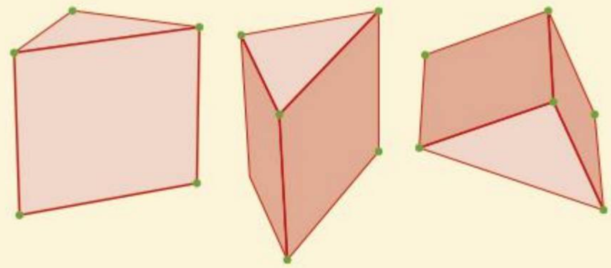
7.3 Representing combined probabilities with diagrams and formulae

In Section 7.2 you found probabilities by representing combined events in a sample space diagram or a Venn diagram. There are other ways to find probabilities of combined events, which can add to your problem-solving skills.

Investigation 8

1 Retrieve your data from the triangular prism investigation in Section 7.1.

2 **Collaborate** in a pair to use your 100 trials to find experimental probabilities of the events A, B, C, D and E of the prism falling face down on these faces respectively.



3 In a pair, each put your own data for your 50 trials into one of the columns of a table:

First outcome	Second outcome	Combined outcome
B	C	BC
A	A	AA
E	C	EC
C	B	CB
...	...	...
...	...	...

4 Would the event AA be as likely as or less likely than the event A? Use subjective probability and reasoning to justify your answer.

5 **Factual** What is the experimental probability of the combined event AA?

6 **Factual** What is the relationship between your answer to **5** and the experimental probability of A? Is this relationship exact or approximate?

7 Repeat steps **4** and **5** for other combined events such as BD etc. If necessary, you may wish to collaborate by sharing data from the whole class on an online spreadsheet.

8 **Conceptual** Can you generalize your findings by completing: "The probability of event X and event Y **both** occurring is ..."

(Your generalization is true only for **independent** events, which are explored later in this section.)

9 **Conceptual** By what other methods, apart from diagrams, can combined probabilities be found?

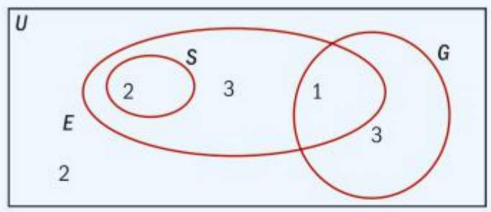
Statistics and probability

In this section you will use Venn diagrams to investigate and represent laws of probability and you will use this language, these symbols and definitions:

Name	Symbol applied to event(s)	Informal language	Formal definition
Intersection	$A \cap B$	"A and B"	Events A and B **both** occur
Union	$A \cup B$	"A or B"	Events A **or** B **or both** occur
Complement	A'	"Not A"	Event A **does not** occur
Conditional	$A\|B$	"A given B"	Event A occurs, **given that** event B **has** occurred

Example 9

A student is chosen at random from this class. If E is the event "the student takes ESS" and G is the event "the student takes geography", then find these probabilities and interpret what they mean:



a $P(E \cap G)$ and $P(G \cap E)$ **b** $P(E \cup G)$ and $P(G \cup E)$

c $P(E')$ **d** $P(E|G)$ and $P(G|E)$.

a $P(E \cap G) = P(G \cap E) = \frac{1}{11}$ is the probability that a randomly chosen student studies **both** ESS and geography.

Only one student takes both ESS and geography. This example illustrates that $E \cap G$ means exactly the same as $G \cap E$. In general, such terms are always the same.

b $P(E \cup G) = P(G \cup E) = \frac{2+3+1+3}{11} = \frac{9}{11}$ is the probability that a randomly chosen student studies ESS **or** geography **or both**.

Similarly, $E \cup G$ means the same as $G \cup E$ hence $P(E \cup G) = P(G \cup E)$ is always true.

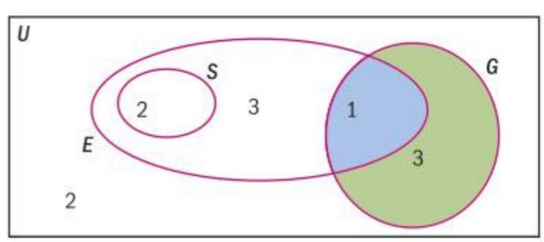
c $P(E') = \frac{5}{11}$ is the probability that a randomly chosen student **does not** study ESS.

There are 5 students outside the ESS oval. $P(E') = 1 - \frac{6}{11} = \frac{5}{11}$ is another way to find the probability required.

d $P(E|G) = \frac{1}{1+3} = \frac{1}{4}$, the probability that a randomly chosen students studies ESS **given** that he/she studies geography,

whereas $P(G|E) = \frac{1}{2+3+1} = \frac{1}{6}$.

Since it is **given** that G has occurred, the sample space is now G, not U.



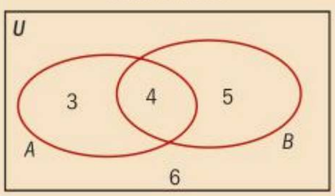
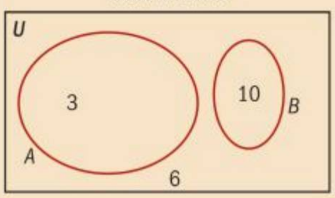
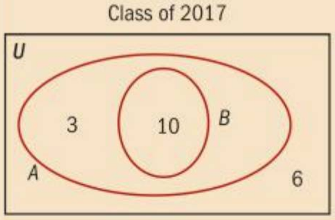
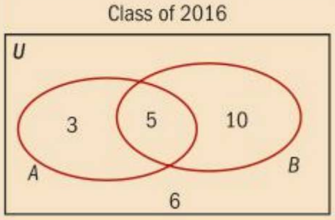
These are not equal since the information **given** changes the sample space. This example shows that the statement $\text{P}(E\mid G) = \text{P}(G\mid E)$ is not generally true.	Only 1 student studies ESS and geography hence $\text{P}(E\mid G) = \frac{1}{4}$. Notice how this contrasts with $\text{P}(E) = \frac{6}{11}$.

Just as areas of mathematics like trigonometry or sequences have formulae, so does probability. In this investigation you will see some relationships that you can generalize as **laws of probability**.

Investigation 9

The following Venn diagrams represent how many students study art or biology in four different classes, using the sets A and B.

Fill in the probabilities for each Venn diagram and investigate your answers.

	1	2	3	4	5	6	7	8
Venn diagram	$\text{P}(A)$	$\text{P}(A')$	$\text{P}(A)+\text{P}(A')$	$\text{P}(B)$	$\text{P}(A)+\text{P}(B)$	$\text{P}(A\cup B)$	$\text{P}(A\cap B)$	$\text{P}(A)+\text{P}(B)-\text{P}(A\cap B)$
Class of 2019 								
Class of 2018 								
Class of 2017 								
Class of 2016 								

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Answer these questions and discuss your answers in a group.

1 What relationship is shown in column 3?
2 This relationship is true in general. Why?
3 What relationship exists between the probabilities in columns 6 and 8?
4 This relationship is true in general. Why?
5 **Factual** How can you tell that two events cannot **both** occur?
6 Which class shows mutually exclusive events?
7 For which class can the relationship you wrote for question **3** be simplified?
8 Complete the table to summarize and justify what you have discovered.

Left-hand side of a law of probability	=	Right-hand side	When true?
$P(A \cup B)$	=	$P(A) + P(B) -$ ____	Always
$P(A \cap B)$	=	0	Only if A and B are ____ events.
Therefore, $P(A \cup B)$	=	$P(A) + P(B)$	

Two events A and B are **mutually exclusive** if they cannot **both** occur. Hence:

- Knowing that A has occurred means you know that B cannot, and that knowing that B occurs means you know that A cannot.
- The occurrence of each event **excludes** the possibility of the other.
- Consequently, $P(A \cap B) = P(B \cap A) = 0$.

A and A' are **complementary** events. This means that

- A and A' are mutually exclusive.
- $P(A) + P(A') = 1$

TOK

When watching crime series, reading a book or listening to the news, the evidence of DNA often closes a case. If it was that simple, no further detection or investigation would be needed. Research the "Prosecutor's fallacy".

How will reason contrast with emotion in making a decision based solely on DNA evidence?

Laws of probability can be used in the calculation of probabilities of combined events and also to make and justify statements about combined events.

Example 10

Catarina explores the names of the students in her class of 15 students.

She sorts through the data from all 15 students and determines the sets A and B:

A = {Students with exactly two vowels in their given name}
= {Clara, Tomas, Fanygu, Rea, James}

B = {Students with exactly three vowels in their given name}
= {Barbora, Achille, Malena, Daniel, Oliver, Rikardo}

a Represent Catarina's information on a Venn diagram.

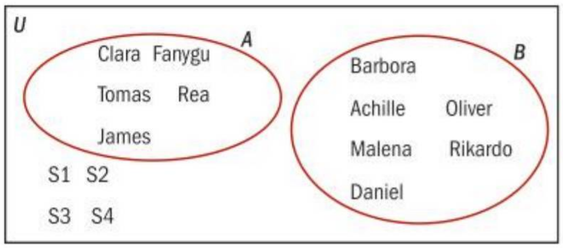
A student from the class is chosen at random.

b Find $P(A)$, $P(B)$ and $P(A \cap B)$.

c State with a reason whether events A and B are mutually exclusive.

d Hence write down $P(A \cup B)$.

a



The sample space has 15 students; however, you are only given 11 names. Hence the remaining students have been denoted S1, S2, S3 and S4 for completeness.

b $P(A) = \dfrac{5}{15} = \dfrac{1}{3}$, $P(B) = \dfrac{6}{15} = \dfrac{2}{5}$ and $P(A \cap B) = 0$

c The events A and B are mutually exclusive since $P(A \cap B) = 0$.

Having exactly two vowels in a given name **excludes** the possibility that there are exactly three vowels.

d $P(A \cup B) = \dfrac{5}{15} + \dfrac{6}{15} = \dfrac{11}{15}$

Apply the addition law of probability for mutually exclusive events.

Example 11

Catarina further explores the number of siblings of the students in her class of 15 students.

She writes down another set in addition to those from Example 10:

C = {Students with at least one sibling} = {Alexandra, Isabella, Lukas, Paula, Oliver, Rikardo}

a Draw sets B and C on a Venn diagram.

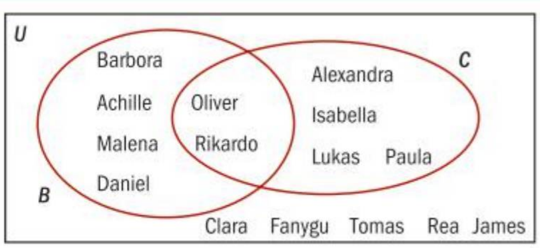
A student from the class is chosen at random.

b Find $P(B)$, $P(C)$ and $P(B \cap C)$.

c State with a reason whether events B and C are mutually exclusive.

d Hence write down $P(B \cup C)$.

a



All 15 students can now be represented in the Venn diagram.

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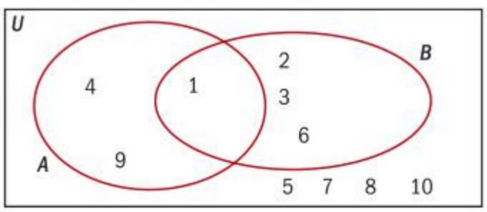
b $P(B) = \frac{6}{15} = \frac{2}{5}$, $P(C) = \frac{6}{15} = \frac{2}{5}$ and $P(B \cap C) = \frac{2}{15}$	
c The events B and C are not mutually exclusive since $P(B \cap C) \neq 0$.	Having three vowels in your given name **does not exclude** the possibility that you have at least one sibling.
d $P(B \cup C) = \frac{6}{15} + \frac{6}{15} - \frac{2}{15} = \frac{10}{15} = \frac{2}{3}$	Apply the addition law for events that are not mutually exclusive.

Exercise 7F

1 A fair decahedral die numbered 1, 2, 3, …, 10 is thrown and the number noted.

The events A, "throw a square number", and B, "throw a factor of six", are represented on the Venn diagram below:



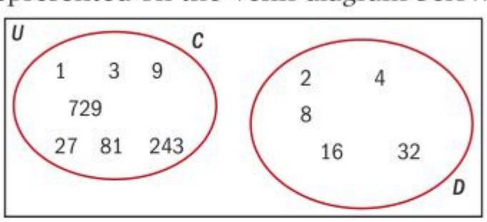
a Find $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$.

b Hence show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

c State with a reason whether events A and B are mutually exclusive.

2 A fair dodecahedral die numbered 1, 2, 3, 4, 8, 9, 16, 27, 32, 81, 243 and 729 is thrown and the number noted.

The events C, "throw an odd number", and D, "throw an even number", are represented on the Venn diagram below:



a Find $P(C)$, $P(D)$, $P(C \cap D)$ and $P(C \cup D)$.

b Hence show that $P(C \cup D) = P(C) + P(D)$.

c State with a reason whether events C and D are mutually exclusive.

3 A school is inspecting 24 student lockers before the start of the new academic year to see if they have been left tidy. It is found that some lockers have some food items left inside and some lockers have stationery items inside. Lockers 2, 5, 7, 8, 11, 17, 18 and 19 all have food items and lockers 1, 3, 4, 11, 13, 15, 17, 20 and 21 all have stationery items.

a Draw this information on a Venn diagram.

b State with a reason whether events "a randomly chosen locker contains foot items" and "a randomly chosen locker contains stationery items" are mutually exclusive.

c Find the probability that a locker selected at random has at least one type of item left inside.

4 Finn explores the ages of the people in his family. He represents his family's ages with the set $U = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 25, 35, 55, 65\}$.

a Draw U and each of the following sets on a Venn diagram:

$A = \{\text{even numbers}\}$, $B = \{\text{multiples of } 3\}$, $C = \{\text{prime numbers}\}$ and $D = \{\text{numbers greater than } 30\}$.

b Finn chooses a family member at random. Use your diagram to determine which, if any, of A, B, C or D can form a mutually exclusive pair of events.

You have learned the concept of **mutually exclusive events**. In the following investigation you learn about **independent events**. These terms are easy to confuse with each other.

Two events A and B are **independent** if the occurrence of each event does **not affect in any way** the occurrence of the other.

Equivalently, **knowing** that A has occurred does not affect the probability that B does, and **knowing** that B occurs does not affect the probability that A does.

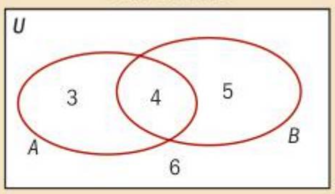
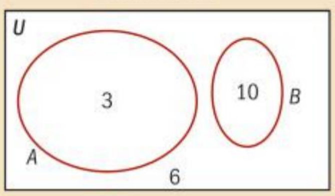
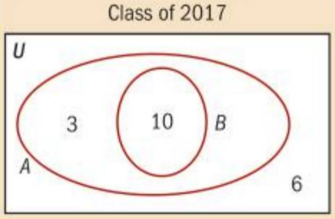
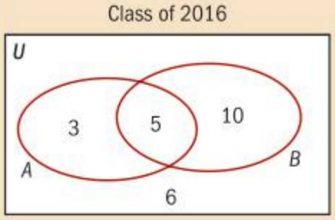
Two events A and B are **dependent** if they are not independent.

TOK

What do we mean by a "fair" game? Is it fair that casinos should make a profit?

Investigation 10

Fill in the probabilities for each Venn diagram and investigate your answers.

	1	2	3	4	5	6
Venn diagram	$P(A)$	$P(B)$	$P(A \cap B)$	$P(A \vert B)$	$P(A \vert B)P(B)$	$\frac{P(A \cap B)}{P(B)}$
Class of 2019 						
Class of 2018 						
Class of 2017 						
Class of 2016 						

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Answer these questions and discuss your answers in a group.

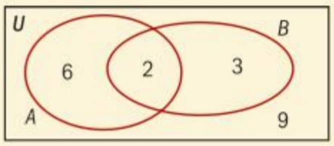
1 What relationship exists between the probabilities in columns 4 and 6? This relationship is true in general.

2 In which of the four cases above does **knowing** that a student studies biology **affect** the probability that they study art? How can you tell?

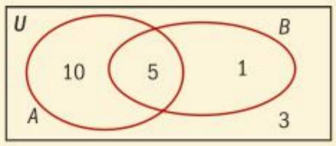
3 In which of the four cases above does **knowing** that a student studies biology **not** affect the probability that they study art? How can you tell?

Use your answers to **2** and **3** to find which of the following Venn diagrams represent pairs of independent events.

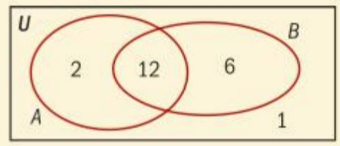
Class of 2015



Class of 2014



Class of 2013



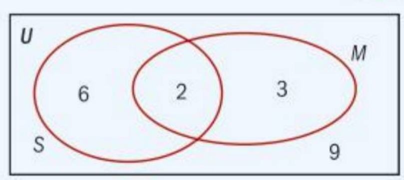
4 Complete the table to summarize and justify what you have discovered.

Left-hand side of a law of probability	=	Right-hand side	When true?
$P(A \mid B)$	=	$\frac{P(A \cap B)}{\_\_\_}$	Always
Therefore, $P(A \cap B)$	=		Always
$P(A \mid B)$	=	$P(A)$	Only if A and B are __________ events
Therefore, $P(A \cap B)$	=	$P(A) \times$ ______	Only if A and B are __________ events

These laws of probability can be used in making and justifying statements about combined events.

Example 12

This Venn diagram shows the number of students in a class who study Spanish and the number of students who study mathematics.



Determine whether S and M are independent events by:

a calculating and considering the values of $P(S)$ and $P(S \mid M)$

b calculating and considering the values of $P(S) \times P(M)$ and $P(S \cap M)$.

a $P(S) = \frac{6+2}{6+2+3+9} = \frac{8}{20} = \frac{2}{5}$ $P(S \mid M) = \frac{2}{2+3} = \frac{2}{5}$	This quantifies the probability that a randomly chosen student from the **entire class** studies Spanish. $P(S \mid M)$ quantifies the probability that a randomly chosen student from the **students who study mathematics** studies Spanish.

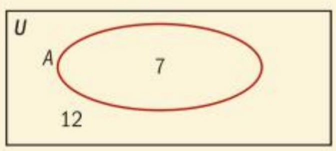
Since $P(S) = P(S\|M)$, S and M are independent events.	Knowing that the randomly chosen student studies mathematics does not affect the probability that the student studies Spanish.
b Since $P(S) \times P(M) = \frac{2}{5} \times \frac{5}{20} = \frac{2}{20} = P(S \cap M)$, S and M are independent events.	This is an equivalent way to show the events are independent.

Now that you have learned about mutually exclusive events and independent events, you can gain knowledge and understanding about how these terms differ.

Investigation 11

At each stage, compare and contrast your ideas and results with a classmate.

1 Find the probabilities:



$P(A) =$		
$P(A') =$		
$P(A \cap A') =$		
$P(A\|A') =$		

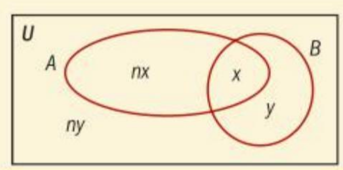
2 **Conceptual** Are complementary events always mutually exclusive? How do you know?

3 **Conceptual** Are complementary events always independent events? How do you know?

4 Identify a Venn diagram from the previous investigation that represents two mutually exclusive events that are not independent.

5 Identify a Venn diagram from the previous investigation that represents two independent events that are not mutually exclusive.

Write down your own three positive integers x, y and n and use them to fill in this Venn diagram, which represents two events A and B.



6 **Factual** Are A and B mutually exclusive? Are they independent?

7 **Conceptual** Can non-mutually exclusive events be independent?

Adapt your Venn diagram by replacing y with $y + 1$ but leaving ny, x and nx alone.

8 **Factual** Are A and B mutually exclusive? Are they independent?

9 **Conceptual** Can non-mutually exclusive events be dependent?

10 **Factual** If C represents the event "throw a 2 on a cubical die" and D the event "throw an odd number on a cubical die", are these events mutually exclusive? Are they independent?

Given the event C, write down your own event that excludes the possibility of C happening.

11 **Factual** Are your events independent?

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Statistics and probability

12 **Conceptual** Can mutually exclusive events be independent?

13 **Factual** If D represents the event "toss a head on a fair coin", would C and D be mutually exclusive? Would they be independent?

Given the event C, write down your own event that is independent of C happening.

14 **Factual** Are your events mutually exclusive?

15 **Conceptual** Must independent events be mutually exclusive?

16 **Factual** Choose the correct word to complete each sentence:

Complementary events are	sometimes / never / always	mutually exclusive events
Dependent events are		independent events
Mutually exclusive events		dependent events
Independent events are		mutually exclusive events

You can use the laws of probability to justify other statements.

Example 13

For a consumer survey, 2371 adults are asked questions. An adult is chosen at random from those taking part.

C is the event "the adult likes coffee".

D is the event "the adult is called David".

a You are given that C and D are independent events. Explain why this is the case.

b Given that $P(C) = 0.8$ and $P(D) = 0.007$, interpret your answers in context in each of the following. Find:

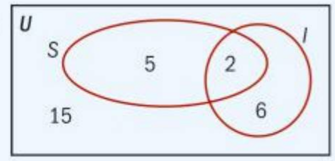
i $P(C \cap D)$ **ii** $P(C \cup D)$.

c Determine whether C and D are mutually exclusive. Justify your answer.

a Having any given name does not affect the probability of liking coffee in any way.	Recall and apply the definition of independent events. You could also say liking coffee does not affect the probability of having any particular name.
b **i** $P(C \cap D) = 0.8 \times 0.007 = 0.0056$ Choosing a coffee drinker called David is less likely than choosing a David and less likely than choosing a coffee drinker.	Since you are given that C and D are independent, you can use $P(C \cap D) = P(C) \times P(D)$.
ii $P(C \cup D) = 0.8 + 0.007 - 0.0056$ $= 0.8014$ Choosing a coffee drinker or someone called David is more likely than choosing a David and more likely than choosing a coffee drinker.	Apply the formula: $P(C \cup D) = P(C) + P(D) - P(C \cap D)$.
c C and D are not mutually exclusive because $P(C \cap D) \neq 0$.	Write a complete and clear reason.

Example 14

The Venn diagram shows the number of students in a class who can speak Spanish, Italian, both these languages and neither language.



A student is chosen at random from the class.

Let S and I be the events "choose a Spanish speaker" and "choose an Italian speaker" respectively.

Find:

a $P(S)$ **b** $P(I)$ **c** $P(S|I)$ **d** $P(I|S)$ **e** $P(I \cap S)$ **f** $P(S) \times P(I)$.

Hence determine whether S and I are independent.

a $P(S) = \frac{n(S)}{n(U)} = \frac{5+2}{15+5+2+6} = \frac{7}{28} = \frac{1}{4}$ **b** $P(I) = \frac{n(I)}{n(U)} = \frac{2+6}{15+5+2+6} = \frac{8}{28} = \frac{2}{7}$	Write down the appropriate formula and fill in the numbers from the Venn diagram to show complete working out that is easy to check.
c $P\left(S\middle|I\right) = \frac{2}{2+6} = \frac{2}{8} = \frac{1}{4}$ **d** $P\left(I\middle|S\right) = \frac{2}{5+2} = \frac{2}{7}$ **e** $P(I \cap S) = \frac{2}{28} = \frac{1}{14}$	Recall that $P(S|I)$ means you **know** the student chosen **does** speak Italian. So there are 8 students to consider, 2 of whom speak Spanish.
f $P(S) \times P(I) = \frac{1}{4} \times \frac{2}{7} = \frac{1}{14}$ Since $P(S|I) = P(S)$, S and I are independent.	Write a complete and clear reason. $P(I|S) = P(I)$ and $P(I \cap S) = P(I) \times P(S)$ are equivalent explanations.

Exercise 7G

1 For these pairs of events, state whether they are mutually exclusive, independent or neither.

a A = throw a head on a fair coin
B = throw a prime number on a fair die numbered 1, 2, 3, 4, 5, 6

b C = it will rain tomorrow
D = it is raining today

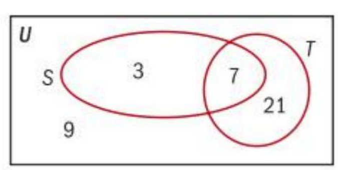
c D = throw a prime number on a fair die numbered 1, 2, 3, 4, 5, 6: E = throw an even number on the same die

d F = throw a prime number on a fair die numbered 1, 2, 3, 4, 5, 6: G = throw an even number on another die

e G = choose a number at random from {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} that is at most 6, H = choose a number from the same set that is at least 7

f M = choose a number at random from {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} that is no more than 5, H = choose a number from the same set that is 4 or more

g S = choose a Spanish speaker at random from a set of students represented by the set below, T = choose a Turkish speaker at random from this set

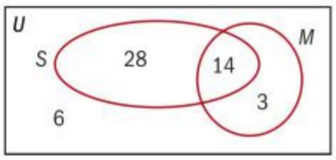


Statistics and probability

2 In a survey carried out in an airport, it is found that the events A: "a randomly chosen person has an Australian passport" and V: "a randomly chosen person has three vowels in their first name" are independent. It is found also that $\mathrm{P}(A) = 0.07$ and $\mathrm{P}(V) = 0.61$.

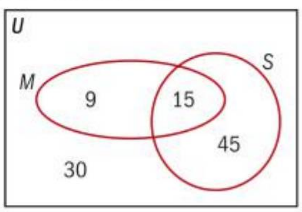
Find $\mathrm{P}(A \cup V)$ and interpret its meaning.

3 A class of undergraduate students were asked in 2016 their major subject and whether they listen to music on their commute to university. S is the set of science majors and M is the set of students who listen to music on their commute.

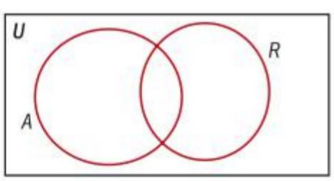


a Find $\mathrm{P}(S) \times \mathrm{P}(M)$ and $\mathrm{P}(S \cap M)$, and hence determine whether S and M are independent events, stating a reason for your answer.

b The same questions were asked in a survey in 2017 with the results given in the Venn diagram below. Find $\mathrm{P}(S)$ and $\mathrm{P}(S|M)$ and hence determine whether S and M are independent events, stating a reason for your answer.



4 Students are surveyed about languages spoken. You are given the following Venn diagram that represents the events A: "a randomly chosen person can speak Arabic" and R: "a randomly chosen person can speak Russian".



You are given also that A and R are independent events with $\mathrm{P}(A) = \frac{1}{8}$, $\mathrm{P}(R) = \frac{2}{7}$ and $n(U) = 56$.

a Draw the Venn diagram showing the numbers in each region.

b Hence find the probability that a student selected at random from this group speaks no more than one of the languages.

5 The letters of the word MATHEMATICS are written on 11 separate cards as shown below:

M	A	T	H	E	M
A	T	I	C	S	

a A card is drawn at random then replaced. Then another card is drawn.

Let A be the event the first card drawn is the letter A.

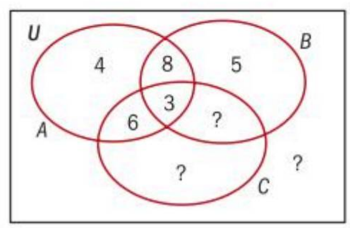
Let M be the event the second card drawn is the letter M. Find:

i $\mathrm{P}(A)$ **ii** $\mathrm{P}(M|A)$

iii $\mathrm{P}(A \cap M)$.

b In a different experiment, a card is drawn at random and **not** replaced. Then another card is drawn. Re-calculate the probabilities that you found in part **a**.

6 A group of 50 investors own properties in north European cities. The following Venn diagram shows how many investors own properties in Amsterdam, Brussels or Cologne. One of the investors is chosen at random.



a Find $\mathrm{P}(B \mid A)$.

b Find $\mathrm{P}(C \mid A)$.

c Interpret your answers for **a** and **b**.

d You are given that $\mathrm{P}(C \mid B) = \frac{10}{23}$ and $\mathrm{P}(A|C) = \frac{1}{3}$. Calculate the remaining regions shown in the Venn diagram.