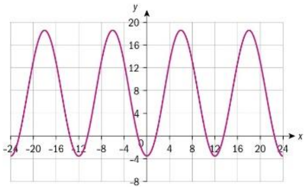
$\theta_2 = 126.9 \quad (x,y) = (-70\sin(126.9), 70\cos(126.9)) = (-56.0, -42.0)$

iii $\theta_3 = 95.7 \quad (x,y) = (-70\sin(95.7), 70\cos(95.7)) = (-69.7, -6.95)$

$\theta_3 = 264.3 \quad (x,y) = (-70\sin(264.3), 70\cos(264.3)) = (69.7, -6.95)$

3 **a** $T(4) = -11\cos(30 \times 4) + 7.5 = 13$

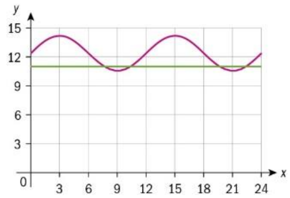
b



The temperature will be zero in the middle of February and November.

4 **a** $D(5.5) = 1.8\sin(30 \times 5.5) + 12.3 = 12.8\,\text{m}$

b



Depth of 10.9 after 7.7 hours, 10.2 hours, 19.7 hours and 22.3 hours.

Exercise 11B

1 **a** Amplitude is 3, period is 90°, y = 1 is principal axis and the range is $\{y : -2 \le y \le 4\}$

b Amplitude is 0.5, period is 720°, principal axis is y = −3 and range is $\{y : -3.5 \le y \le -2.5\}$

c Amplitude is 7.1, period is 120°, principal axis is y = 1 and the range is $\{y : -6.1 \le y \le 8.1\}$

d Amplitude is 5, period is 720°, principal axis is 7 = 8.1 and the range is $\{y : 3.1 \le y \le 13.1\}$

2 **a** a = 2

b a = 1, b = 2

c a = 3, b = 0.5, d = −1

3 **a** b = 2

b a = −3, d = 1

c a = 2.5, b = 0.5, d = 0

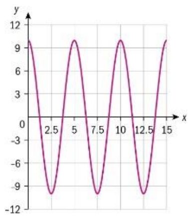
4 **a** $t = \frac{5.2+1}{2} = 3.1.$ $s = -\frac{5.2-1}{2} = -2.1.$ $r = \frac{90}{2} = 45.$

b **i** t **ii** s **iii** $\frac{360}{r} = 8$

5 **a** $d = \frac{4.6-6.2}{2} = -0.8.$ $a = \frac{4.6+6.2}{2} = 5.4.$ $b = \frac{180}{2} = 90.$

b a is the highest amount above sea level. b is the number of cycles in 6 hours. d is sea level.

6 **a**



b r = 10, $s = \frac{360}{5} = 72$

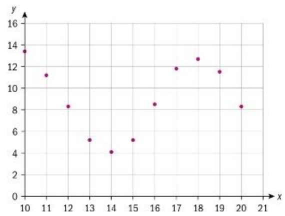
7 You can draw a sample space diagram for the sum of *a* and *d*, the only parameters to affect the maximum and minimum of the function, as follows:

d \ a	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7

There are three possibilities in a sample space of 12 with a maximum greater than five, hence the probability required is $\frac{3}{12} = 0.25$

Exercise 11C

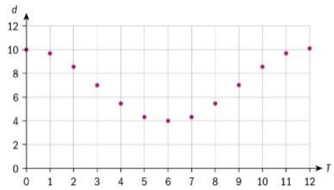
1 a



b $a = \dfrac{13.4 - 4.1}{2} = 4.65.$ $\quad d = \dfrac{13.4 + 4.1}{2} = 8.75.$ $\quad b = \dfrac{360}{20 - 12} = 45$

$y = 4.65\sin(45x) + 8.75$

2 a

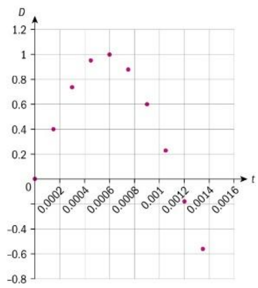


b $a = \dfrac{10.1 - 4}{2} = 3.05,$ $\quad d = \dfrac{10.1 + 4}{2} = 7.05,$ $\quad b = \dfrac{360}{12} = 30.$

The data is modelled by $y = 3.05\cos(30x) + 7.05$

c $\dfrac{1}{30}\cos^{-1}\left(-\dfrac{21}{61}\right) = 3.67$, between 0000 and 0340 hours and between 0820 and 1200 hours.

3 a



b $a = 0.9961,\ b = \dfrac{360}{6\times10^{-4}\times4} = 150000, \quad d = 0.$

$D(t) = 0.9961\sin(150000t)$

c $D(0.00011) = 0.9961\sin(150000\times0.00011) = 0.283$ decibels

d $D(0.002) = 0.9961\sin(150000\times0.002) = -0.863$ decibels

e Part **c** is more reliable as 0.00011 falls within the given data range while 0.002 is outside of it.

4 a $a = \dfrac{10.1-5.31}{2} = 2.395, \quad d = \dfrac{(10.1+5.31)}{2} = 7.705, \quad b = 1$

$y = 2.395\cos(\alpha) + 7.705.$

b $AD^2 = AB^2 + BD^2 - 2(AB)(BD)\cos(180-\alpha-\beta)$

$AD = \sqrt{65.05 - 36.96\cos(180-\alpha-\beta)}$

c Model is a reasonable fit, but cannot be accurate as the cosine rule depends on two angles and not just one.

5 a

Month(t)	**Rise**	**Set**
0	11.33	15.7
1	10.15	17.22
2	8.62	18.72
3	6.78	20.27
4	5.02	21.82
5	3.4	23.45
6	3.08	23.95
7	4.57	22.57
8	6.15	20.78
9	7.58	19.00
10	9.15	17.22
11	10.73	15.82

This is necessary so that they can be graphed on the cartesian plane, which uses decimal notation.

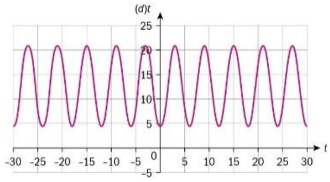
b $r: \quad a = \dfrac{(11.33-3.08)}{2} = 4.125, \quad b = \dfrac{360}{12} = 30, \quad d = \dfrac{11.33+3.08}{2} = 7.205$

$r(t) = 4.125\cos(30t) + 7.205$

$s: \quad a = \dfrac{(23.95-15.7)}{2} = 4.125, \quad b = \dfrac{360}{12} = 30, \quad d = \dfrac{23.95+15.7}{2} = 19.825$

$s(t) = -4.125\cos(30t) + 19.825$

c



i Around 3 months or 90 days

ii Around 6 months have at least 12 hours while around 4 have no more than 18. So around $\frac{1}{4}$ of the year has at least 12 hours but no more than 18

6 a $a = \frac{6.87 - 4.88}{2} = 0.995, \quad b = \frac{360}{12} = 30, \quad d = \frac{6.87 + 4.88}{2} = 5.875$

$r(t) = 0.995\cos(30t) + 5.875$

b Karim can arrive at least 1 hour before sunset without arriving earlier than 0500 between the dates of 7th October through to the following year on 24th March.

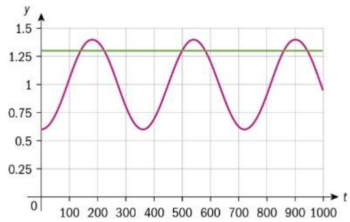
7 a $a = \frac{30 - 0}{2} = 15, \quad b = \frac{360}{0.06} = 6000, \quad d = \frac{30}{2} = 15$

$y = -15\cos(6000t) + 15$

b The dot travels $30\pi cm$ in 0.06s. So the speed of the fan is $\frac{30\pi}{0.06} = 500\pi cms^{-1}$.

Chapter review

1



Solutions are t = 499 and 581

2 a $y = -\cos(3x) - 1$. **b** $y = 3\sin(0.5x) - 1$ **c** $y = -\sin(x) + 3$

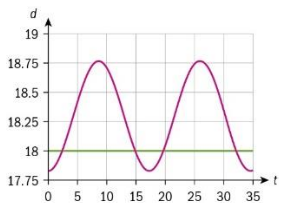
3 $a(x)$ is not periodic. $b(x)$ is not periodic. $c(x) = 3(-2\cos(x)) = -6\cos(x)$.
Period is 360°, amplitude is 6 and the principal axis is y = 0.
$d(x) = -2\cos(3x)$. Period is 120°, amplitude is 2, the principal axis is y = 0.

4 a $a = \frac{(3+1)}{2} = 2, b = \frac{360}{(270-180)\times 2} = 2, \ d = \frac{3-1}{2} = 1. \ y = -2\cos(2x) + 1.$

b $\beta = -1$

5 a Maximum value is 18.77m and the minimum value is 17.83m

b



First time after 19s to reach 18m is 19.63s.

6 **a** $a=\frac{6+16-6}{2}=8, b=\frac{360}{2\times 102}=\frac{30}{17}, d=\frac{6+16+6}{2}=14.$ $y=8\sin\left(\frac{30}{17}x\right)+14$ or

$y=-8\sin\left(\frac{30}{17}x\right)+14$

b F = (204,14), A = (255,22)

7 **a** Minimum is 1.8 A1

Maximum is 6.4 A1

b Using GDC M1

$x=268^\circ$ A1

$x=452^\circ$ A1

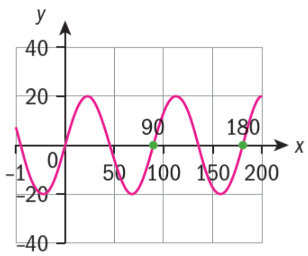
8 The principal axis is $\frac{5.5+1.5}{2}(=3.5)$. Hence $p=3.5$ M1A1

The amplitude is $\frac{5.5-1.5}{2}=2$. Hence $q=2$ M1A1

The period is 120° **:** $120^\circ=\frac{360^\circ}{r}$ M1

Hence $r=3$, So $y=3.5+2\cos 3x$ A1

9 **a**



M1A1A1

b $y=0$ A1

c $f(x)=-5$ when $x=48.6^\circ, 86.4^\circ, 137^\circ, 176^\circ$ M1A1

Solution is therefore $48.6^\circ<x<86.4^\circ$ and $137^\circ<x<176^\circ$ A1A1

10 The amplitude is 3 M1

Hence $p=3$ A1

The period is $81^\circ-(-9^\circ)=90^\circ$ M1

$90^\circ=\frac{360^\circ}{q}$ A1

Hence $q=4$ A1

So $y=3\sin(4x+r)$

$y=0$ when $x=36^\circ$ (equidistant from -9° and 81°) M1

So $0=3\sin(144^\circ+r)$ M1

So $\sin(144^\circ+r)=0$ and the first positive root is when $r=36^\circ$ A1

Therefore $y=3\sin(4x+36^\circ)$

11 **a** 0.3 A1

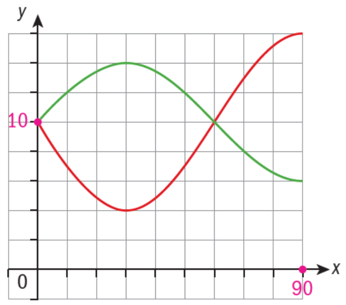
b $y_{MIN}=5.4$ A1

First occurs when $12.5x=180$ M1

$x = \frac{180}{12.5} = 14.4$ A1

c Period $= \frac{360}{12.5} = 28.8$ M1A1

12 a



M1A1

b The principal axis is $\frac{16+4}{2} = (10)$. Hence $p = 10$ M1A1

The amplitude is $\frac{16-4}{2}(=6)$.Hence $q = 6$ M1A1

The period is $2 \times 60^\circ = 120^\circ$ M1

$120^\circ = \frac{360^\circ}{r}$ A1

Hence $r = 3$. So $y = 10 - 6\sin 3x$ A1

13 Amplitude $= 2$, so $b = 2$ A1

At $(60^\circ, 5)$, $5 = a + 2$ M1A1

So $a = 3$ A1

Therefore $y = 3 + 2\sin cx$

By symmetry, the curve goes through the point $(180^\circ, 1)$ M1

So $1 = 3 + 2\sin(180c)$ A1

$-1 = \sin(180c)$

$180c = 270$ A1

Therefore $c = \frac{3}{2}$ A1

Therefore $y = 3 + 2\sin\left(\frac{3x}{2}\right)$

14 a $D = \frac{22+12}{2} = 17$ M1A1

$A = \frac{22-12}{2} = 5$ M1A1

The period is $\frac{360}{B} = 24$ M1

Therefore $B = 15$ A1

So $T = 5\sin(15(t-C))+17$

At $(3,12)$, $12 = 5\sin(15(t-C))+17$ M1

$-1 = \sin(15(3-C))$

$15(3-C) = -90$ A1

$C = 9$ A1

Therefore $T = 5\sin(15(t-9))+17$

b Solving $T = 5\sin(15(t-9))+17$ and $T = 20$ by GDC M1

Solutions are $T = 18.54$ and $T = 11.46$ A1A1

$18.54 - 11.46$

$= 7.08$ hours $($7 hours 5 minutes$)$ A1

12 Analysing rates of change: differential calculus

Skills check

1 **a** $(x-5)(3x+2) = 3x^2 - 13x - 10$ **b** $6x^3 + 7x^2 - 5x$

2 $y = \frac{1}{2}(11 - x)$

3 Volume: $32\pi\ \text{cm}^3$, surface area: $16\pi + 8\sqrt{13}\pi\ \text{cm}^2$

4 **a** x^{-5} **b** x^{-1}

5 20

Exercise 12A

a **i** $y' = 0$

ii $y' = 0$ at $x = 1$ (the tangent to y is stationary at $x = 1$).

iii The function is stationary for any value of x.

b **i** $y' = 4$

ii $y' = 4$ at $x = 1$ (the tangent to y is increasing at $x = 1$).

iii Since $y' = 4 > 0$, the function is increasing for any value of x.

c **i** $\frac{df}{dx} = 3 \times 2x^{2-1} = 6x$

ii $\frac{df}{dx} = 6$ at $x = 1$ (the tangent to $f(x)$ is increasing at $x = 1$).

iii The function is increasing when $6x > 0$. This is equivalent to $x > 0$.

d **i** $\frac{df}{dx} = 5 \times 2x^{2-1} - 3 = 10x - 3$

ii $\frac{df}{dx} = 10 \times 1 - 3 = 7$ at $x = 1$ (the tangent to $f(x)$ is increasing at $x = 1$).

iii The function is increasing when $10x - 3 > 0 \Leftrightarrow x > \frac{3}{10}$.

e **i** $\frac{df}{dx} = 3 \times 4x^{4-1} + 7 = 12x^3 + 7$

ii $\frac{df}{dx} = 12 \times 1^3 + 7 = 19$ at $x = 1$ (the tangent to $f(x)$ is increasing at $x = 1$).

iii The function is increasing when $12x^3 + 7 > 0$. This inequality can re-arranged to $x^3 > -\frac{7}{12}$,

or $x > \sqrt[3]{-\frac{7}{12}} = -0.836$ (3 s.f.)

f **i** $\frac{df}{dx} = 5 \times 4x^{4-1} - 3 \times x^{2-1} + 2 = 20x^3 - 6x + 2$

ii $\frac{df}{dx} = 20 \times 1^3 - 6 \times 1 + 2 = 16$ at $x = 1$ (the tangent to $f(x)$ is increasing at $x = 1$).

iii The function is increasing when $20x^3 - 6x + 2 > 0$. The function $g(x) = 20x^3 - 6x + 2$ has a single root at $x = -0.670$ (which is found by solving $g(x) = 0$). Therefore, the derivative $f'(x)$ is increasing when $x > -0.670$.

g **i** First note that $y = 2x^2 - 3x^{-1}$, so $y'(x) = 2\times 2x^{2-1} - 3\times(-1)x^{-1-1} = 4x + 3x^{-2} = 4x + \frac{3}{x^2}$

ii $y' = 4\times 1 + \frac{3}{1^2} = 4 + 3 = 7$ at $x = 1$ (the tangent to y is increasing at $x = 1$).

iii The function is increasing when $4x + \frac{3}{x^2} > 0$. Solving $4x + \frac{3}{x^2} = 0$ (for $x \neq 0$) shows that the equation $y' = 0$ has a single root at $x = x_r = -\sqrt[3]{\frac{3}{4}}$. When $x > x_r, 4x + \frac{3}{x^2} > 0$, when $x < x_r, 4x + \frac{3}{x^2} < 0$ (this can be verified using, for example, a graphical calculator). Therefore, the function $y = 2x^2 - \frac{3}{x}$ is increasing when $x > -\sqrt[3]{\frac{3}{4}}$.

h **i** First write $y = 6x^{-3} + 4x - 3$. Therefore, $y' = 6\times(-3)x^{-3-1} + 4 = -18x^{-4} + 4 = -\frac{18}{x^4} + 4$

ii $y' = -\frac{18}{1^4} + 4 = -14$ at $x = 1$ (the tangent to y is decreasing at $x = 1$).

iii The function $g(x) = -\frac{18}{x^4} + 4$ has roots ($g(x) = 0$) at $x = \pm 1.46$. When $x\langle -1.46, g(x)\rangle 0$, when $-1.46 < x < 1.46, g(x) < 0$, and when $x > 1.46, g(x) > 0$. Therefore, the function $y = \frac{6}{x^3} + 4x - 3$ is increasing when $x > 1.46$ and when $x < -1.46$.

i **i** First expand: $y = (2x-1)(3x+4) = 6x^2 + 5x - 4$, so $y' = 12x + 5$.

ii $y' = 12\times 1 + 5 = 17$ at $x = 1$ (the tangent to y is increasing at $x = 1$).

iii y is increasing when $y' = 12x + 5 > 0$, which may be re-arranged to $x > -\frac{5}{12}$.

j **i** Expand: $f(x) = 2x^4 - 8x^2 - 10x$, so $f'(x) = 8x^3 - 16x - 10$.

ii $f'(1) = 8\times 1^3 - 16\times 1 - 10 = -18$ (the tangent to $f(x)$ is decrease\ng at $x = 1$).

iii The only solution of $f'(x) = 0$ is $x = 1.66$. Therefore, $f'(x) > 0$ (f is increasing) when $x > 1.66$.

k **i** First write $y = 7x^{-3} + 8x^4 - 6x^2 + 2$. Then
$y' = -3\times 7\times x^{-3-1} + 8\times 4x^{4-1} - 6\times 2x^{2-1} = -21x^{-4} + 32x^3 - 12x$.

ii $y' = -\frac{21}{1^4} + 32\times 1^3 - 12\times 1 = -1$ at $x = 1$ (the tangent to y is decreasing at $x = 1$).

iii $y' = 0$ has a single solution $x = 1.01$, with $y' > 0$ when $x > 1.01$ and $y' < 0$ when $x < 1.01$. Therefore y is increasing when $x > 1.01$.

Exercise 12B

1 **a** $\frac{dA}{dr} = 2\pi r^{2-1} = 2\pi r$

b $\frac{dA}{dr} = 2\pi\times 2 = 4\pi$ when $r = 2$.

2 **a** $\frac{dP}{dc} = -0.056\times 2c^{2-1} + 5.6 = -0.112c + 5.6$

b When $c = 20, \frac{dP}{dc} = -0.112\times 20 + 5.6 = 3.36$, when $c = 60, \frac{dP}{dc} = -0.112\times 60 + 5.6 = -1.12$.

c At the larger number of sales, selling more cupcakes will actually decrease profit, whilst it will increase profit at the lower value.

3 **a** $f'(t) = 80\times 2t^{2-1} - 160 = 160t - 160 = 160(t-1)$

b The function $f'(t)$ represents the velocity of the bungee jumper.

c $f'(0.5) = 160(0.5-1) = -80, f'(1.5) = 160(1.5-1) = 80$. At these times, the bungee jumper is travelling at the same speed, but in opposite directions (moving away from start point at $t = 0.5$ and towards the start point at $t = 1.5$).

d $f(2) = 160(2-1) = 160 = f(0)$. The bungee jumper passes through the start point at the same speed that he left at – this is unrealistic; some energy will be lost overcoming, for example, air resistance.

4 $f'(x) = 3x^2 + 2x + 2$. The gradient at A and B is 3, so the x-co-ordinates of these points satisfy $3 = f'(x) = 3x^2 + 2x + 2 \Rightarrow 3x^2 + 2x - 1 = 0$. This equation has solutions $x_1 = -1, x_2 = \frac{1}{3}$.

The corresponding y co-ordinates are $y_1 = f(-1) = (-1)^3 + (-1)^2 + 2(-1) = -2$, and $y_2 = f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 + \frac{2}{3} = \frac{22}{27}$.

The co-ordinates of points A and B are $(-1,-2)$ and $\left(\frac{1}{3}, \frac{22}{27}\right)$.

5 The pink line has a gradient of $m = \tan^{-1} 45° = 1$. Also note that $h'(x) = 2 - 0.2x$. Therefore, the pink and purple lines meet where the gradient of the purple line equals the gradient of the pink line: $m = 2 - 0.2x \Rightarrow x = 5$. The point of intersection has y co-ordinate of $h(5) = 2 \times 5 - 0.1 \times 5^2 = 7.5$ - the point is 7.5 m above the ground.

Exercise 12C

1 At $x = 3, y = f(3) = 2 \times 3^2 - 4 = 14$. Also, $f'(x) = 4x$ so the gradient of the tangent at $x = 3$ is $m = f'(3) = 12$. Therefore, the tangent at $x = 3$ has equation $y - 14 = 12(x-3) \Rightarrow y = 12x - 22$.

2 The y-co-ordinate of the point of contact is $y = f(1) = -1^2 + 2 = 1$. Also, $f'(x) = -2x + 2$, so the gradient of the tangent at $x = 1$ is $f'(1) = -2 + 2 = 0$. Therefore, the tangent at $x = 1$ (i.e. the equation of the plank) is simply the constant function $y = 1$.

3 **a** $f'(x) = -4x \Rightarrow f'(1) = -4$; the gradient of the wheel at $x = 1$ is -4.

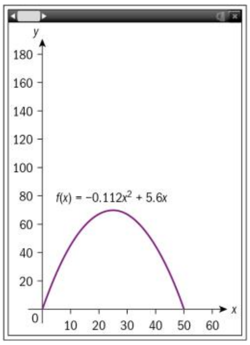
b The gradient of the spoke is therefore $-\frac{1}{-4} = \frac{1}{4}$.

4 First find the gradient m of the tangent to $f(x)$ at $x = 1$: $f'(x) = 6x - 4$, so $m = f'(1) = 2$. The normal at this point has gradient of $-\frac{1}{m} = -\frac{1}{2}$. Therefore, the equation of the normal at $(1,4)$ is $y - 4 = -\frac{1}{2}(x-1) \Rightarrow y = -\frac{x}{2} + \frac{9}{2}$.

5 At $x = 2, y = 2^4 - 6 \times 2 + 3 = 7$. The derivative of y is $y'(x) = 4x^3 - 6$, so the gradient of the tangent at $x = 2$ is $m = 4(2^3) - 6 = 26$. The tangent, therefore, has equation $y - 7 = 26(x-2) \Rightarrow y = 26x - 45$ and the normal (which has gradient $-1/m$) has equation $y - 7 = -\frac{1}{26}(x-2) \Rightarrow y = \frac{92}{13} - \frac{x}{26}$.

6 $f'(x) = 2x$, so the gradient of the tangents at $x = 2, x = -2$ are $m_1 = 4, m_2 = -4$ (respectively). The equations of the normals are, therefore, $y - 2^2 = -\frac{1}{4}(x-2) \Rightarrow y = -\frac{1}{4}x + \frac{9}{2}$ and $y - 22 = \frac{1}{4}(x+2) \Rightarrow y = \frac{1}{4}x + \frac{9}{2}$. The normals therefore meet at $x = 0$ (by setting the two normals equal to each other); at this point $y = \frac{9}{4}$: the fountain will be placed at $\left(0, \frac{9}{4}\right)$.

7 **a**



b $f(15) = 58.8 = f(35)$. Also, $f'(x) = -0.224x + 5.6$, so $f'(15) = 2.24, f'(35) = -2.24$. Therefore, the normal at $x = 15$ has equation $y - f(15) = -\frac{1}{f'(15)}(x - 15) \Rightarrow y = 65.5 - 0.446x$ and the normal at $x = 35$ has equation $y - f(35) = -\frac{1}{f'(35)}(x - 35) \Rightarrow y = 43.1 + 0.446x$.

c The normal meet where $43.1 + 0.446x = 65.5 - 0.446x \Rightarrow 22.4 = 0.892x \Rightarrow x = 25.1$ (3 s.f.). At this point, $y = 43.1 + 0.446 \times 25 = 54.3$.

d Yes, position is within the park.

8 $f'(x) = 2ax + 3$. Since $f'(2) = 7$, then $4a + 3 = 7 \Rightarrow a = 1$. Then $b = f(2) = a \times 2^2 + 3 \times 2 - 1 = 9$.

9 First find k using the fact that $f'(1) = 2 \times 1 + k = 3 \Rightarrow k = 1$ (since $f'(x) = 2x + k$). Then $b = f(1) = 1^2 + k + 3 = 5$.

10 Since $y = -2$ when $x = 1$, then we must have $-2 = a + b + 1$. Also, the gradient of the tangent at $x = 1$ is $y'(1) = 2a + b$ (since $y'(x) = 2ax + b$). Therefore, the normal at $x = 1$ has gradient $-\frac{1}{y'(1)} = -\frac{1}{2a + b}$ and hence $1 = -\frac{1}{2a + b} \Rightarrow 2a + b = -1$. We need to simultaneously solve $-2 = a + b + 1$ and $2a + b = -1$; the solution is $a = 2, b = -5$.

Exercise 12D

1 **a** $y'(3) = \frac{3}{4}$ **b** $y'(3) = 1 + \ln 3 = 2.10$ (3 s.f.). **c** $f'(3) = \frac{55}{36}$

d $y'(3) = \frac{-47}{196}$ **e** $y'(3) = 7e^6 = 2824$ **f** $g'(3) = 672$

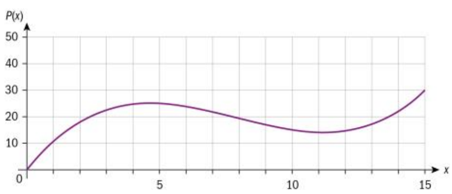
Exercise 12E

1 **a** $f'(t) = 7.25 - 2 \times 1.875t = 7.25 - 3.75t$

b At the stationary point, $f'(t) = 7.25 - 3.75t = 0 \Rightarrow t = 1.93$ (3 s.f.). At this time, $f(t) = 1 + 7.25 \times 1.93 - 1.875 \times (1.93)^2 = 8.01$ (3 s.f.): the stationary point is at $(1.93, 8.01)$.

c When $t < 1.93$, then $3.75t < 7.25$ so $f'(t) > 0$ and when $t > 1.93$, then $3.75t > 7.25$, so $f'(t) < 0$, hence $t = 1.93$ s is a maximum.

2 **a**



b $P(2) = 0.08 \times 2^3 - 1.9 \times 2^2 + 15.2 \times 2 = 18.04, P(3) = 0.08 \times 3^3 - 1.9 \times 3^2 + 15.2 \times 3 = 22.56$. The gradient of the chord between $(2, P(2))$ and $(3, P(3))$ is therefore $m = \frac{22.56 - 18.04}{3 - 2} = 4.52$; the average rate of change between $x = 2$ and $x = 3$ is 4.52 thousands of dollars per million units sold.

c $P'(x) = 0.24x^2 - 3.8x + 12.5$, so $P'(3) = 3.26, P'(8) = -2.54, P'(13) = 3.66$. These represent the instantaneous rate of change of profit with respect to units sold.

d The instantaneous rate of change is negative when $P'(x) < 0$ and positive when $P'(x) > 0$. The equation $P'(x) = 0$ has solutions $x = 4.66, 11.2$. Therefore, the instantaneous rate of change is negative when $4.66 < x < 11.2$ and instantaneous rate of change is positive when $x < 4.66$ and $x > 11.2$.

This means that profit increases with more sales when $x < 4.66$ and $x > 11.2$ but profit will decrease with more sales when $4.66 < x < 11.2$.

e The instantaneous rate of change is zero when $x = 4.66, 11.2$.

f At the points where $P'(x) = 0$ (i.e. $x = 4.66, 11.2$), then $P(x) = 25.1, 14.1$ (respectively). We can see from the sketch that the gradient function $P'(x)$ changes sign at these points, i.e. they are indeed (local) maxima and minima.

3 The maximum height is $f = 4$ at $t = 6$.

4 Find the stationary points of the profit function by solving $P'(n) = -0.112n + 5.6 = 0 \Rightarrow n = 50$. (Note that when $n \langle 50, P'(n) \rangle 0$ and when $n > 50, P'(n) < 0$ so this is indeed a maximum). At this point, the profit is $P(50) = \text{US\$}120$.

5 a i $P(n) = 0.5n + 1.5 + \frac{4}{n+1}$: the stationary points of $P(n)$ occur (for $0 < n < 5$) at $n = 1.82$. This is a local minimum and there are no other stationary points for $0 < n < 5$, so the maximum profit occurs at either $n = 0$ or $n = 5$ (in this range). Since $P(0) = 5.5, P(5) = 4.67$ then, under this model, they should buy no parts to maximise profit! (The profit will be 55000 EUR).

ii $P(n) = \frac{n^3}{3} - \frac{5n^2}{2} + 6n - 4$: the stationary points of $P(n)$ occur (for $0 < n < 5$) at $n = 2$ (local maximum) and $n = 3$ (local minimum), with $P(2) = \frac{2}{3}, P(3) = \frac{1}{2}$. Also, $P(0) = -4, P(5) = \frac{31}{6}$. Therefore, under this model, the profit is maximised by buying 5000 parts, which gives a profit of 51667 EUR.

iii $P(n) = \frac{n^3}{24} - \frac{5n^2}{8} + 3n$: the stationary point of $P(n)$ occurs (for $0 < n < 5$) at $n = 4$ (local maximum), with $P(4) = \frac{14}{3}$. There are no other turning points in $0 < n < 5$ so this local maximum is at the maximum value of $P(n)$ on $0 < n < 5$. Therefore, under this model, the factory should by 4000 parts, which gives a profit of 46666 EUR.

b They should adopt the first strategy.

6 $y'(x) = -0.324x^3 + 2.67x^2 - 5.74x + 3$. By solving $y'(x) = 0$, we find stationary points at $x = 0.776$ (local max), $x = 5.15$ (local max) and $x = 2.31$ (local min). Since $y(0.776) = 0.986, y(5.15) = 3.92$ then we determine that the maximum height on the route is 392.

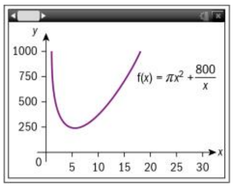
Exercise 12F

1 **a** The volume of a cylinder is the product of its cross sectional area (in this case πr^2) and its height h, therefore, as the volume is 400 cm^3, we have $400 = \pi r^2 h$.

b The surface area of the curved surface is $A_c = 2\pi rh$ and the area of the base is $A_b = \pi r^2$. Hence the total surface area is $A = A_c + A_h = 2\pi rh + \pi r^2$.

c Using part **a**, we can expressed A_c as $A_c = \frac{2(\pi r^2 h)}{r} = 2 \times \frac{400}{r} = \frac{800}{r}$. Using this form, the total surface area is $A = \pi r^2 + \frac{800}{r}$.

d



e The minimum area is $\min A = 239$ at $r = 5.03$.

f This can be verified graphically.

2 Using the same method as q1: the surface area of a closed cylinder of radius r and height h is $A = 2\pi rh + 2\pi r^2$, and the volume is $V = \pi r^2 h$. If we're given that the total surface area is 5000 cm^2, we can express h in terms of r: $A = 5000 = 2\pi rh + 2\pi r^2 \Rightarrow h = \frac{5000 - 2\pi r^2}{2\pi r}$. Hence, the volume can be expressed only in terms of the radius as $V(r) = \frac{r(5000 - 2\pi r^2)}{2}$.

$V(r)$ has a maximum of $V = 27145$ at $r = 16.3$, at 16 the gradient is positive and at 17 it is negative, so $r = 16.3$ is a maximum.

3 **a** The perimeter is $p = 100 = 2x + 2l$, where l is the length of the garden. Therefore, $l = 50 - x$.

b The area is the product of the length and the width, i.e. $A = xl = x(50 - x)$ m^2.

c $\frac{dA}{dx} = 50 - 2x$

d $\frac{dA}{dx} = 0$ when $x = 25$. This is indeed a maximum as $\frac{dA}{dx} > 0$ when $x < 25$ and $\frac{dA}{dx} < 0$ when $x > 25$. Therefore, the maximum area of the grass is $A = 25^2 = 625\text{m}^2$, which occurs when the garden is a $25\text{ m} \times 25\text{ m}$ square.

4 The volume of a cone with radius r and height $h = 18 - r$ is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (18 - r)$.

$V'(r) = \frac{1}{3}\pi(36r - 3r^2)$, so $V'(r) = 0$ has solutions $r = 0, \pm 12$. We restrict ourselves to $r > 0$, so the only turning point is $r = 12$. This is a maximum because $V'(r) < 0$ when $r > 12$ and $V'(r) > 0$ when $0 < r < 12$.

Therefore, the maximum volume of the cone is $V = V(12) = 288\pi = 905\text{cm}^3$, when $r = 12$ (3 s.f.).

5 **a** We can imagine that, after removing the squares, the sides of the rectangle are split into three pieces, which have length x, x and $20 - 2x$ on one side (the first two correspond to the removed sections, and the latter to the remaining section), and x, x and $24 - 2x$ on the other side. The resulting box therefore has a base of size $20 - 2x \times 24 - 2x$ and height x; the volume of the box is $V = x(20 - 2x)(24 - 2x)$.

b Expanding, we have $V = 4x^3 - 88x^2 + 480x$, and hence $V'(x) = 12x^2 - 176x + 480$. The stationary points of V are at $V'(x) = 0$, which has two solutions in the range $0 < x < \frac{24}{2}$ at $x = 3.62, 11.0$. The second of these corresponds to a negative volume, so is ignored. The former is a local maximum with $V(3.62) = 774.16$. Since $V(0) = 0 = V(24)$, this is a maximum on the interval of interest: $0 < x < 12$.

The value of x which maximises the volume is $x = 3.62\text{cm}$ which provides a volume of $V = 774\text{cm}^3$.

6 a The volume is $V = \pi r^2 h$

b The surface area of the curved part is $A_c = 2\pi rh$ and the surface area of the ends are each $A_e = \pi r^2$. Therefore, the total surface area is $A = A_c + 2A_e = 2\pi r^2 + 2\pi rh$.

We can use the volume constraint to write the r in terms of h: $300 = \pi r^2 h \Rightarrow h = \frac{300}{\pi r^2}$, so

$A = 2\pi r^2 + \frac{600\pi r}{\pi r^2} = 2\pi r^2 + \frac{600}{r}$

c $A'(r) = 4\pi r - \frac{600}{r^2}$, so the only stationary point is at $r = r_1 = \sqrt[3]{\frac{150}{\pi}}$, and this is a local minimum (this can be seen by, for example, plotting the graph of $A(r)$). Therefore, the minimum surface area is $A(r_1) = 248\ \text{cm}^2$ which occurs at $r = r_1 = 3.63\ \text{cm}$ and $h = 7.25$.

7 $P'(n) = -0.184n + 33.3$, so the only stationary point of P in $n > 0$ occurs at $n = \frac{-33.3}{0.184} = 180.9$.

This is a maximum point of the function P (a quadratic with a negative leading co-efficient has a single global maximum at its turning point), but the quantity n can only take integer values. The maximum profit is therefore attained at the next largest or next smallest integer to $n = 180.9$. We calculate $P(180) = 2700.2 < 2700.29 = P(181)$: the maximum profit is \$2700.29, when $n = 181$ goods are sold per day.

8 $f'(x) = -1.8x + 52$, so the only turning point of f occurs when $x = x_1 = \frac{52}{1.8} = 28.9$. This is a maximum because $f'(x) > 0$ for $x < x_1$ and $f'(x) > 0$ for $x > x_1$. However, x is an integer, so the maximum is attained at $x = 28$ or $x = 29$. Since $f(28) = 390.4 < 391.1 = f(29)$, we conclude the maximum profit is $f(29) = 391.10$ USD, when 29 units are sold.

Chapter Review

1 a $f'(x) = 0$, the tangent to f at $x = 1$ has gradient $m = 0$.

b $y'(x) = 3$, the tangent to y at $x = 1$ has gradient $m = 3$.

c $g'(x) = 4x - 4$, the tangent to f at $x = 1$ has gradient $m = 0$.

d $y'(x) = 18x^2 - 6x + 1$, the tangent to f at $x = 1$ has gradient $m = 13$.

e $f'(x) = -\frac{2}{x^2} + 3$, the tangent to f at $x = 1$ has gradient $m = -2 + 3 = 1$.

f $f'(x) = -\frac{18}{x^4} + 4x$, the tangent to f at $x = 1$ has gradient $m = -18 + 4 = -14$.

2 First note $f(4) = 0.5 \times 4^2 - 3 \times 4 + 2 = -2$, so a point on both the normal and the tangent is $P(4, -2)$. Since $f'(t) = t - 3$, then $f'(4) = 1$; the gradient of the tangent at P is 1 and the gradient of the normal at P is −1. Hence, the normal at P has equation $y + 2 = -1(x - 4) \Rightarrow y = 2 - x$ and the tangent at P has equation $y + 2 = x - 4 \Rightarrow y = x - 6$.

3 Since $f'(x) = 2x - 5$, to find the x-coordinate of the point A, when the gradient of the tangent to f is 1, we need to solve $f'(x) = 2x - 5 = 1 \Rightarrow x = 3$. The corresponding y co-ordinate is $f(3) = -10$, so A has co-ordinates of $(3, -10)$.

4 Let (x_1, y_1) be the co-ordinates of B. Note that $f'(x) = 6x + 4$ so the normal to the curve at B has gradient of $-\dfrac{1}{6x_1 + 4}$, and we're given that this has to equal $\dfrac{1}{2}$, so $6x_1 + 4 = -2 \Rightarrow x_1 = -1$.

Then $y_1 = f(x_1) = 3 - 4 - 3 = -4$.

5 **a** $f'(t) = -1.667 + 0.0834t$

b $f'(12) = -0.666$ travelling downhill, $f'(32) = 1.00$ travelling uphill

c The stationary points are where $f'(t) = 0 \Rightarrow 0.0834t = 1.667 \Rightarrow t = 20.0$ s (this is the time at which Jacek is at the minimum point on the track). This is a minimum point because $f'(t) < 0$ when $t < 20$ and $f'(t) > 0$ when $t > 20$.

6 **a** The volume of a cylinder of radius r and height h is $V = \pi r^2 h$. We're given that $V = 300$ cm^3, so $300 = \pi r^2 h$.

b $h = \dfrac{300}{\pi r^2}$, The surface area of the curved part is $A_c = 2\pi rh$ and the surface area of each of the ends is $A_e = \pi r^2$. Hence, the total surface area is $S = A_c + 2A_e = 2\pi rh + 2\pi r^2$.

c Using the expression from part **a**, $S = 2\pi r \times \dfrac{300}{\pi r^2} + 2\pi r^2 = 2\pi r^2 + \dfrac{600}{r}$.

d $\dfrac{dS}{dr} = 4\pi r - \dfrac{600}{r^2}$

e The only stationary point of S is where $4\pi r^3 = 600 \Rightarrow r = \sqrt[3]{\dfrac{150}{\pi}} = 3.62$ cm. The corresponding surface area is $S = 248$ cm^3 and $h = \dfrac{300}{\pi r^2}$ = 7.26 cm. (all 3 s.f.)

7 **a** $\dfrac{dy}{dx} = -0.1x + 1.5$	M1A1
b Setting $\dfrac{dy}{dx} = 0$	M1
$-0.1x + 1.5 = 0$	
$x = 15$	A1
$y = -0.05 \times 15^2 + 1.5 \times 15 + 82 = 93.25$ m	A1
c Evaluating $\dfrac{dy}{dx}$ at $x = 14.5$ and $x = 15.5$	M1
$\dfrac{dy}{dx} = 0.05$ and $\dfrac{dy}{dx} = -0.05$	A1
Sign goes from positive to negative, therefore a maximum point	R1
8 **a** $V = x(40 - 2x)(30 - 2x)$	M1A1
$= x(1200 - 60x - 80x + 4x^2) = x(1200 - 140x + 4x^2)$	A1
$= 1200x - 140x^2 + 4x^3$	
b $\dfrac{dV}{dx} = 1200 - 280x + 12x^2$	M1A1

c Setting $\frac{dV}{dx} = 0$ M1

$1200 - 280x + 12x^2 = 0$ A1

$12x^2 - 280x + 1200 = 0$

$3x^2 - 70x + 300 = 0$

$x^2 - \frac{70}{3}x + 100 = 0$

d Using GDC to solve $x^2 - \frac{70}{3}x + 100 = 0$ M1

$x = 5.66\text{ cm}$ A1

$V_{max} = 1200 \times 5.657 - 140 \times 5.657^2 + 4 \times 5.657^3$ M1

$= 3032\ \text{cm}^3$ (3030 to 3 s.f.) A1

9 **a** Use of GDC (demonstrated by one correct value) M1

$a = -0.0534$ A1

$b = 1.09$ A1

$c = 7.48$ A1

$T = -0.0534h^2 + 1.09h + 7.48$

b $\frac{dT}{dh} = -0.107h + 1.09$ M1A1

c Setting $\frac{dT}{dh} = 0$ M1

$h = 10.2$ A1

d The maximum temperature usually occurs after midday, whereas this is only 10 hours after midnight. R1

10 $(-1.59, -13.2)$ A1A1

$(0.336, 3.81)$ A1A1

$(1.17, 1.97)$ A1A1

11 **a** $\frac{dy}{dx} = 2x^2 - 7x + 2$ M1A1

$\frac{dy}{dx} = -3$ M1

$2x^2 - 7x + 2 = -3$

$2x^2 - 7x + 5 = 0$

$(2x - 5)(x - 1) = 0$ M1

$x = \frac{5}{2}$ $y = -\frac{35}{24}$ A1

$x = 1$ $y = \frac{25}{6}$ A1

b $0.314 < x < 3.19$ A1A1

12 At $x = 1$, $y = \frac{3}{2}$ A1

$\frac{dy}{dx} = -2x^3$ M1A1

At $x = 1$ $\frac{dy}{dx} = -2$ A1

gradient of the normal is therefore $\frac{1}{2}$ M1

Equation of the normal is therefore $y-\frac{3}{2}=\frac{1}{2}(x-1)$ M1

$y-\frac{3}{2}=\frac{1}{2}x-\frac{1}{2}$

$y=\frac{1}{2}x+1$ A1

13 Substituting $(2,-1)$ gives M1

$-1=4a+2b+3$ A1

$4a+2b=-4$

$2a+b=-2$

$\frac{dy}{dx}=2ax+b$ M1

$8=4a+b$ A1

Solving simultaneously gives M1

$a=5$ A1

$b=-12$ A1

14 a $y=20-x$

$xy=x(20-x)=20x-x^2$ M1

Differentiate and set to zero: M1

$20-2x=0$ A1

$x=10$ A1

So $xy_{MAX}=100$ A1

b $x^2+y^2=x^2+(20-x)^2$ M1

$=x^2+x^2-40x+400$

$=2x^2-40x+400$

Differentiate and set to zero: M1

$4x-40=0$ A1

$x=10$ A1

So $(x^2+y^2)_{MAX}=10^2+10^2=200$ A1

c $4\times 9.5-40=-2$ A1

$4\times 10.5-40=+2$ A1

The derivative goes from negative to positive, therefore this is a maximum R1