7.2 Representing combined probabilities with diagrams

You have taken the first steps in the quantification of probabilities, experienced random experiments and investigated how to make predictions in the world of chance by applying formulae.

Probability situations themselves have a structure that you can represent in different ways, for example in problems where two or more sets are combined in some way.

Investigation 6

For each situation, think about how best to represent the situation with a diagram.

Compare and contrast your diagrams with others in your class **then** solve the problems.

Situation 1

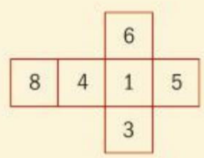
In a class survey on subject choices, Isabel, Clara, Coco, Anastasiia and Fangyu all state that they study biology. Isabel, Clara, Fangyu and Tomas all study chemistry whereas Barbora, Coco and Achille study neither biology nor chemistry.

1 **Factual** Find the probability that a student chosen randomly from this class studies both biology and chemistry.

2 Create your own probability question using your representation of the situation and have another student answer it.

Situation 2

One example of a Sicherman die is a fair cubical die with this net:



It is thrown together with a fair octahedral die whose faces are numbered 1, 2, 3, 4, 5, 6, 7 and 8.

3 **Factual** Find the probability that the number obtained by adding the two numbers thrown on each die is prime.

4 Find and describe a pattern in your representation of this situation and acquire some knowledge from your pattern.

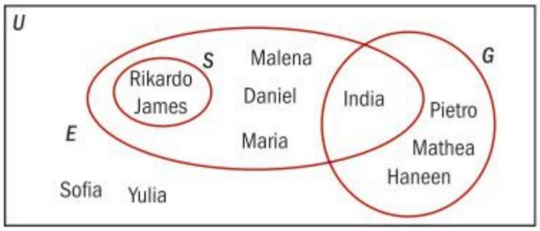
5 **Conceptual** What advantages are there in using a diagram in problem-solving with combined probabilities?

Two frequently used representations of probability problems are Venn diagrams and sample space diagrams.

A **Venn diagram** represents the sample space in a rectangle. Within the rectangle, each event is represented by a set of outcomes in a circle or an oval shape and is labelled accordingly.

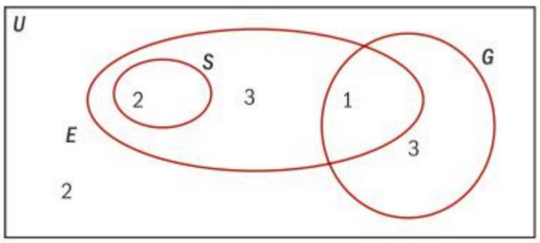
Example 4

In a class survey, Rikardo, Malena, Daniel, Maria, India and James reported that they study environmental systems and societies (ESS). India, Pietro, Mathea and Haneen said that they study geography. Rikardo and James were the only ones who reported that they studied Spanish whereas Sofia and Yulia studied none of the subjects mentioned in the survey. Represent the data in a Venn diagram.



Each event is represented by an italic capital letter.

U represents the entire sample space. In set terminology, this is called the universal set.



This diagram can be simplified to show the number of students in each region.

Example 5

Use the Venn diagram in Example 4 to find the probabilities that a student chosen randomly from this class:

a studies ESS

b studies ESS but not Spanish

c studies all three subjects

d studies exactly two of the subjects.

a $P(E) = \frac{n(E)}{n(U)} = \frac{2+3+1}{11} = \frac{6}{11}$

P(E) represents the probability of the event "the student chosen at random is from the set E".

b $\frac{4}{11}$

There are a total of four students within the ESS oval but outside the Spanish oval.

c 0

The diagram clearly shows that there are no students who study all three subjects.

d $\frac{3}{11}$

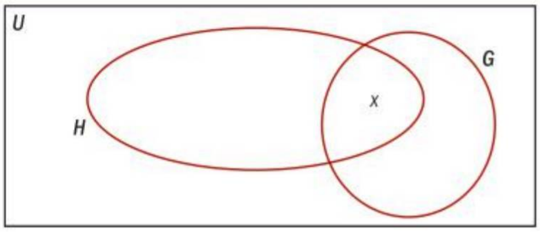
The diagram clearly shows that two students—Rikardo and James—study Spanish and ESS, whereas one student—India—studies both geography and ESS. These are the only three students who study exactly two of the subjects surveyed.

Example 6

In a class of 26 students, it is found that 10 study geography, 16 study history and 4 study neither history nor geography.

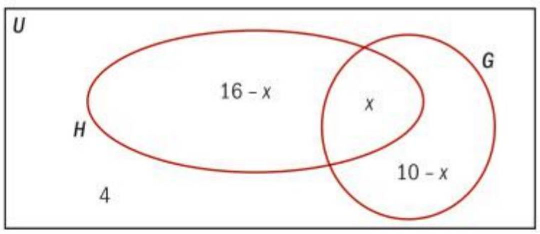
a Calculate the number of students who study both history and geography.

b Hence find the probability that a student selected at random from this class studies exactly one of these subjects.



Since the information given asks for the number who study both subjects, you can draw a Venn diagram with two intersecting sets as shown.

Representing the unknown with x is an appropriate problem-solving strategy.



Completely filling in the Venn diagram with all the information given is another problem-solving strategy.

a $(16 - x) + x + (10 - x) = 22$

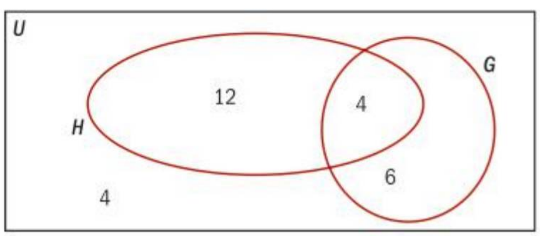
$\Rightarrow 26 - x = 22$

$\Rightarrow x = 4$

Since 4 study neither history nor geography the sum of these three regions must be 22.

Simplify and solve the equation.

b



Modify the Venn diagram using your answer to part **a**.

The probability that a randomly selected student studies exactly one of the subjects is $\frac{12+6}{26} = \frac{18}{26} = \frac{9}{13}$

From the Venn diagram read off the information needed to calculate the answer.

Exercise 7C

1 A survey of 117 consumers found that 81 had a tablet computer, 70 had a smartphone and 29 had both a smartphone and a tablet computer.

a Find the number of consumers surveyed who had neither a smartphone nor a tablet.

b Find the probability that when choosing one of the consumers surveyed at random, a consumer who only has a smartphone is chosen.

c In a population of 10 000 consumers, predict how many would have only a tablet computer.

2 In a class of 20 students, 12 study biology, 15 study history and 2 students study neither biology nor history.

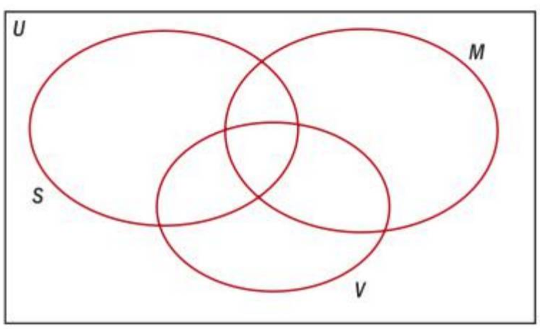
a Find the probability that a student selected at random from this class studies both biology and history.

b Given that a randomly selected student studies biology, find the probability that this student also studies history.

c In an experiment, a student is selected at random from this class and the student's course choices are noted. If the experiment is repeated 60 times, find the expected number of times a student who studies both biology and history is chosen.

3 In a survey, 91 people were asked about what devices they use to listen to music. In total, 59 people used a streaming service to listen to music, 44 used a mobile device and 29 used vinyl. Also, 22 people used both a streaming service and a mobile device, 9 used both a mobile device and vinyl and 20 people used only a streaming service. Finally, 8 people said they used all three devices.

a Draw this Venn diagram with numbers assigned to the eight regions.



Hence find the probability that a person chosen at random from those surveyed:

b listens to music on exactly one device

c listens to music on exactly two devices.

4 A garage keeps records of the last 94 cars tested for roadworthiness. The main reasons for failing the test are faulty tyres, steering or bodywork. In total, 34 failed for tyres, 40 failed for steering and 29 failed for bodywork. 11 cars failed for other reasons. 7 cars failed for both tyres and steering, 6 for steering and bodywork and 11 for bodywork and tyres. The owner of the garage wishes to calculate the number of cars that failed for all three reasons.

a Draw a Venn diagram to represent the information, using x to represent the number of cars failing for all three reasons.

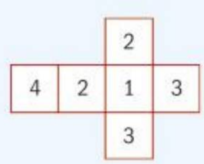
b Hence calculate the value of x.

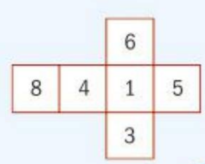
c Hence find the probability that a car selected at random from this data set failed for at least two reasons.

A **sample space diagram** is a useful way to represent the whole sample space and often takes the form of a table.

Example 7

It is claimed that when this pair of Sicherman dice is thrown, and the two numbers obtained added together, the probability of each total is just the same as if the two dice were numbered with 1, 2, 3, 4, 5 and 6. Verify this claim.





Sample space diagram for the total of two dice numbered 1, 2, 3, 4, 5 and 6:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Sample space diagram for the two Sicherman dice:

	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Form a sample space diagram for each experiment. Enter each total in the table as shown.

In both tables, $P(T=2) = P(T=12) = \frac{1}{36}$

$P(T=3) = P(T=11) = \frac{2}{36} = \frac{1}{18}$

$P(T=4) = P(T=10) = \frac{3}{36} = \frac{1}{12}$

$P(T=5) = P(T=9) = \frac{4}{36} = \frac{1}{9}$

$P(T=6) = P(T=8) = \frac{5}{36}$

and $P(T=7) = \frac{6}{36} = \frac{1}{6}$

Then find the probability of each outcome in the sample space, representing the total as T.

The probability of each total is the same for each pair of dice, so the claim is true.

State your conclusion.

Once time has been invested in drawing a diagram, it can be used to quantify many different probabilities.

Example 8

Use the sample space diagram in Example 7 to find the probability that the total found by throwing two Sicherman dice is:

a at most 4 **b** a factor of 24 **c** at least 8.

Continued on next page

TOK

How does a knowledge of probability theory affect decisions we make?

Statistics and probability

a $\frac{6}{36} = \frac{1}{6}$	The sample space diagram for the total of two dice numbered 1, 2, 3, 4, 5 and 6 is easier to use.
b $\frac{17}{36}$	
c $\frac{15}{36} = \frac{5}{12}$	

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Shade in or highlight the outcomes required to help count them.

Exercise 7D

1 Jakub designs a fair cubical die numbered with the first six prime numbers. He throws his die and a fair tetrahedral (four-sided) die numbered with the first four square numbers and writes down the difference between the two numbers, D. Find the probability that D is:

a a prime number **b** a square number.

2 a Two fair cubical dice are rolled in a game. The score is the greater of the two numbers. If the same number appears on both dice, then the score is that number. Find the probability that the score is at most 4.

b In 945 trials of this game, find the expected frequency of the event "the score is at least 4".

3 Alex throws a fair tetrahedral die numbered 1, 2, 3 and 4 and an octahedral (eight-sided) die numbered 1, 2, 3, …, 8. He defines M as the product of his two numbers. Find:

a P(M is odd) **b** P(M is prime)

c P(M is both odd and prime).

Bethany has two fair six-sided dice that she throws. She defines N as the product of her numbers. Find:

d P(N is odd) **e** P(N is more than 13)

f P(N is a factor of 36).

g Bethany and Alex can see that the probability that M is odd equals the probability that N is odd and try to find more events that have the same probabilities in each of their experiments. Find at least one such event.

4 Genetic material contained in pairs of human chromosomes determine whether a child is male or female. Males have the pair XY chromosomes and females XX. Inheriting the XY combination causes male characteristics to develop and XX causes female characteristics to develop. Sperm contain an X or Y with equal probability and an egg always contains an X. An infant inherits one chromosome determining gender from each parent.

a Complete the following *Punnet square* to show the possible outcomes of pairs of chromosomes that can be inherited:

		Chromosome inherited from mother	
		X	X
Chromosome inherited from father	X	XX	
	Y	XY	

b Hence show that the probability a child is born female is 0.5.

Investigation 7

1 In a class of 11 students, students 1, 2, 3, 4, 5 and 6 all study art. Students 3, 4, 7, 8, 9 and 10 all study biology. Choose a diagram to represent this information and find the probability that a student selected randomly from this class studies biology but not art.

2 Two fair cubical dice A and B have faces numbered 1, 2, 3, 4, 5 and 6 and 3, 4, 7, 8, 9 and 10 respectively. The dice are thrown and the total noted. Choose a diagram to represent this information and find the probability that the total is a square number.

Ask yourself: from what I have learned in this section, what are all the similarities and all the differences between Venn diagram and sample space diagram representations of combined events involving sets?

Share your ideas in a group.

3 Construct and share a list of similarities and differences.

		Do you agree with the statement for each type of representation?	
Statement about representation		Venn diagram	Sample space diagram
	Involves construction of a rectangular figure and you filling in (calculated) numbers	Yes	Yes
	Represents how two or more sets relate to each other		
	Enables you to combine elements of two sets in many ways		
	Enables you to combine elements of three sets in many ways		
	...	...	...
	...	...	...

4 **Conceptual** How can we decide how best to represent events given in the context of combined events involving sets?

International-mindedness

A well-known French gambler, Chevalier de Méré consulted Blaise Pascal in Paris in 1650 with questions about some games of chance. Pascal began to correspond with his friend Pierre Fermat about these problems which began their study of probability.

Exercise 7E

1 Diego inspects the music collection in his study. He finds that he has albums by artists A, B, C, G, H, R, Q, L and M on CD and albums by artists B, C, V, D, E, H, I, J and M on vinyl.

Diego chooses an artist at random. Find the probability that he does not have a choice of formats on which to listen to his artist's music.

2 Two boxes A and B contain numbered cards as shown below:

A			B		
3	8	0	7	6	7

Two cards are chosen at random, one from each box and the number obtained from box B is raised to the power of the number obtained from box A. Find the probability that the number obtained is a factor of 5764801.

3 Assuming that the probabilities of a kitten being born male or female are the same, what is the probability that in a litter of four kittens there are two males and two females?

4 Pierfranco designs a game.

- He throws a fair coin with one side labelled X and the other labelled Y.
- If the outcome is X, a number is chosen at random from the set {5, 10, 15, 20, 25, 30, 45}.
- If the outcome is Y, the computer simulation chooses from the set {30, 40, 50, 60, 70, 80, 90}.

Pierfranco scores a point if the outcome is a multiple of 6.

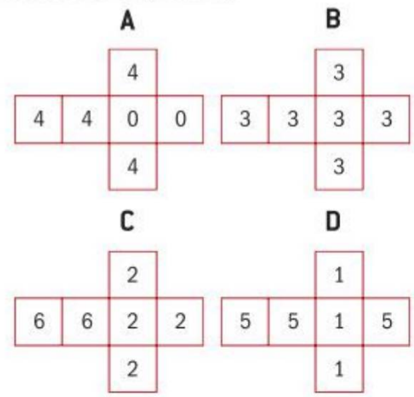
Find the expected number of points Pierfranco would score if he played the game 54 times.

5 Rea throws two fair cubical dice numbered 1, 2, 3, …, 6. Let R be the total of the numbers obtained by Rea.

Teodora throws a fair tetrahedral die numbered 1, 2, 4 and 8 and a fair five-sided die numbered 1, 2, 3, 4 and 5. Let the total of the numbers obtained by Teodora be T.

Determine which event is more likely: $R = 5$ or $T = 5$.

6 These dice compete in the "Dice world cup". A pair of dice is thrown and the highest number wins. The semifinals are A vs B and C vs D. The winners of each semifinal go in to the world cup final.



Construct sample space diagrams to find the probabilities of the outcomes of each semifinal.

Developing inquiry skills

In the first opening scenario, imagine 100 trials. How many outcomes would you expect in each area shown on this diagram?

		Cab yellow?	
		Yes	No
Witness correct?	Yes	??	??
	No	??	??