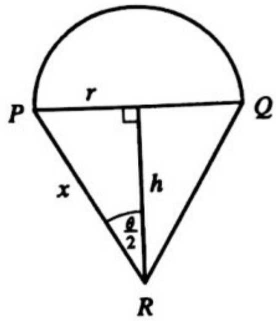
42. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x:

$\sin\frac{\theta}{2} = \frac{r}{x} \Rightarrow r = x\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2} = \frac{h}{x} \Rightarrow h = x\cos\frac{\theta}{2}$. Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$
$$= \frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0.$$



3.5 The Chain Rule

1. Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(4) = 4\cos 4x$.

2. Let $u = g(x) = 4 + 3x$ and $y = f(u) = \sqrt{u} = u^{1/2}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}(3) = \frac{3}{2\sqrt{u}} = \frac{3}{2\sqrt{4+3x}}$.

3. Let $u = g(x) = 1 - x^2$ and $y = f(u) = u^{10}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (10u^9)(-2x) = -20x(1-x^2)^9$.

4. Let $u = g(x) = \sin x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\cos x) = \sec^2(\sin x)\cdot\cos x$, or equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$.

Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = e^x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(e^x) = e^x\cos e^x$.

7. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \Rightarrow$

$$F'(x) = \tfrac{1}{4}(1+2x+x^3)^{-3/4}\cdot\frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}}\cdot(2+3x^2)$$
$$= \frac{2+3x^2}{4(1+2x+x^3)^{3/4}} = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

8. $F(x) = (x^2-x+1)^3 \Rightarrow F'(x) = 3(x^2-x+1)^2(2x-1)$

9. $g(t) = \frac{1}{(t^4+1)^3} = (t^4+1)^{-3} \Rightarrow g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \frac{-12t^3}{(t^4+1)^4}$

10. $f(t) = \sqrt[3]{1+\tan t} = (1+\tan t)^{1/3} \Rightarrow f'(t) = \frac{1}{3}(1+\tan t)^{-2/3}\sec^2 t = \frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$

11. $y = \cos(a^3+x^3) \Rightarrow y' = -\sin(a^3+x^3)\cdot 3x^2$ [a^3 is just a constant] $= -3x^2\sin(a^3+x^3)$

12. $y = a^3 + \cos^3 x \Rightarrow y' = 3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $= -3\sin x\cos^2 x$

13. $y = e^{-mx} \Rightarrow y' = e^{-mx}\frac{d}{dx}(-mx) = e^{-mx}(-m) = -me^{-mx}$

14. $y = 4\sec 5x \Rightarrow y' = 4\sec 5x\tan 5x(5) = 20\sec 5x\tan 5x$

15. $f(x) = xe^{-x^2} \Rightarrow f'(x) = xe^{-x^2}(-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(-2x^2 + 1) = e^{-x^2}(1 - 2x^2)$

16. $g(x) = e^{-5x} \cos 3x \Rightarrow g'(x) = e^{-5x}(-3\sin 3x) + (\cos 3x)(-5e^{-5x}) = -e^{-5x}(3\sin 3x + 5\cos 3x)$

17. $G(x) = (3x-2)^{10}(5x^2 - x + 1)^{12} \Rightarrow$

$$\begin{aligned} G'(x) &= (3x-2)^{10}(12)(5x^2 - x + 1)^{11}(10x - 1) + (5x^2 - x + 1)^{12}(10)(3x-2)^9(3) \\ &= 6(3x-2)^9(5x^2 - x + 1)^{11}[2(3x-2)(10x-1) + 5(5x^2 - x + 1)] \\ &= 6(3x-2)^9(5x^2 - x + 1)^{11}[(60x^2 - 46x + 4) + (25x^2 - 5x + 5)] \\ &= 6(3x-2)^9(5x^2 - x + 1)^{11}(85x^2 - 51x + 9) \end{aligned}$$

18. $g(t) = (6t^2 + 5)^3(t^3 - 7)^4 \Rightarrow$

$$\begin{aligned} g'(t) &= (6t^2+5)^3(4)(t^3-7)^3(3t^2) + (t^3-7)^4(3)(6t^2+5)^2(12t) \\ &= 12t(6t^2+5)^2(t^3-7)^3[t(6t^2+5) + 3(t^3-7)] \\ &= 12t(6t^2+5)^2(t^3-7)^3(9t^3 + 5t - 21) \end{aligned}$$

19. $y = e^{x\cos x} \Rightarrow y' = e^{x\cos x} \cdot \dfrac{d}{dx}(x\cos x) = e^{x\cos x}[x(-\sin x) + (\cos x)\cdot 1] = e^{x\cos x}(\cos x - x\sin x)$

20. Using Formula 5 and the Chain Rule, $y = 10^{1-x^2} \Rightarrow$

$y' = 10^{1-x^2}(\ln 10) \cdot \dfrac{d}{dx}(1 - x^2) = -2x(\ln 10)10^{1-x^2}$.

21. This function is a quotient raised to a power. To find its derivative, we use the Chain Rule by differentiating the power first and then multiplying by the derivative of the quotient. $F(y) = \left(\dfrac{y-6}{y+7}\right)^3 \Rightarrow$

$$F'(y) = 3\left(\frac{y-6}{y+7}\right)^2 \frac{(y+7)(1) - (y-6)(1)}{(y+7)^2} = 3\left(\frac{y-6}{y+7}\right)^2 \frac{13}{(y+7)^2} = \frac{39(y-6)^2}{(y+7)^4}$$

22. $s(t) = \sqrt[4]{\dfrac{t^3+1}{t^3-1}} = \left(\dfrac{t^3+1}{t^3-1}\right)^{1/4} \Rightarrow$

$$\begin{aligned} s'(t) &= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{(t^3-1)(3t^2) - (t^3+1)(3t^2)}{(t^3-1)^2} = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{3t^2(t^3 - 1 - t^3 - 1)}{(t^3-1)^2} \\ &= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{3t^2(-2)}{(t^3-1)^2} = \frac{1}{2}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{-3t^2}{(t^3-1)^2} \end{aligned}$$

23. $y = \dfrac{r}{\sqrt{r^2+1}} \Rightarrow$

$$\begin{aligned} y' &= \frac{\sqrt{r^2+1}(1) - r\cdot\frac{1}{2}(r^2+1)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2} = \frac{\sqrt{r^2+1} - \dfrac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} = \frac{\dfrac{\sqrt{r^2+1}\sqrt{r^2+1} - r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} \\ &= \frac{(r^2+1) - r^2}{\left(\sqrt{r^2+1}\right)^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } (r^2+1)^{-3/2} \end{aligned}$$

(continued)

Another solution: Write y as a product and make use of the Product Rule.

$$y = r\left(r^2+1\right)^{-1/2} \;\Rightarrow\; y' = r \cdot -\tfrac{1}{2}\left(r^2+1\right)^{-3/2}(2r) + \left(r^2+1\right)^{-1/2}\cdot 1$$

$$= \left(r^2+1\right)^{-3/2}\left[-r^2 + \left(r^2+1\right)^1\right] = \left(r^2+1\right)^{-3/2}(1) = \left(r^2+1\right)^{-3/2}$$

The step that students usually have trouble with is factoring out $\left(r^2+1\right)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

24. $y = \dfrac{e^{2u}}{e^u+e^{-u}} \;\Rightarrow$

$$y' = \frac{\left(e^u+e^{-u}\right)\left(e^{2u}\cdot 2\right) - e^{2u}\left(e^u - e^{-u}\right)}{\left(e^u+e^{-u}\right)^2} = \frac{e^{2u}\left(2e^u + 2e^{-u} - e^u + e^{-u}\right)}{\left(e^u+e^{-u}\right)^2} = \frac{e^{2u}\left(e^u+3e^{-u}\right)}{\left(e^u+e^{-u}\right)^2}$$

Another solution: Eliminate negative exponents by first changing the form of y.

$$y = \frac{e^{2u}}{e^u+e^{-u}}\cdot\frac{e^u}{e^u} = \frac{e^{3u}}{e^{2u}+1} \;\Rightarrow$$

$$y' = \frac{\left(e^{2u}+1\right)\left(3e^{3u}\right) - e^{3u}\left(2e^{2u}\right)}{\left(e^{2u}+1\right)^2} = \frac{e^{3u}\left(3e^{2u}+3-2e^{2u}\right)}{\left(e^{2u}+1\right)^2} = \frac{e^{3u}\left(e^{2u}+3\right)}{\left(e^{2u}+1\right)^2}$$

25. Using Formula 5 and the Chain Rule, $y = 2^{\sin \pi x} \;\Rightarrow$

$$y' = 2^{\sin\pi x}(\ln 2)\cdot\frac{d}{dx}(\sin \pi x) = 2^{\sin\pi x}(\ln 2)\cdot\cos\pi x\cdot\pi = 2^{\sin\pi x}(\pi\ln 2)\cos\pi x$$

26. $y = \tan^2(3\theta) = (\tan 3\theta)^2 \;\Rightarrow\; y' = 2(\tan 3\theta)\cdot\dfrac{d}{d\theta}(\tan 3\theta) = 2\tan 3\theta\cdot\sec^2 3\theta\cdot 3 = 6\tan 3\theta\sec^2 3\theta$

27. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \;\Rightarrow$

$$y' = 2\,[\cot(\sin\theta)]\cdot\frac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta)\cdot\left[-\csc^2(\sin\theta)\cdot\cos\theta\right] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

28. $y = \sin(\sin(\sin x)) \;\Rightarrow\; y' = \cos(\sin(\sin x))\,\dfrac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x))\,\cos(\sin x)\,\cos x$

29. $y = \sin\left(\tan\sqrt{\sin x}\right) \;\Rightarrow$

$$y' = \cos\left(\tan\sqrt{\sin x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{\sin x}\right) = \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x}\cdot\frac{d}{dx}(\sin x)^{1/2}$$

$$= \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x}\cdot\tfrac{1}{2}(\sin x)^{-1/2}\cdot\cos x$$

$$= \cos\left(\tan\sqrt{\sin x}\right)\left(\sec^2\sqrt{\sin x}\right)\left(\frac{1}{2\sqrt{\sin x}}\right)(\cos x)$$

30. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \;\Rightarrow\; y' = \tfrac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\tfrac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\tfrac{1}{2}x^{-1/2}\right)\right]$

31. $y = \sin(\sin x) \;\Rightarrow\; y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -$ and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x+\pi$.

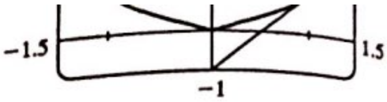
32. $y = x^2e^{-x} \;\Rightarrow\; y' = x^2\left(-e^{-x}\right) + e^{-x}(2x) = 2xe^{-x} - x^2e^{-x}$. At $\left(1, \frac{1}{e}\right)$, $y' = 2e^{-1} - e^{-1} = \frac{1}{e}$. So an equation of the tangent line is $y - \frac{1}{e} = \frac{1}{e}(x-1)$ or $y = \frac{1}{e}x$.

34. (a) For $x > 0$, $|x| = x$, and ...

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)\left(2-x^2\right)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1, 1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

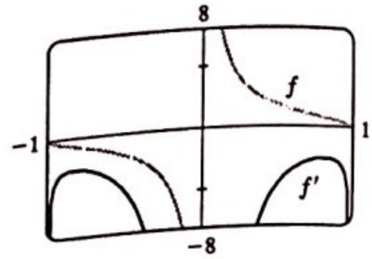


35. (a) $f(x) = \dfrac{\sqrt{1-x^2}}{x} \Rightarrow$

$$f'(x) = \frac{x \cdot \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) - \sqrt{1-x^2}\,(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$= \frac{-x^2 - \left(1-x^2\right)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

(b)

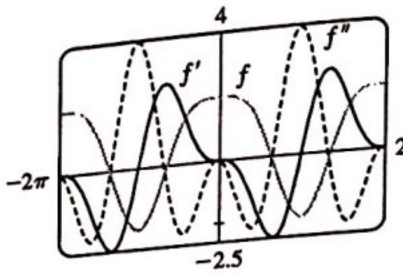


Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

36. (a) $f(x) = 2\cos x + \sin^2 x \Rightarrow$

$$f'(x) = 2(-\sin x) + 2\sin x(\cos x)$$
$$= \sin 2x - 2\sin x \Rightarrow$$
$$f''(x) = 2\cos 2x - 2\cos x$$
$$= 2(\cos 2x - \cos x)$$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

37. $F(x) = f(g(x)) \Rightarrow$
$F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 7 \cdot 4 = 28$. Notice that we did not use $f'(3) = 2$.

38. $w = u \circ v \Rightarrow w(x) = u(v(x)) \Rightarrow w'(x) = u'(v(x)) \cdot v'(x)$, so
$w'(0) = u'(v(0)) \cdot v'(0) = u'(2) \cdot v'(0) = 4 \cdot 5 = 20$. The other pieces of information, $u(0) = 1$, $u'(0) = 3$, and $v'(2) = 6$, were not needed.

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39. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

40. (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

41. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\frac{3-4}{6-2} = -\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\frac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = (-\frac{1}{4})(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\frac{2-0}{5-2} = \frac{2}{3}$. Thus, $g'(3)g'(1) = (\frac{2}{3})(-3) = -2$.

42. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$.
So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$.
So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(1.5) = 6$.

43. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$. So $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1 = \frac{14.8 - 12.6}{0.1 - 0} = 22$, $m_2 = \frac{18.4 - 14.8}{0.2 - 0.1} = 36$. So $f'(0.1) \approx \frac{m_1 + m_2}{2} = \frac{22 + 36}{2} = 29$.

For $g'(0.5)$: $m_1 = \frac{0.10 - 0.17}{0.5 - 0.4} = -0.7$, $m_2 = \frac{0.05 - 0.10}{0.6 - 0.5} = -0.5$. So $g'(0.5) \approx \frac{m_1 + m_2}{2} = -0.6$.

Hence, $h'(0.5) = f'(0.1)g'(0.5) \approx (29)(-0.6) = -17.4$.

44. $g(x) = f(f(x)) \Rightarrow g'(x) = f'(f(x))f'(x)$. So $g'(1) = f'(f(1))f'(1) = f'(2)f'(1)$.

For $f'(2)$: $m_1 = \frac{3.1 - 2.4}{2.0 - 1.5} = 1.4$, $m_2 = \frac{4.4 - 3.1}{2.5 - 2.0} = 2.6$. So $f'(2) \approx \frac{m_1 + m_2}{2} = 2$.

For $f'(1)$: $m_1 = \frac{2.0 - 1.8}{1.0 - 0.5} = 0.4$, $m_2 = \frac{2.4 - 2.0}{1.5 - 1.0} = 0.8$. So $f'(1) \approx \frac{m_1 + m_2}{2} = 0.6$.

Hence, $g'(1) = f'(2)f'(1) \approx (2)(0.6) = 1.2$.

45. (a) $f(x) = \sqrt{x} \Rightarrow f'(x) = 1/(2\sqrt{x})$, so f is not differentiable at $x = 0$. Since h is differentiable on $[0, \infty)$ and $\sqrt{x}$ is differentiable on $(0, \infty)$, it follows that $G(x) = h(\sqrt{x})$ is differentiable on $(0, \infty)$.

(b) By the Chain Rule, $G'(x) = h'(\sqrt{x})\frac{d}{dx}\sqrt{x} = h'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{h'(\sqrt{x})}{2\sqrt{x}}$.

46. (a) $F(x) = f(x^\alpha) \Rightarrow F'(x) = f'(x^\alpha)\frac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha - 1}$

(b) $G(x) = [f(x)]^\alpha \Rightarrow G'(x) = \alpha[f(x)]^{\alpha - 1} f'(x)$

47. (a) $F(x) = f(e^x) \Rightarrow F'(x) = f'(e^x)\frac{d}{dx}(e^x) = f'(e^x)e^x$

(b) $G(x) = e^{f(x)} \Rightarrow G'(x) = e^{f(x)}\frac{d}{dx}f(x) = e^{f(x)}f'(x)$