c Estimate the area of this region using six trapezoids.

d Find the percentage error made with the estimation made in part **c**.

8 Consider the region enclosed by the graph of the function $f(x) = 1 + e^x$, the x-axis and the vertical lines $x = 0$ and $x = 2$.

a Sketch the function f and shade the region.

b **i** Write down a definite integral that represents the area of this region.

ii Find the area of this region. Give your answer correct to four significant figures.

c Estimate the area of this region using five trapezoids. Give your answer correct to four significant figures.

d Find the percentage error made with the estimation found in part **c**.

Developing inquiry skills

In the opening scenario for this chapter you looked at how to estimate the area of an island.

How could you improve your estimation of the area of the island using what you have studied in this section?

How close is your estimate to the claimed area? Why is your answer an estimate?

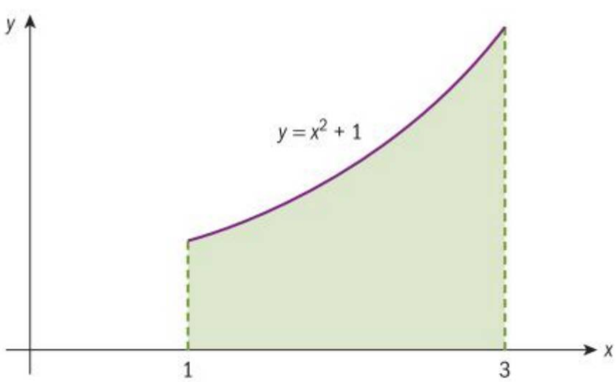
TOK

Galileo said that the universe is a grand book written in the language of mathematics.

Where does mathematics come from? Does it start in our brains or is it part of the universe?

13.2 Integration: the reverse process of differentiation

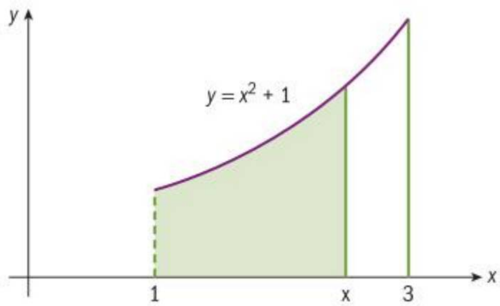
You have so far found areas under curves between two given values. For example, you have seen how to find the area enclosed between $y = x^2 + 1$, the x-axis and the vertical lines $x = 1$ and $x = 3$.



HINT

The area of the shaded region is $\int_1^3 (x^2 + 1)\,dx = \frac{32}{3}$.

You are now going to find expressions for areas when one of the limits is fixed and the other is variable.



HINT

As the value of *x* increases from 1 to 3, the area of the shaded region also increases.

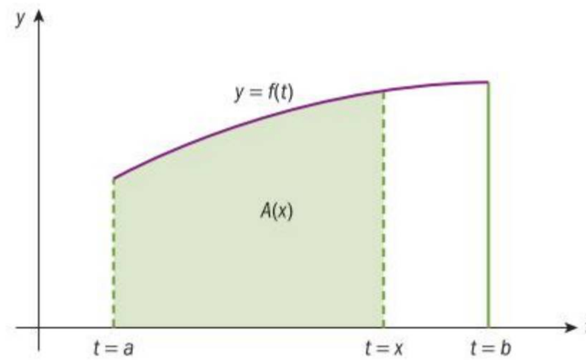
There is a function, the **area function,** which maps every value of *x* to the area of the shaded region.

What is the formula of the area function?

And what would the area function be in a different region?

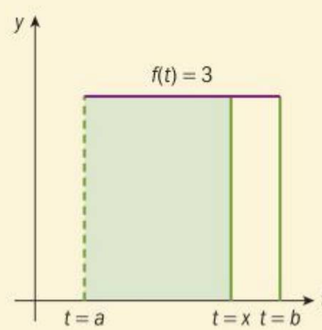
Consider a positive and continuous function $y = f(t)$ over the interval $a \le t \le b$. The area enclosed between the graph of $y = f(t)$, the *t*-axis and the vertical lines $t = a$ and $t = x$ where $a \le x \le b$ is defined by

$$A(x) = \int_a^x f(t)\,\mathrm{d}t.$$



In this investigation you will find the relationship between the two major branches of calculus: integration and differentiation.

Investigation 3



1 Consider the area under the graph of $f(t) = 3$ between $t = a$ and $t = x$, where $a < x$.

Show that the area function is $A(x) = 3x - 3a$.

2 Consider the area under the graph of $f(t) = t$ between $t = a$ and $t = x$, where $a < x$. Show that the area function can be written as $A(x) = \frac{x^2}{2} - \frac{a^2}{2}$. Draw the graph of the function *f* and shade the area enclosed between this graph and the *t*-axis over the interval $a \le t \le x$.

3 Consider the area under the graph of $f(t) = 2t$ between $t = a$ and $t = x$, where $a < x$. Show that the area function can be written as $A(t) = x^2 - a^2$. Draw the graph of the function *f* and shade the area enclosed between this graph and the *t*-axis over the interval $a \le t \le x$.

4 The results from **1** to **3** are summarized in the first two columns from the table shown below. The last two columns will be completed later.

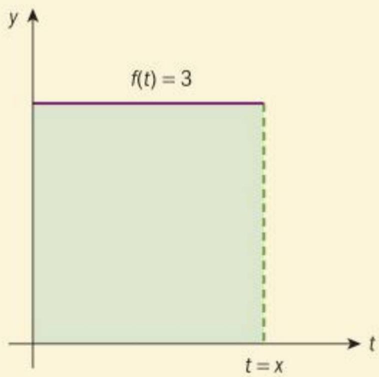
$f(t)$	$A(x)$	$F(x)$	$A(x)$
$f(t) = 3$	$A(x) = 3x - 3a$	$F(x) = 3x$	$F(x) - F(a)$
$f(t) = t$	$A(x) = \frac{x^2}{2} - \frac{a^2}{2}$		
$f(t) = 2t$	$A(x) = x^2 - a^2$		

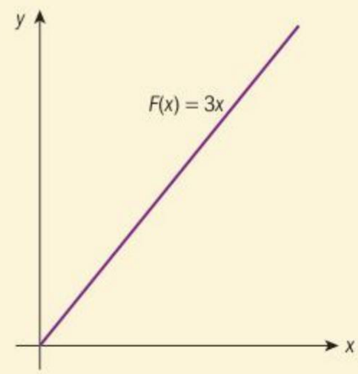
Continued on next page

- The expressions for $A(x)$ have been written as the difference between two terms. The first term is a function of x and the second term is constant. What do the two terms have in common?

Let the first term in each of the expressions for $A(x)$ be a new function $F(x)$.

- Complete the third column of the table with the corresponding expressions for $F(x)$. The first row has been completed for you.
- For which value of a are the expressions for $F(x)$ and $A(x)$ equal? What area would be represented by $F(x)$ when a takes this value? How would you represent this area using definite integrals?
- Complete the fourth column of the table by writing $A(x)$ in terms of $F(x)$. The first row has been completed for you.
- Below are shown the graphs of $f(t) = 3$ and $F(x) = 3x$.



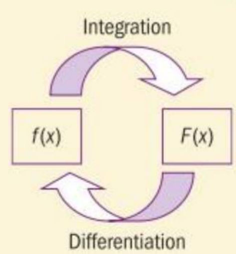


- What is the gradient of the graph of F at every x? How does this relate to the graph of f?
- For each of the two remaining functions f, find an expression for the gradient of the function F at every x. What can you say about the relationship between the gradient function of F and the function f?
- Now consider the function $f(t) = 3t + 1$. What would be $F(x)$ in this case? What is the relationship between the graph of $F(x)$ and the graph of f?
- If you are given that the area function is $F(x) = x^3$, how would you find the formula of f?

5 **Conceptual** How does integration relate to differentiation?

If $f(t) \geq 0$ is a continuous function, the area enclosed between the graph of f and the x-axis over the interval $a \leq t \leq x$ can be found with the definite integral $\int_a^x f(t)\,dt$.

Also $\int_a^x f(t)\,dt = F(x) - F(a)$ where $F'(x) = f(x)$.



If $F(x)$ is a function where $F'(x) = f(x)$, we say that $F(x)$ is an **antiderivative** of f.

The process of finding an antiderivative is called **antidifferentiation.**

International-mindedness

The **fundamental theorem of calculus** shows the relationship between the derivative and the integral and was developed in the 17th century by Gottfried Leibniz and Isaac Newton.

For example,

$F(x) = 3x$ is an antiderivative of $f(x) = 3$ because $F'(x) = 3$.

$F(x) = \dfrac{x^2}{2} - 3$ is an antiderivative of $f(x) = x$ because $F'(x) = x$.

$F(x) = x^2 + 1$ is an antiderivative of $f(x) = 2x$ because $F'(x) = 2x$.

Can you think of an antiderivative for $f(x) = 3x^2$?

Reflect What is an antiderivative of a function?

How is the antiderivative related to the definite integral?

TOK

How do "believing that" and "believing in" differ?

How does belief differ from knowledge?

Example 11

a Find an antiderivative of $y = 4 + 2x$.

b Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = 3x$.

a $F(x) = 4x + x^2$	An antiderivative of $y = 4 + 2x$ is a function F whose derivative is $4 + 2x$. The derivative of $4x$ is 4. The derivative of x^2 is $2x$. The derivative of $4x + x^2$ is $4 + 2x$.
b $g(x) = 1.5x^2$	If the gradient function of $y = g(x)$ is $3x$ then $g'(x) = 3x$. Function g is a function whose derivative is $3x$. The derivative of $1.5x^2$ is $3x$.

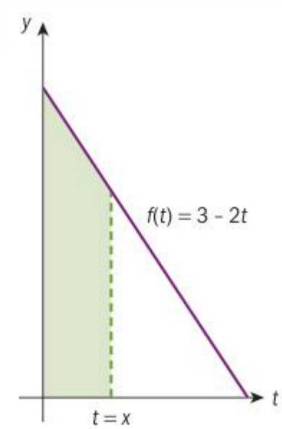
Example 12

Consider the definite integral $\int_0^x (3 - 2t)\,dt$ where $0 \le x \le 1.5$.

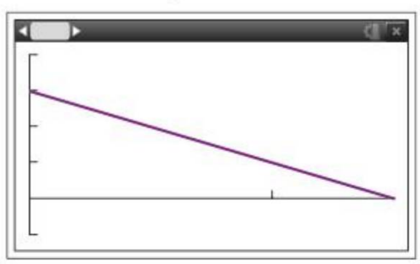
a Draw on a diagram the function $f(t) = 3 - 2t$ and shade the area represented by $\int_0^x (3 - 2t)\,dt$.

b By using antiderivatives find an expression for $\int_0^x (3 - 2t)\,dt$ in terms of x.

a



You can use your GDC to draw the graph.



HINT

Remember that $0 \le x \le 1.5$.

Shade the area under the graph of f between $t = 0$ and $t = x$.

Continued on next page

b $\int_0^x (3-2t)\,dt = 3x - x^2$

Find an antiderivative of $f(t) = 3 - 2t$, a function whose derivative is $f(t) = 3 - 2t$.

The derivative of $3t$ is 3.

The derivative of t^2 is $2t$.

The derivative of $3t - t^2$ is $3 - 2t$.

$F(x) = 3x - x^2$

$F(0) = 0$

$F(x) - F(0) = 3x - x^2$

HINT

Use the formula for the area of a trapezoid to check that this answer is correct.

Exercise 13F

1 a For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 1$ and $F(x) = x$

ii $f(x) = 1$ and $F(x) = x + 3$

iii $f(x) = 1$ and $F(x) = x - 6$

b Find two further antiderivatives for $f(x) = 1$.

c Write down the general form of an antiderivative of $f(x) = 1$.

2 a For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 2x$ and $F(x) = x^2$

ii $f(x) = 2x$ and $F(x) = x^2 + 1$

iii $f(x) = 2x$ and $F(x) = x^2 - 4$

b Find two further antiderivatives for $f(x) = 2x$.

c Write down the general form of an antiderivative of $f(x) = 2x$.

3 a For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = x$ and $F(x) = \frac{x^2}{2}$

ii $f(x) = x$ and $F(x) = \frac{x^2}{2} + 2.5$

iii $f(x) = x$ and $F(x) = \frac{x^2}{2} - 12$

b Write down the general form of an antiderivative of $f(x) = x$.

4 a For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 2x + 1$ and $F(x) = x^2 + x$

ii $f(x) = 2x + 1$ and $F(x) = x^2 + x - 3.2$

iii $f(x) = 2x + 1$ and $F(x) = x^2 + x + 4$

b Write down the general form of an antiderivative of $f(x) = 2x + 1$.

5 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = 3$.

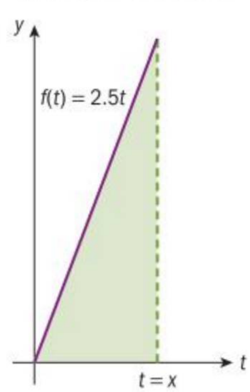
6 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = -2$.

7 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = \frac{x}{2}$.

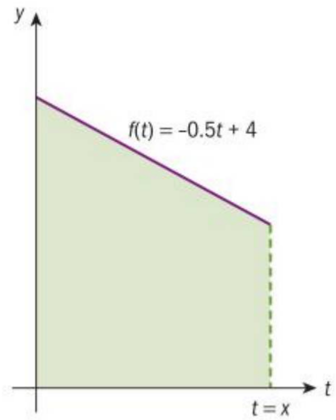
8 Find an antiderivative of $f(x) = x^2$.

9 Find the area function, $A(x)$, in each of the following situations.

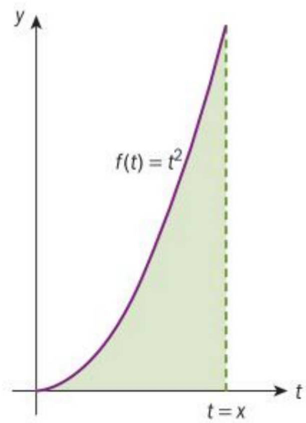
a



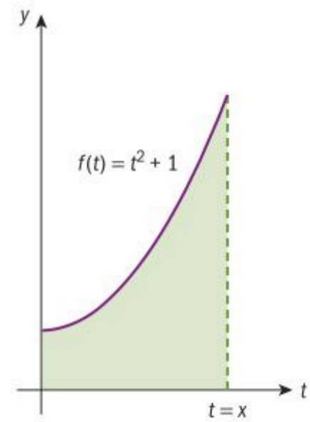
b



c



d



10 Consider the graph of the function $f(t) = 4$ over the interval $0 \leq t \leq x$. Find $\int_0^x 4\,\mathrm{d}t$.

11 Consider the graph of the function $f(t) = 2t + 1$ over the interval $0 \leq t \leq x$. Find $\int_0^x (2t+1)\,\mathrm{d}t$.

12 Consider the graph of the function $f(t) = 4t$ over the interval $2 \leq t \leq x$. Find $\int_2^x 4t\,\mathrm{d}t$.

Investigation 4

1 On the same set of axes, sketch the graphs of

$y = x^2$ $\quad$ $y = x^2 + 1$ $\quad$ $y = x^2 - 2$ $\quad$ $y = x^2 + 3$

over the interval $-4 \leq x \leq 4$.

- How can you describe their relative position?
- Find the gradient of each of these curves at $x = 1$. What do you notice?
- Now find the gradient of each of these curves at $x = -2$. What can you say about these answers?
- What is the gradient of each of these curves at any x?
- Find another curve for which the gradient at x is the same as the gradient of any of the curves from **1**.
- How would you write the formula of any curve with the gradient the same as the gradient of the curves from **1**?
- All these curves make up a family of functions that are antiderivatives of $2x$. Why is this?

The notation used to indicate this family of functions is $\int 2x\,\mathrm{d}x = x^2 + c$, where c is a constant.

Continued on next page

Calculus

This is read as "the integral of $2x$ with respect to x is $x^2 + c$".

- What is the value of c when the antiderivative is $y = x^2 + 1$? What is the constant of integration when the antiderivative is $y = x^2$?
- What is the derivative of $x^2 + c$ with respect to x?

2 What is the family of functions that are antiderivatives of $f(x) = 3$? What do their graphs have in common? Write this family of functions using integral notation.

3 What does the integral $\int x \, dx$ represent? Calculate it.

4 **Conceptual** How does an indefinite integral define a family of antiderivatives?

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$ where $c \in \mathbb{R}$.

The expression $\int f(x)\,dx$ is called an **indefinite integral** and $\int f(x)\,dx$ is read as "the integral of f with respect to x".

Note: c is called the **constant of integration**.

Reflect What does an indefinite integral represent?
How is an antiderivative related to an indefinite integral?

Example 13

a Find the family of antiderivatives of $f(x) = 5$.

b Find $\int 4x\,dx$.

a $F(x) = 5x + c$ where $c \in \mathbb{R}$	These are the functions F for which $F'(x) = 5$.
b $2x^2 + c$ where $c \in \mathbb{R}$	This represents the family of antiderivatives of $4x$, the functions whose derivatives are $4x$. This is $2x^2 + c$.

Exercise 13G

1 Find the family of antiderivatives of 2.

2 Find the family of antiderivatives of $x + 1$.

3 Find all the functions whose derivatives are equal to -1.

4 Find all the functions whose derivatives are equal to $2x$.

5 Calculate the following indefinite integrals.

a $\int 1\,dx$ **b** $\int 6\,dx$ **c** $\int \frac{1}{2}\,dx$

6 Calculate the following indefinite integrals.

a $\int x\,dx$ b $\int 2x\,dx$

c $\int 5x\,dx$ d $\int \frac{1}{2}x\,dx$

e $\int ax\,dx$ (a is a non-zero constant)

7 Calculate the following indefinite integrals.

a $\int x^2\,dx$ b $\int 3x^2\,dx$ c $\int 4x^2\,dx$

d $\int \frac{1}{2}x^2\,dx$ e $\int \frac{x^2}{3}\,dx$ f $\int \frac{3}{2}x^2\,dx$

g $\int ax^2\,dx$ (a is a non-zero constant)

8 Calculate the following indefinite integrals.

a $\int x^3\,dx$ b $\int 4x^3\,dx$ c $\int 2x^3\,dx$

d $\int \frac{4}{5}x^3\,dx$ e $\int \frac{x^3}{3}\,dx$ f $\int -x^3\,dx$

g $\int ax^3\,dx$ (a is a non-zero constant)

In questions **5** to **8** you may have noticed that all the functions were of the form ax^n.

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ where a and n are constants, $a \neq 0$ and $n \neq -1$. n is an integer.

This is an integration rule and is called the **power rule.**

Reflect Why is the condition $n \neq -1$ given in the power rule?

Example 14

a Find $\int \frac{x^5}{4}\,dx$. b Find $\int (\frac{x^2}{2} + 3x - 1)\,dx$. c Find $\int \frac{4}{x^2}\,dx$.

a $\int \frac{x^5}{4}\,dx = \frac{x^6}{24} + c$	Apply the power rule with $a = \frac{1}{4}$ and $n = 5$: $\int \frac{x^5}{4}\,dx = \int \frac{1}{4}x^5\,dx = \frac{1}{4} \times \frac{x^{5+1}}{5+1} + c = \frac{1}{4} \times \frac{x^6}{6} + c$
b $\int (\frac{x^2}{2} + 3x - 1)\,dx = \frac{x^3}{6} + \frac{3x^2}{2} - x + c$	As the derivative of a sum is the sum of the derivatives, calculate an antiderivative for each term by applying the power rule and then add the three terms up. $\int \frac{x^2}{2}\,dx = \frac{1}{2} \times \frac{x^{2+1}}{2+1} + c_1 = \frac{1}{6}x^3 + c_1$

HINT

Do not forget the constant of integration.

Continued on next page

	$\int 3x\,dx = 3 \times \frac{x^{1+1}}{1+1} + c_2 = \frac{3x^2}{2} + c_2$
	$\int -1\,dx = -x + c_3$
c $\int \frac{4}{x^2}dx = -\frac{4}{x} + c$	Apply the power rule $a = 4;\ n = -2$
	$\int \frac{4}{x^2}dx = \int 4x^{-2}\,dx = 4 \times \frac{x^{-2+1}}{-2+1} + c$

HINT

c_1, c_2, c_3 are three constants that added up give another constant c.

Reflect What is the integral of a sum of multiples of powers of x?
How can you tell that the indefinite integrals are correct?

Exercise 13H

1 Find the following indefinite integrals.

a $\int 10\,dx$ **b** $\int 0.6x^2 dx$

c $\int x^5 dx$ **d** $\int (7-2x)dx$

e $\int (1+2x)dx$ **f** $\int (5+x-\frac{1}{3}x^2)dx$

g $\int (-x+\frac{3x^2}{4}+0.5)dx$ **h** $\int (1-x+\frac{x^3}{2})dx$

i $\int (x^2-\frac{1}{2}x+4)dx$ **j** $\int \frac{1}{2}x^{-2}\,dx$

k $\int \frac{2}{x^4}dx$ **l** $\int (4x+\frac{5}{x^3})dx$

2 For $f(x) = x^2 - \frac{x}{3} + 4$, find:

a $f'(x)$ **b** $\int f(x)dx$

3 Find $\int (t-3t^2)dt$.

4 Find $\int (4t^3-3t+1)dt$.

5 Find **all** the functions F for which the gradient equals $3+x-\frac{x^2}{4}$.

6 Find **all** the functions F for which the gradient is $y = -x + 0.5x^2$.

7 **a** Expand $(x+1)(x-2)$.

b Hence, find all the functions $y = f(x)$ for which $\frac{dy}{dx} = (x+1)(x-2)$.

8 Find the indefinite integral of the function $g(x) = x^3 - \frac{2}{x^2} + 1$.

Example 15

a The curve $y = f(x)$ passes through the point $(1, 3)$. The gradient of the curve is given by $f'(x) = 2 + \frac{x}{3}$. Find the equation of the curve.

b If $\frac{dy}{dx} = 2x - 4x^2$ and $y = -1$ when $x = 3$, find y in terms of x.

a $f(x)=\int(2+\frac{x}{3})\mathrm{d}x = 2x+\frac{1}{3}\times\frac{x^2}{2}+c$	Apply the power rule to find an antiderivative of $f'(x)$.
$f(x)=2x+\frac{x^2}{6}+c$	
$f(1)=2\times1+\frac{1^2}{6}+c$	If the curve passes through the point $(1, 3)$ then $f(1)=3$.
$3=2+\frac{1}{6}+c$	
$c=\frac{5}{6}$	
$f(x)=2x+\frac{x^2}{6}+\frac{5}{6}$	
b $y=\int(2x-4x^2)\mathrm{d}x = x^2-\frac{4x^3}{3}+c$	Integrate $\frac{\mathrm{d}y}{\mathrm{d}x}$ to find y in terms of x.
$-1=3^2-\frac{4\times3^3}{3}+c$ $-1=9-36+c$	Use the fact that $y=-1$ when $x=3$ to find the value of the constant c.
$c=26$	
$y=x^2-\frac{4x^3}{3}+26$	

Example 16

Find the cost function, $C(x)$, when the marginal cost is $M(x) = 1 + 2x$ and the fixed cost is US$40.

$C(x)=x+x^2+40$	The cost function is an antiderivative of the marginal cost function. To find $C(x)$, integrate $M(x)$ $\int(1+2x)\,dx = x+x^2+c$ The fixed cost is used to find the constant of integration. $C(0)=40$ then $0+0^2+c=40$ and $c=40$

Exercise 13I

1 The derivative of the function f is given by $f'(x) = 3x + 4x^2$. The point $(-1, 0)$ lies on the graph of f. Find an expression for f.

2 It is given that $\frac{dy}{dx} = x + \frac{x^2}{5} + 2$ and that $y = 3$ when $x = 4$. Find an expression for y in terms of x.

3 It is given that $f'(x) = 3 - x$ and $f(2) = 1$. Find $f(x)$.

4 It is given that $f'(x) = 3x^2 - \frac{x^4}{3}$ and $f(0) = 2$. Find $f(x)$.

5 It is given that $\frac{dy}{dx} = 0.2x + 3x^2 + 1$ and that $y = -1$ when $x = 1$. Find the value of y when $x = 0.5$.

6 The derivative of the function f is given by $f'(x) = -x + 3$. The point $(-2, 0)$ lies on the graph of f.

a Find an expression for f.

b Find $f(2)$.

7 A company's marginal cost function is $M(x) = x - \frac{2}{x^2} + 1$ and the fixed cost is US\$145. Find the cost function.

8 A company's marginal cost function is $M(x) = 3 - 4x + x^2$ and the fixed cost for the company is US\$1000. Find the cost function.

9 A refrigerator factory's marginal revenue function is $M(t) = t^2 - 80$, where t is the number of refrigerators produced by the factory and the revenue is given in US\$.

The factory earns US\$ 567,000 in revenue from selling 120 refrigerators.

a Find the revenue function, $R(t)$.

b Find the factory's total revenue from producing 150 refrigerators.

International-mindedness

Ibn Al Haytham, born in modern-day Iraq in the 10th century, was the first mathematician to calculate the integral of a function in order to find the volume of a paraboloid.

Developing your toolkit

Now do the Modelling and investigation activity on page 584.

TOK

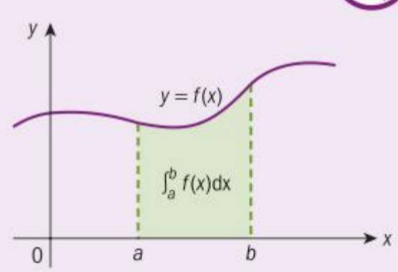
Why do we study mathematics?

What's the point?

Can we do without it?

Chapter summary

- When f is a non-negative function for $a \le x \le b$, $\int_a^b f(x)\,dx$ gives the area under the curve from $x = a$ to $x = b$.



- If $v(t)$ is a velocity–time function and $v(t) \ge 0$ over the interval of time $a \le t \le b$ then
  distance travelled $= \int_a^b v(t)\,dt$.
- The trapezoid rule is
  $$\int_a^b f(x)\,dx \cong \frac{1}{2} \times \frac{b-a}{n} \times \{y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})\}$$
  where the interval $a \le x \le b$ is divided into n intervals of equal width.
- Consider a positive and continuous function $y = f(t)$ over the interval $a \le t \le b$. The area enclosed between the graph of $y = f(t)$, the t-axis and the vertical lines $t = a$ and $t = x$ where $a \le x \le b$ is defined by $A(x) = \int_a^x f(t)\,dt$.