3.6 Implicit Differentiation

1. (a) $\frac{d}{dx}\left(xy+2x+3x^2\right)=\frac{d}{dx}(4) \Rightarrow (x\cdot y'+y\cdot 1)+2+6x=0 \Rightarrow xy'=-y-2-6x \Rightarrow$

$y'=\frac{-y-2-6x}{x}$ or $y'=-6-\frac{y+2}{x}$.

(b) $xy+2x+3x^2=4 \Rightarrow xy=4-2x-3x^2 \Rightarrow y=\frac{4-2x-3x^2}{x}=\frac{4}{x}-2-3x$, so $y'=-\frac{4}{x^2}-3$.

(c) From part (a), $y'=\frac{-y-2-6x}{x}=\frac{-(4/x-2-3x)-2-6x}{x}=\frac{-4/x-3x}{x}=-\frac{4}{x^2}-3$.

2. (a) $\frac{d}{dx}\left(4x^2+9y^2\right)=\frac{d}{dx}(36) \Rightarrow 8x+18y\cdot y'=0 \Rightarrow y'=-\frac{8x}{18y}=-\frac{4x}{9y}$

(b) $4x^2+9y^2=36 \Rightarrow 9y^2=36-4x^2 \Rightarrow y^2=\frac{4}{9}(9-x^2) \Rightarrow y=\pm\frac{2}{3}\sqrt{9-x^2}$, so

$y'=\pm\frac{2}{3}\cdot\frac{1}{2}(9-x^2)^{-1/2}(-2x)=\mp\frac{2x}{3\sqrt{9-x^2}}$

(c) From part (a), $y'=-\frac{4x}{9y}=-\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9-x^2}\right)}=\mp\frac{2x}{3\sqrt{9-x^2}}$.

3. $\frac{d}{dx}\left(x^3+x^2y+4y^2\right)=\frac{d}{dx}(6) \Rightarrow 3x^2+\left(x^2y'+y\cdot 2x\right)+8yy'=0 \Rightarrow x^2y'+8yy'=-3x^2-2xy$

$\Rightarrow (x^2+8y)y'=-3x^2-2xy \Rightarrow y'=-\frac{3x^2+2xy}{x^2+8y}=-\frac{x(3x+2y)}{x^2+8y}$

4. $\frac{d}{dx}\left(x^2-2xy+y^3\right)=\frac{d}{dx}(c) \Rightarrow 2x-2(xy'+y\cdot 1)+3y^2y'=0 \Rightarrow 2x-2y=2xy'-3y^2y' \Rightarrow$

$2x-2y=y'(2x-3y^2) \Rightarrow y'=\frac{2x-2y}{2x-3y^2}$

5. $\frac{d}{dx}\left(x^2y+xy^2\right)=\frac{d}{dx}(3x) \Rightarrow (x^2y'+y\cdot 2x)+(x\cdot 2yy'+y^2\cdot 1)=3 \Rightarrow$

$x^2y'+2xyy'=3-2xy-y^2 \Rightarrow y'(x^2+2xy)=3-2xy-y^2 \Rightarrow y'=\frac{3-2xy-y^2}{x^2+2xy}$

6. $\frac{d}{dx}\left(y^5+x^2y^3\right)=\frac{d}{dx}\left(1+ye^{x^2}\right) \Rightarrow 5y^4y'+\left(x^2\cdot 3y^2y'+y^3\cdot 2x\right)=0+y\cdot e^{x^2}\cdot 2x+e^{x^2}\cdot y' \Rightarrow$

$y'\left(5y^4+3x^2y^2-e^{x^2}\right)=2xye^{x^2}-2xy^3 \Rightarrow y'=\frac{2xy\left(e^{x^2}-y^2\right)}{5y^4+3x^2y^2-e^{x^2}}$

7. $\sqrt{xy}=1+x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x \Rightarrow \frac{x}{2\sqrt{xy}}y'+\frac{y}{2\sqrt{xy}}=x^2y'+2xy$

$\Rightarrow y'\left(\frac{x}{2\sqrt{xy}}-x^2\right)=2xy-\frac{y}{2\sqrt{xy}} \Rightarrow y'\left(\frac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\frac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \Rightarrow y'=\frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

8. $\sqrt{1+x^2y^2} = 2xy \Rightarrow \frac{1}{2}(1+x^2y^2)^{-1/2}(x^2 \cdot 2yy' + y^2 \cdot 2x) = 2(xy' + y \cdot 1) \Rightarrow$

$\dfrac{2x^2y}{2\sqrt{1+x^2y^2}}y' + \dfrac{2xy^2}{2\sqrt{1+x^2y^2}} = 2xy' + 2y \Rightarrow y'\left(\dfrac{x^2y}{\sqrt{1+x^2y^2}} - 2x\right) = 2y - \dfrac{xy^2}{\sqrt{1+x^2y^2}} \Rightarrow$

$y'\left(\dfrac{x^2y - 2x\sqrt{1+x^2y^2}}{\sqrt{1+x^2y^2}}\right) = \dfrac{2y\sqrt{1+x^2y^2} - xy^2}{\sqrt{1+x^2y^2}} \Rightarrow$

$y' = \dfrac{2y\sqrt{1+x^2y^2} - xy^2}{x^2y - 2x\sqrt{1+x^2y^2}} = \dfrac{y\left(2\sqrt{1+x^2y^2} - xy\right)}{x\left(xy - 2\sqrt{1+x^2y^2}\right)} = -\dfrac{y}{x}$

Another method: Since $1 + x^2y^2$ is positive, we can square both sides first and then differentiate implicitly.

9. $\dfrac{d}{dx}(4\cos x \sin y) = \dfrac{d}{dx}(1) \Rightarrow 4[\cos x \cdot \cos y \cdot y' + \sin y \cdot (-\sin x)] = 0 \Rightarrow$

$y'(4\cos x \cos y) = 4\sin x \sin y \Rightarrow y' = \dfrac{4\sin x \sin y}{4\cos x \cos y} = \tan x \tan y$

10. $x\cos y + y\cos x = 1 \Rightarrow x(-\sin y)y' + \cos y + y(-\sin x) + \cos x \cdot y' = 0 \Rightarrow y' = \dfrac{y\sin x - \cos y}{\cos x - x\sin y}$

11. $\cos(x-y) = xe^x \Rightarrow -\sin(x-y)(1-y') = xe^x + e^x \Rightarrow 1 - y' = -\dfrac{(x+1)e^x}{\sin(x-y)} \Rightarrow$

$y' = 1 + \dfrac{(x+1)e^x}{\sin(x-y)}$

12. $\sin x + \cos y = \sin x \cos y \Rightarrow \cos x - \sin y \cdot y' = \sin x(-\sin y \cdot y') + \cos y \cos x \Rightarrow$

$(\sin x \sin y - \sin y)y' = \cos x \cos y - \cos x \Rightarrow y' = \dfrac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$

13. $\dfrac{d}{dx}\left(\dfrac{x^2}{16} - \dfrac{y^2}{9}\right) = \dfrac{d}{dx}(1) \Rightarrow \dfrac{x}{8} - \dfrac{2yy'}{9} = 0 \Rightarrow y' = \dfrac{9x}{16y}$. When $x = -5$ and $y = \frac{9}{4}$ we have

$y' = \dfrac{9(-5)}{16(9/4)} = -\dfrac{5}{4}$, so an equation of the tangent line is $y - \frac{9}{4} = -\frac{5}{4}(x+5)$ or $y = -\frac{5}{4}x - 4$.

14. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1 \Rightarrow \dfrac{2x}{9} + \dfrac{yy'}{18} = 0 \Rightarrow y' = -\dfrac{4x}{y}$. When $x = -1$ and $y = 4\sqrt{2}$ we have

$y' = -\dfrac{4(-1)}{4\sqrt{2}} = \dfrac{1}{\sqrt{2}}$, so an equation of the tangent line is $y - 4\sqrt{2} = \frac{1}{\sqrt{2}}(x+1)$ or $y = \frac{1}{\sqrt{2}}(x+9)$.

15. $y^2 = x^3(2-x) = 2x^3 - x^4 \Rightarrow 2yy' = 6x^2 - 4x^3 \Rightarrow y' = \dfrac{3x^2 - 2x^3}{y}$. When $x = y = 1$,

$y' = \dfrac{3(1)^2 - 2(1)^3}{1} = 1$, so an equation of the tangent line is $y - 1 = 1(x-1)$ or $y = x$.

16. $x^{2/3} + y^{2/3} = 4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \Rightarrow \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \Rightarrow y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$

and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}} = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is

$y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.