13 Approximating irregular spaces: integration

Concepts

- Space
- Approximation

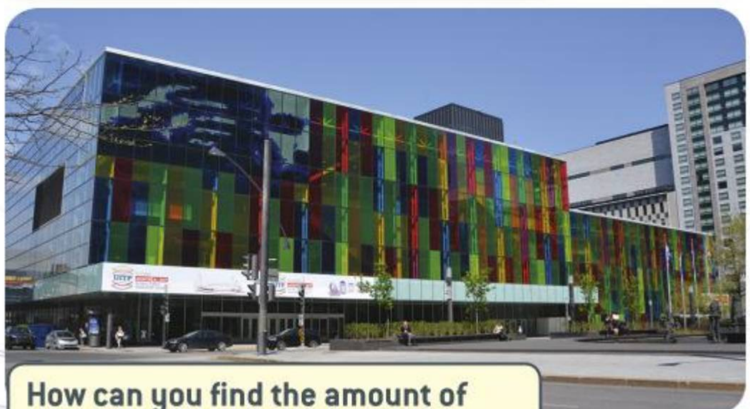
Microconcepts

- Lower limit
- Upper limit
- Definite integrals
- Area under a curve
- Trapezoidal rule
- Numerical integration
- Antiderivatives
- Indefinite integral
- Constant of integration

What do the graphs of functions that have the same derivative have in common? How do they differ?

This chapter explores integration, the reverse of differentiation. The area of an island and the amount of glass needed for the windows of a building can be represented by integrals. Integrals give you a way to estimate the values of areas that cannot be found using existing area formulae.

How can you estimate the area covered by oil spills out at sea?



How can you find the amount of glass in this building?

How can you estimate the area affected by a hurricane?

San Cristóbal is the easternmost island of the Galapagos. Here is a map of the island.

It is claimed that the total area of the island is 558 km². How can you test this value?

Use a rectangle to estimate the area of the island.

How did you use the map scale?

Does your result underestimate or overestimate the claimed area? Why?

How could you improve your estimate?

Developing inquiry skills

Write down any similar inquiry questions you might ask to model the area of something different; for example, the area of a national part, city of lake in your country.

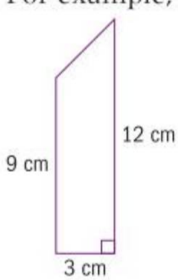
Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Use geometric formulae to find area

For example, the area of a trapezoid:



$$A = \frac{1}{2}(b_1 + b_2)h$$

$$= \frac{1}{2}(9 + 12) \times 3$$

$$= 31.5 \text{ cm}^2$$

2 Find the derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$ where all the exponents of x are integers

For example, the derivative function of

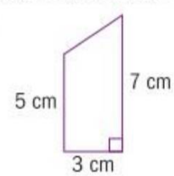
$f(x) = 3x^2 + \frac{x}{2} - 0.5$ is $f'(x) = 6x + \frac{1}{2}$

Skills check

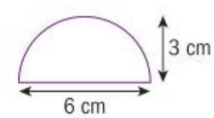
Click here for help with this skills check

1 Find the areas.

a



b



2 Find the derivative of each of these functions.

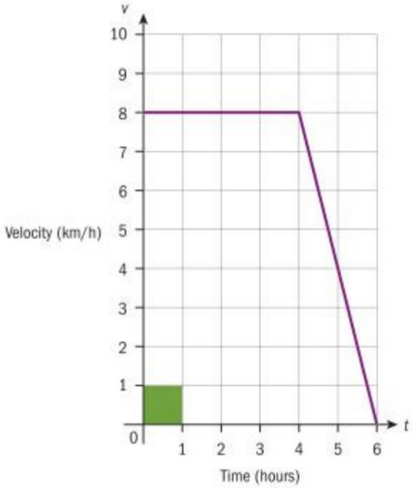
a $f(x) = x^3 + \frac{5x}{2} - 3$

b $g(x) = 4x^2 - x$

13.1 Finding areas

Velocity–time graphs

The graph shown below is a **velocity–time graph** for the journey of an object. The time taken from the start of the journey is represented on the horizontal axis and the velocity of the object is represented on the vertical axis.



This object travels at a constant velocity of 8 km/h for 4 hours and then it travels a further 2 hours with a steadily decreasing velocity until it reaches 0 km/h.

What distance did this object travel during the entire journey?

The graph is a piecewise linear function. To answer the question each part will be considered separately.

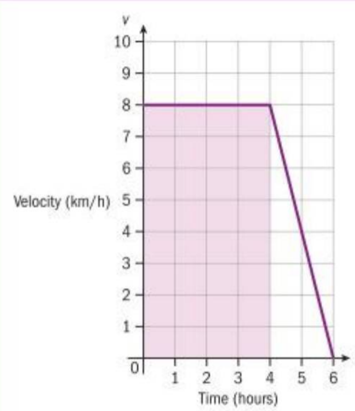
First 4 hours

The velocity is 8 km/h.

In 1 hour it travels 8 km therefore in 4 hours it travels $8 \times 4 = 32$ km.

What is the relationship between 32 and **the area under the graph** in this first part?

The green square shown in the graph above represents 1 km. Why?



Last 2 hours

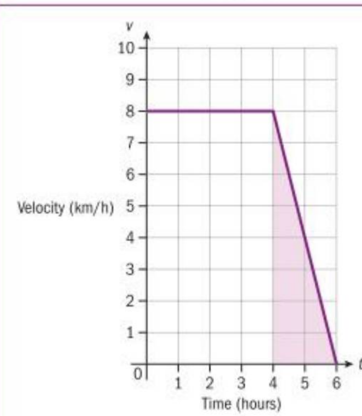
How many green squares will fit under the second part of the piecewise function?

Find the area under the graph.

The shape is triangular. Applying the formula for the area of a triangle:

$$\frac{8\,\text{km/h} \times 2\,\text{h}}{2} = 8\,\text{km}$$

Total distance travelled = 32 km + 8 km = 40 km



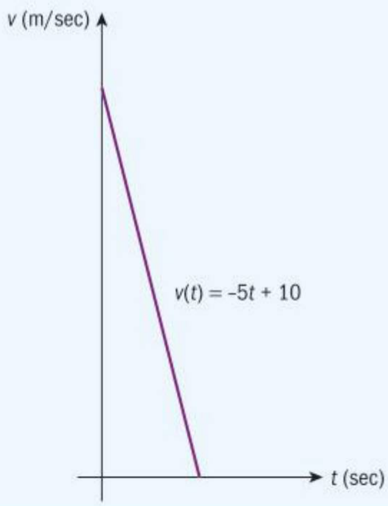
Reflect How does the area under a velocity–time graph relate to the distance travelled?

Now, if the velocity–time graph were curved, how would you calculate the area?

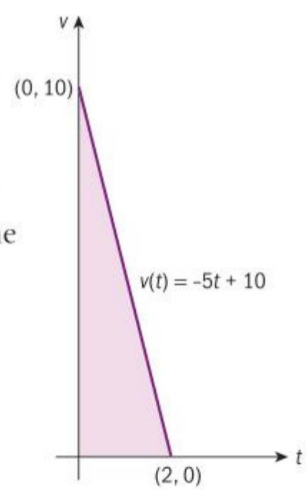
In this chapter you will study different methods to find or approximate areas between the graph of a function and the x-axis.

Example 1

For this velocity–time graph, find the distance travelled.



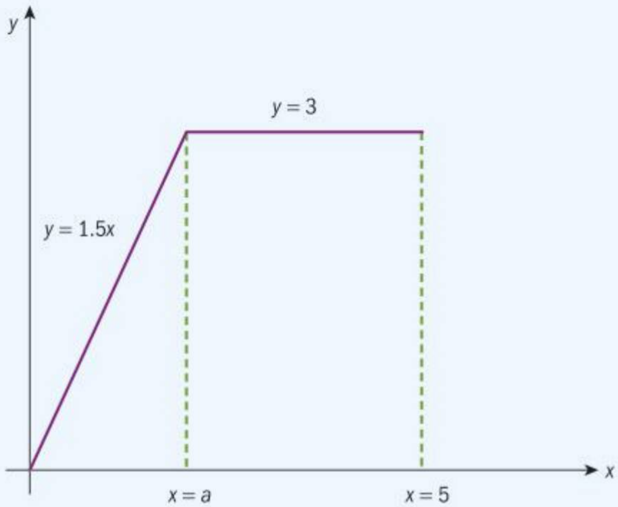
	The distance travelled is equal to the area under the velocity–time graph.
When $v = 0$: $-5t + 10 = 0$ then $t = 2$	To calculate the width of the triangle, find the point where the graph cuts the t-axis.
When $t = 0$: $v(0) = 10$	To calculate the height, find the point where the graph cuts the v-axis. At this point $t = 0$.
Area $= \frac{1}{2} \times 10 \times 2 = 10$ Distance travelled is 10 m.	Substitute into the area formula using $b = 2$ and $h = 10$.



Calculus

Example 2

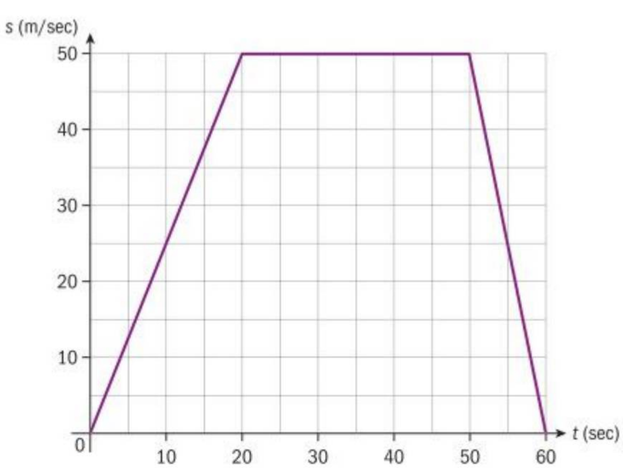
Find the area under the graph.



$1.5a = 3 \Rightarrow a = 2$	Find the value of a, where $y = 1.5x$ and $y = 3$ intersect.
Area of triangle $= \dfrac{2 \times 3}{2} = 3$	
Area of rectangle $= 3 \times 3 = 9$	Add the area of the triangle and the area of the rectangle.
Area under graph $= 3 + 9 = 12$	

Exercise 13A

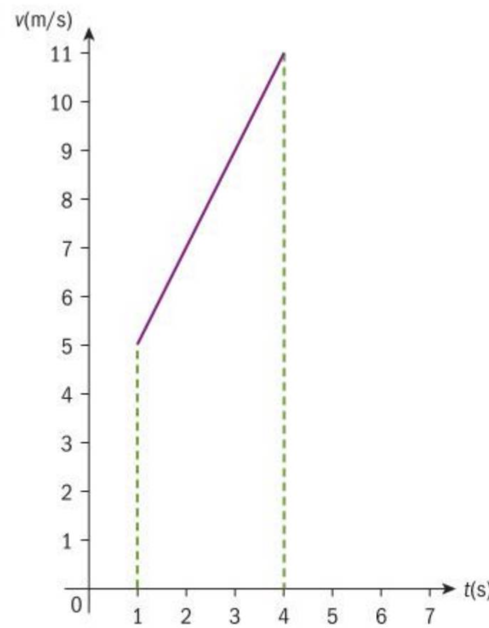
1 The graph below shows how the velocity of a car changes during the first 60 seconds of a journey.



Find the distance travelled during the 60 seconds.

2 For each velocity–time graph, find the distance travelled.

a



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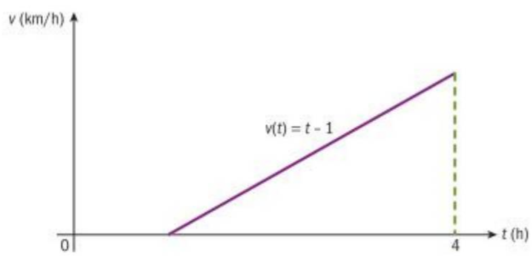

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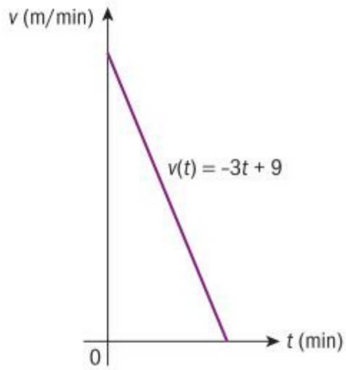
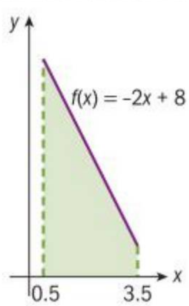


3 For each graph, find the shaded area.

a



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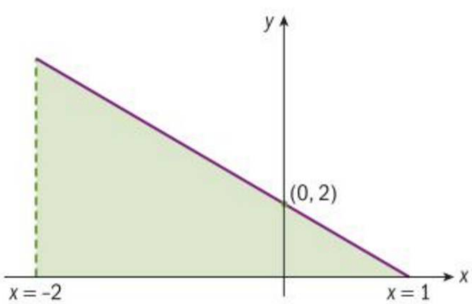

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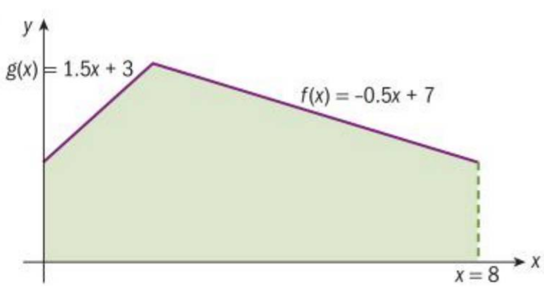

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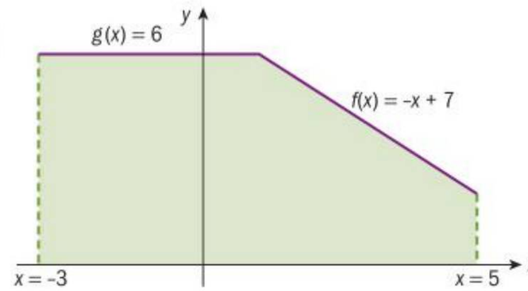


4 **a** Sketch the graph of the function $f(x) = 0.5x + 3$.

b Shade the region enclosed between the graph of f, the x-axis and the vertical lines $x = 1$ and $x = 6$.

c Find the area of the shaded region.

5 **a** Sketch the graph of the function $f(x) = -2x + 6$.

b Shade the region enclosed between the graph of f, the vertical line $x = 0$ and the x-axis.

c Find the area of the shaded region.

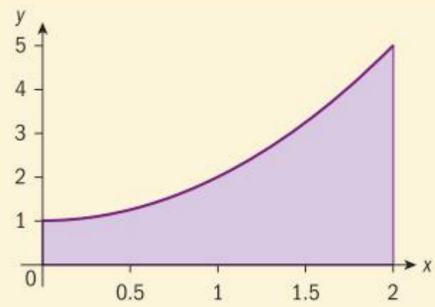
6 **a** Sketch the graph of the piecewise linear function

$$f(x) = \begin{cases} x, & 0 \le x \le 5 \\ 5, & 5 < x \le 9 \end{cases}$$

b Shade the region under the graph of $f(x)$ and above the x-axis.

c Find the area of the shaded region.

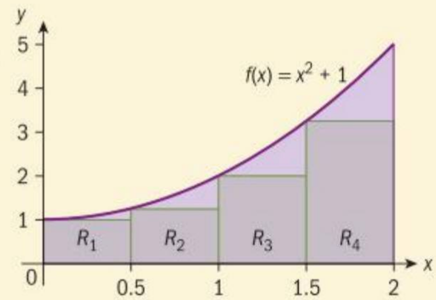
Investigation 1



1 Consider the area bounded by the graph of the function $f(x) = x^2 + 1$, the vertical lines $x = 0$, $x = 2$ and the x-axis.

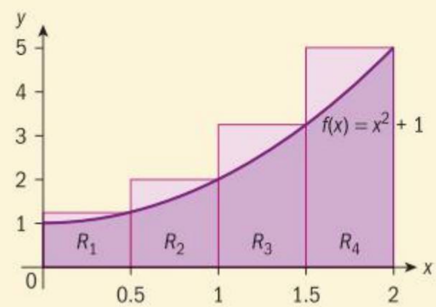
Estimate the area of the shaded region. Is your estimate an overestimate or an underestimate of the actual area? Discuss your method with a classmate.

2 In this investigation you will be using rectangles or vertical strips to estimate this area. At the end of the investigation you can check how close your estimate was to the actual area.



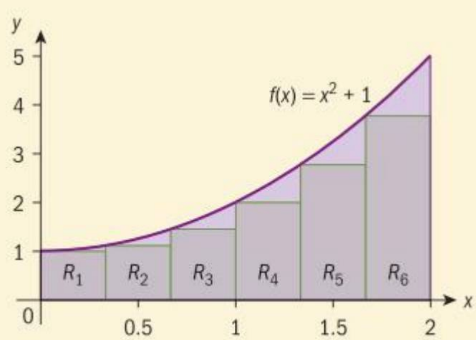
- The graph shows four rectangles with the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ is also shown.
- What is the width of the rectangles? How do you calculate it?
- What is the relationship between the height of each rectangle and the graph of the function?
- Find the heights of each of these rectangles.
- Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.
- Is this an underestimate or an overestimate of the actual area? Why? The sum of these areas will be a **lower bound** of the area of the shaded region. This will give an **underestimate** of the area.

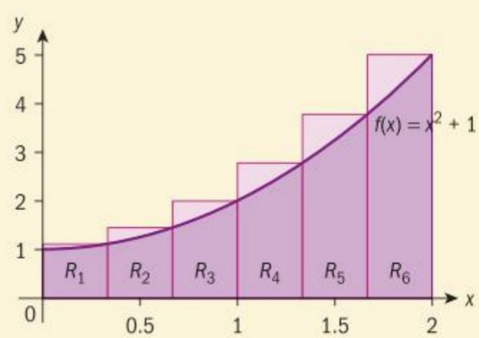
3 This graph also shows four rectangles with the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ is also shown.



- What is the width of each of these rectangles?
- Find the heights of each of these rectangles.
- Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.
- Is this an underestimate or an overestimate of the actual area? Why?
- If L_S represents the lower bound, A represents the actual area and U_S represents the upper bound, write an inequality relating L_S, U_S and A.

4 In each of the following graphs there are six rectangles. The area under the graph of $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ and the x-axis is also shaded.





- Approximate the area under the curve by considering a lower bound and an upper bound. Why do you think that more rectangles are being used?
- Complete the following tables to organize the information. You can create a table with your GDC to calculate the heights of the rectangles. How would you calculate the widths? Remember that they are all equal.
- Lower bound with six rectangles:

Rectangle	Width	Height	Area
R_1			
R_2			
R_3			
R_4			
R_5			
R_6			
			$L_S =$

- Upper bound with six rectangles:

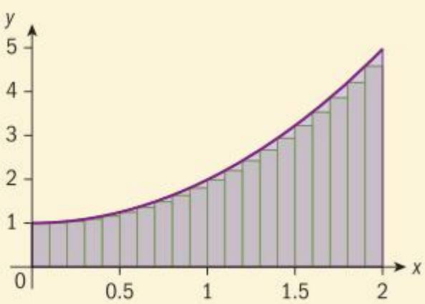
Rectangle	Width	Height	Area
R_1			
R_2			
R_3			
R_4			
R_5			
R_6			
			$U_S =$

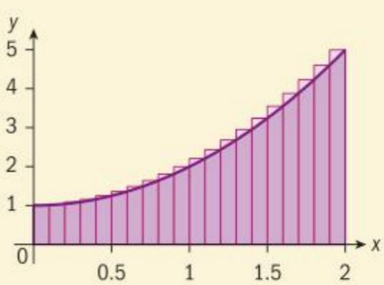
- What do you notice? Are the lower and upper bounds closer to each other than when you had four rectangles? How do you think these estimates can be improved?
- Write a new inequality relating L_S, U_S and A.

5 Look at the following graphs. The number of rectangles, n, has been increased in each case.

L_S and U_S are also given.

$n = 20,\ L_S = 4.47,\ U_S = 4.87$





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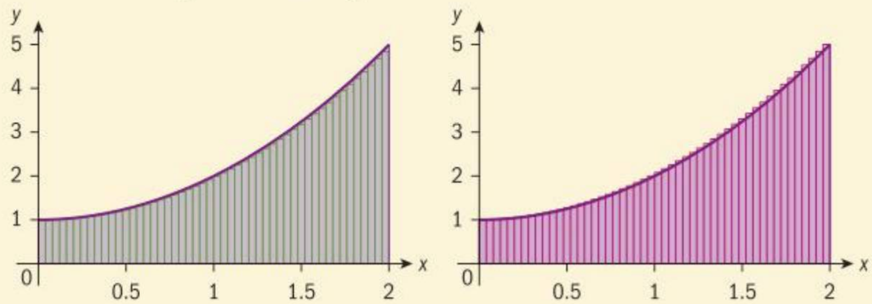
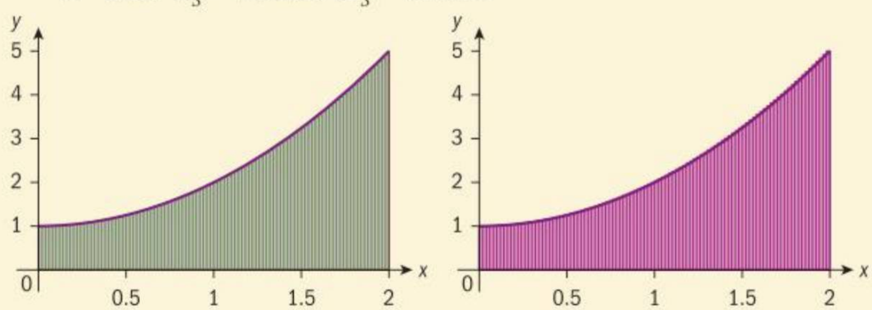
Calculus

$n=50,\ L_S=4.5872,\ U_S=4.7472$



$n=100,\ L_S=4.6268,\ U_S=4.7068$



- What can you say about the values of L_S as n increases?
- What can you say about the values of U_S as n increases?
- Here are some more values for n, L_S and U_S.

n	L_S	U_S
500	4.65867	4.67467
1000	4.66267	4.67087
10000	4.66627	4.66707

- What happens when *n* tends to infinity (gets larger and larger)? What can you say about the value of *A*?

International-mindedness

A Riemann sum, named after 19th century German mathematician Bernhard Riemann, approximates the area of a region, obtained by adding up the areas of multiple simplified slices of the region.

TOK

We are trying to find a method to evaluate the area under a curve.

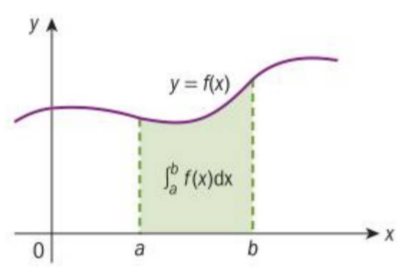
"The main reason knowledge is produced is to solve problems."

To what extent do you agree with this statement?

When f is a non-negative function for $a \le x \le b$, $\int_a^b f(x)\,\mathrm{d}x$ gives the area under the curve from $x=a$ to $x=b$.

$\int_a^b f(x)\,\mathrm{d}x$ is read as "the definite integral between $x=a$ and $x=b$."

The number a is called the **lower limit** of integration and b is called the **upper limit** of integration.



Investigation 1 (continued)

6 In the investigation, what is the lower limit of integration? What is the upper limit of integration? Is the function positive between the lower and the upper limits? How would you represent A using definite integral notation?

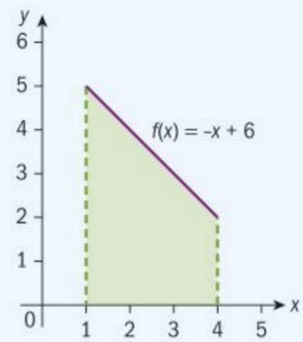
7 **Conceptual** How do areas under curves within a given interval relate to the definite integral and to lower and upper rectangle sums on the same interval?

Reflect What is a definite integral?

Example 3

a Write down a definite integral that gives the area of the shaded region.

b Find the value of the definite integral.



a $\int_{1}^{4}(-x+6)\mathrm{d}x$	The lower limit is $x = 1$. The upper limit is $x = 4$. The function is $f(x) = -x + 6$.
b $\int_{1}^{4}(-x+6)\mathrm{d}x = \frac{1}{2}\times(2+5)\times 3 = 10.5$	The shape is trapezoidal. Bases are: $b_1 = f(1) = -1 + 6 = 5$ $b_2 = f(4) = -4 + 6 = 2$ Height = 3 Substitute into the area of a trapezoid formula.

Example 4

For the definite integral $\int_{-1}^{3}(x+4)\mathrm{d}x$:

a State clearly the function being integrated, the lower limit of integration and the upper limit of integration.

b Sketch the graph of the function. Shade the region whose area the definite integral represents.

c Find the value of the definite integral by using an area formula.

a The function is $f(x) = x + 4$. The lower limit is $x = -1$ and the upper limit is $x = 3$.	Given that this is a linear function, you should expect the graph to be a line segment in the interval $-1 \le x \le 3$.

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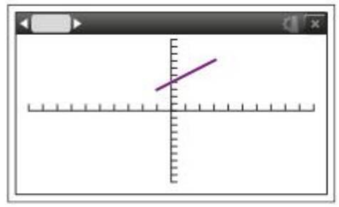
Calculus

You can use technology to see the graph.



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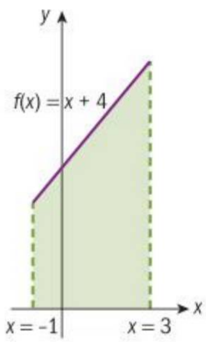


As the function is positive in the given interval, the definite integral represents the area between the graph and the x-axis.

The shape is trapezoidal. The bases of the trapezoid are b_1 and b_2.

$b_1 = f(-1) = -1 + 4 = 3$

$b_2 = f(3) = 3 + 4 = 7$

Height $= 3 - (-1) = 4$

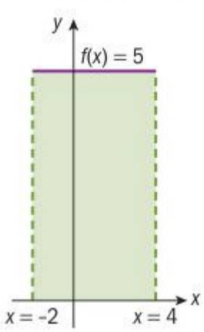
b $\int_{-1}^{3}(x+4)\,dx = \frac{1}{2}\times(3+7)\times 4 = 20$

Substitute into the area of a trapezoid formula.

Exercise 13B

1 a Write down a definite integral that gives the area of each of the following regions.

i



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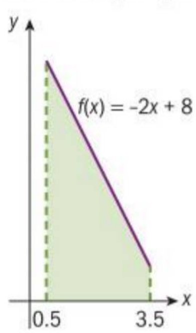

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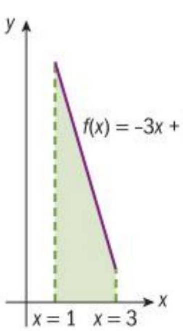

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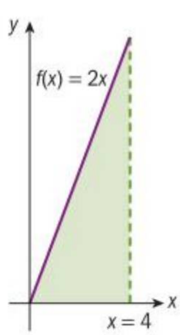


b Calculate the definite integrals from part **a** using existing area formulae.

2 Find the value of the following definite integrals by using existing area formulae. In each case sketch the function in an appropriate interval and shade the region whose area the definite integral represents.

a $\int_{2}^{6}(x+1)\,dx$

b $\int_{0}^{4}(-2x+8)\,dx$

c $\int_{-2}^{0}(-0.5x+4)\,dx$

Example 5

Calculate the definite integral $\int_0^2 (x^2+1)\,dx$.

Compare your answer with the value found for A in Investigation 1.

TOK

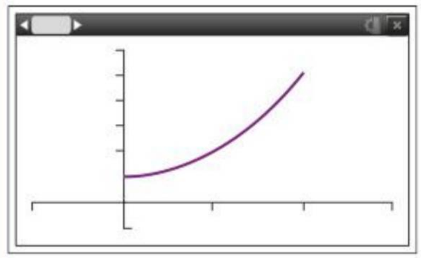
Is imagination more important than knowledge?

$\int_0^2 (x^2+1)\,dx = \frac{14}{3}$ (4.67 to 3 sf).

This value is the same as the one found for A, the area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$ and $x = 2$, in Investigation 1.

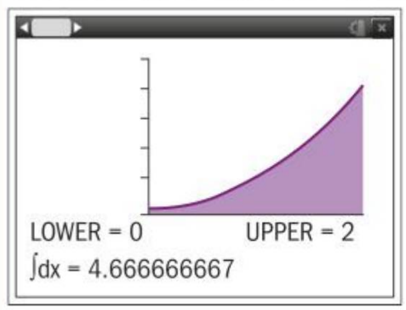
Method 1

Draw the graph of the function with your GDC.



Find the area.

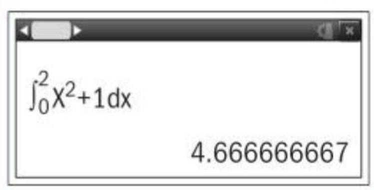
Then enter the lower and upper bounds.



The definite integral is equal to 4.67.

Method 2

Evaluate the definite integral.



Example 6

Consider the region A enclosed between the curve $y = -x(x-3)$ and the x-axis.

a Write down the definite integral that represents the area of A.

b Find the area of A.

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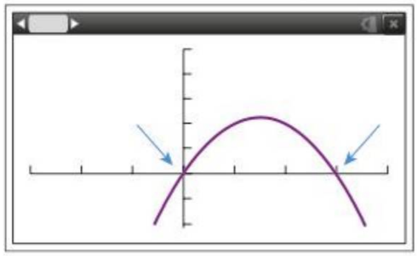
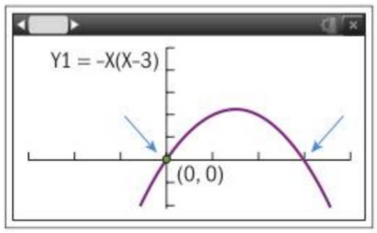
a $\int_0^3 -x(x-3)\,dx$

You first have to identify the region using your GDC.

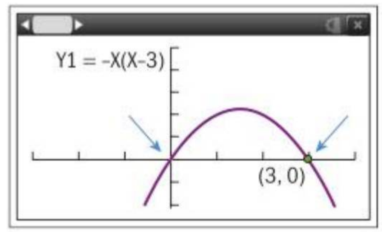


In this case the lower and upper bounds are not given but from the graph it can be seen that these are the **roots** of the parabola.

Find the roots.



$x = 0$ is one of the roots, the lower bound of the definite integral.



$x = 3$ is the other root, the upper bound of the definite integral.

b $A = 4.5$

Once you have identified the region, write down the definite integral.

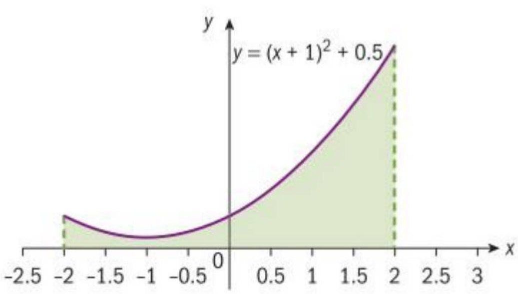
Exercise 13C

1 For each of the following diagrams:

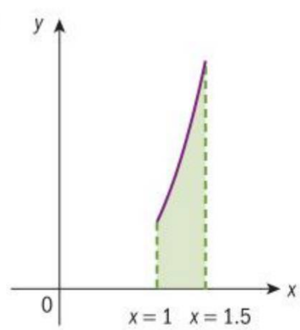
i Write down the definite integral that represents the area of the shaded region.

ii Find the area of the shaded region.

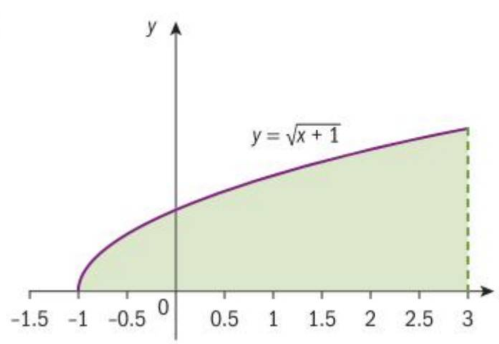
a



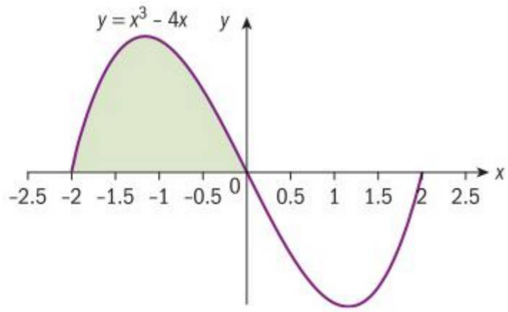
b



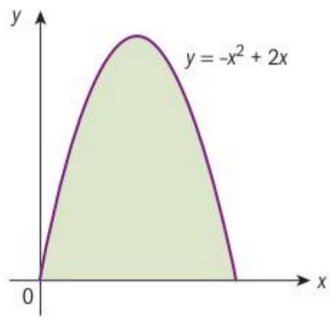
c



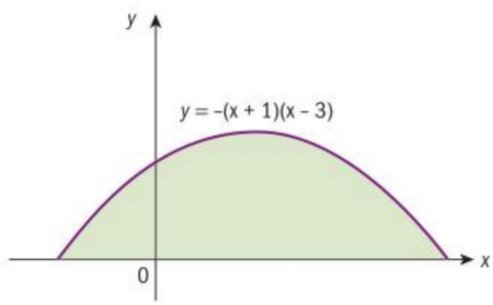
d



e



f



2 In each of the following:

i Write down the definite integral that represents the enclosed area.

ii Find the area.

a $y = x^2$, the x-axis and the vertical lines $x = 2$ and $x = 4$

b $y = 2x$, the x-axis and the vertical lines $x = -1$ and $x = 1$

c $y = \dfrac{1}{1+x^2}$ and the x-axis in the interval $-1 \le x \le 1$

d $y = \dfrac{1}{x}$ and the x-axis in the interval $0.5 \le x \le 3$

e $f(x) = -(x-3)(x+2)$, the vertical axis and the vertical line $x = 1$

f $f(x) = -(x-3)(x+2)$, the vertical axis and the horizontal axis

g $f(x) = -(x-3)(x+2)$ and the horizontal axis

h $f(x) = -x^2 + 2x + 15$ and the vertical lines $x = -2$ and $x = 4.5$

i $f(x) = -x^2 + 2x + 15$ and the line $y = 0$

j $f(x) = 3 - e^x$, the vertical line $x = -1$ and the x-axis

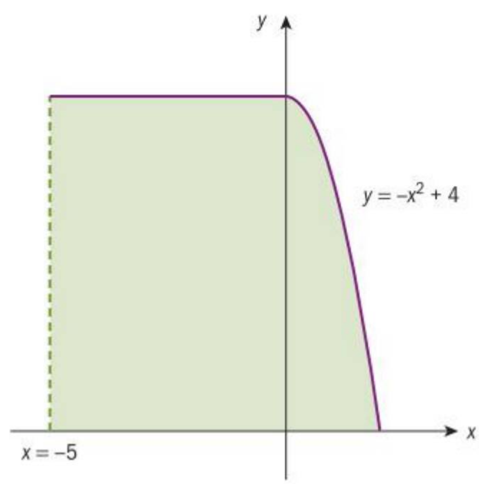
k $y = (x+2)^3 + 5$ and the coordinate axes.

3 Consider the curve $y = -x^2 + 4$.

a Find the zeros of this curve.

b Find the point where this curve cuts the y-axis.

Below is shown the graph of a piecewise function f made up by a horizontal line segment and part of the parabola $y = -x^2 + 4$. The area under the graph of f and above the x-axis has been shaded.



c Find the area under the graph of f in the interval $-5 \le x \le 0$.

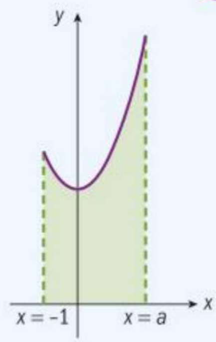
d **i** Write down an expression for the area under the graph of f and above the x-axis for $x > 0$.

ii Find the area.

e Find the area of the whole shaded region.

Example 7

The area of the region bounded by the graph of $f(x) = x^2 + 3$, the x-axis and the vertical lines $x = -1$ and $x = a$ with $a > -1$ is equal to 12. Find the value of a.



$a = 2$	The unknown, a, is the upper bound of this area. First write down the definite integral $\int_{-1}^{a}(x^2+3)\,dx = 12.$ Use technology to find the value of a.

Exercise 13D

1 The area of the region bounded by the graph of $f(x) = -(x+1)(x-5)$, the x-axis, the y-axis and the vertical line $x = a$ where $a > 0$ is equal to 24. Find the value of a.

2 The area of the region bounded by the graph of $f(x) = 2^{-x}$ and the x-axis between $x = -3$ and $x = a$ where $a > -3$ is equal to 9. Find the value of a. Give your answer correct to four significant figures.

3 The area of the region bounded by the graph of $f(x) = x + \frac{1}{x}$ and the x-axis between $x = a$ and $x = 3$ where $0 < a < 3$ is equal to 6.

- **a** Describe this region using a partially shaded diagram.
- **b** Find the value of a. Give your answer correct to four significant figures.

4 Given that $\int_{-2}^{a} x^2\,dx = \frac{7}{3}$ where $a > -2$:

- **a** Find the value of a.
- **b** Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.

5 Given that $\int_{-1}^{b}(1+x^3)\,dx = 2$ where $b > -1$:

- **a** Find the value of b.
- **b** Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.

6 Given that $\int_{-1}^{t}\sqrt{x+1}\,dx = \frac{16}{3}$ where $t > -1$:

- **a** Find the value of t.
- **b** Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.

Numerical integration

You will now study a new **numerical method** to estimate areas between a curve and the x-axis in a given interval. Numerical integration is used, among other techniques, when we do not have a mathematical function to describe the area. Instead we have a set of points.

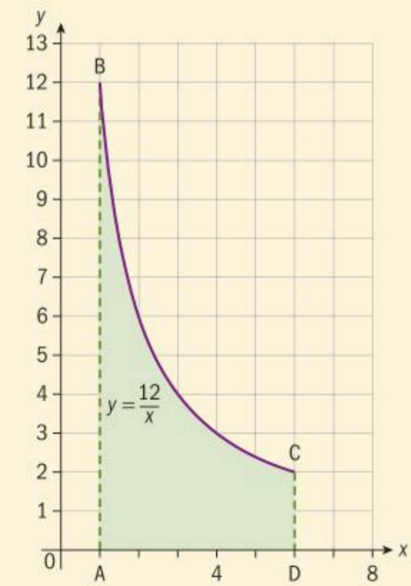
However, throughout the investigation we will use a function to illustrate this new method.

Investigation 2

1 Consider the curve $y = \dfrac{12}{x}$, where $1 \le x \le 6$.

The area of the region enclosed between the graph of $f(x) = \dfrac{12}{x}$ and the x-axis in the interval $1 \le x \le 6$ will be called S.

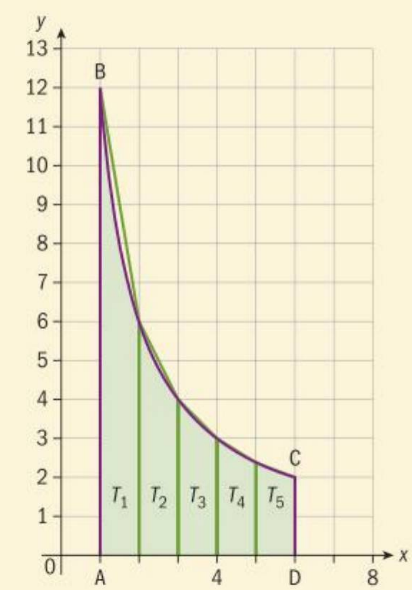
Write down an expression for S and find its value. Give your answer correct to two decimal places.



2 Consider the trapezoid ABCD.

- What is the relationship between the base AB of the trapezoid ABCD and the graph of the function $f(x) = \dfrac{12}{x}$?
- What is the relationship between the base DC of the trapezoid ABCD and the graph of the function $f(x) = \dfrac{12}{x}$?
- Find the area of the trapezoid ABCD. Is this value an overestimate or an underestimate of S? What is the **error** in this approximation?

3 To find a better approximation to the value of S you can subdivide the shaded area into strips with **equal width**. The area of every strip can be approximated with the area of a trapezoid and then all these areas added.



The graph shows the shaded area subdivided into five strips. First, you will find the area of trapezoids T_1, T_2, T_3, T_4 and T_5.

You will approximate the value of S by adding up the areas of these five trapezoids.

- The height of each of these trapezoids (or the width of the strips) is equal to 1 unit. How is it found?
- Let the parallel sides of the trapezoids be y_0, y_1, y_2, y_3, y_4 and y_5 where y_0 is the length of AB and y_5 is the length of DC. How can you calculate the lengths of y_0, y_1, y_2, y_3, y_4 and y_5?
- The table on the next page will help you organize the calculations. Use a table on your GDC to complete it.

Continued on next page

Trapezoid	Base 1 (b_1)	Base 2 (b_2)	h	$A=\frac{1}{2}(b_1+b_2)h$
T_1	$y_0=f(1)=\frac{12}{1}=12$	$y_1=$	1	
T_2	$y_1=$	$y_2=$	1	
T_3	$y_2=$	$y_3=$	1	
T_4	$y_3=$	$y_4=$	1	
T_5	$y_4=$	$y_5=f(6)=\frac{12}{6}=2$	1	
		Sum of the areas of the five trapezoids		

- Is this estimation better than the estimate for S found in **1**? Why? How could you improve this value? Why?

4 Now, you will approximate the value of S by adding up the area of $n=8$ trapezoids.

- What is the height of each of the trapezoids now? How did you find this value?
- Show that the sum of the area of the eight trapezoids is now equal to 21.87, correct to two decimal places. Draw a table similar to the one from part **3** to organize the calculations.
- What is the error made with this approximation?

5 What can you say about the error made in the approximation when the number of trapezoids, n, tends to infinity?

6 **Conceptual** How does the sum of the areas of trapezoids defined by a curve approximate the area under the curve within a given interval?

Trapezoidal rule

In the previous investigation you saw that the definite integral $\int_a^b f(x)\,dx$, the area of the region bounded by the curve $y=f(x)$ and the x-axis over the interval $a \le x \le b$, can be approximated by the sum of the areas of trapezoids.

For example, when $n=4$, the height of each of the trapezoids is $h=\frac{b-a}{4}$.

Recall that

$y_0=f(a)$, $y_1=f(x_1)$, $y_2=f(x_2)$, $y_3=f(x_3)$ and $y_4=f(b)$ so that

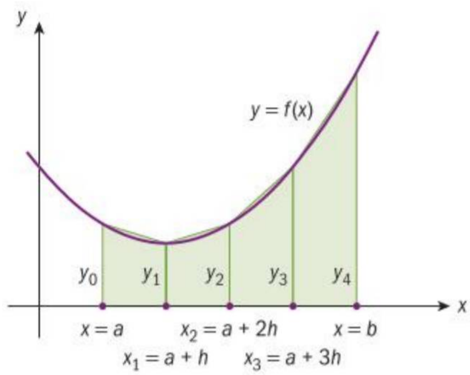
$$\int_a^b f(x)\,dx \cong \underbrace{\frac{1}{2}(y_0+y_1)\times\frac{b-a}{4}}_{\text{Area of } T_1}+\underbrace{\frac{1}{2}(y_1+y_2)\times\frac{b-a}{4}}_{\text{Area of } T_2}+\underbrace{\frac{1}{2}(y_2+y_3)\times\frac{b-a}{4}}_{\text{Area of } T_3}+\underbrace{\frac{1}{2}(y_3+y_4)\times\frac{b-a}{4}}_{\text{Area of } T_4}$$

The right-hand side can be simplified because there is a common factor $\frac{1}{2}\times\frac{b-a}{4}$:

$$\int_a^b f(x)\,dx \cong \frac{1}{2}\times\frac{b-a}{4}\left[(y_0+y_1)+(y_1+y_2)+(y_2+y_3)+(y_3+y_4)\right]$$

The sum inside the square brackets can be simplified to:

$$\int_a^b f(x)\,dx \cong \frac{1}{2}\times\frac{b-a}{4}\left\{y_0+2(y_1+y_2+y_3)+y_4\right\}$$



More generally,

> The trapezoid rule is $\int_a^b f(x)\,dx \cong \frac{1}{2}\times\frac{b-a}{n}\times\left\{y_0+y_n+2(y_1+y_2+\ldots+y_{n-1})\right\}$
>
> where the interval $a \le x \le b$ is divided into n intervals of equal width.

What is the value of x_i when $i = 0$?

What is the value of x_i when $i = n$?

What is the meaning of $\frac{b-a}{n}$ in this formula?

Reflect What geometric methods can be used to approximate integrals?

Example 8

Estimate the area under a curve over the interval $4 \le x \le 12$, with x- and y-values given in the following table.

x	4	6	8	10	12
y	5	13	10	3	4

Area of trapezoid 1 $= \frac{1}{2}(5+13)\times 2 = 18$	In this example, there is no formula of the form $y = f(x)$. You need to find the area of each trapezoid and then sum these areas.
Area of trapezoid 2 $= \frac{1}{2}(13+10)\times 2 = 23$	The table suggest that there are four trapezoids.

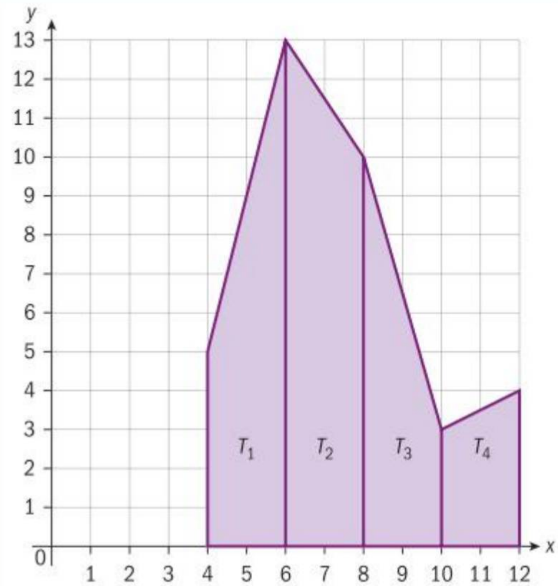
Continued on next page

Calculus

Area of trapezoid 3 $= \frac{1}{2}(10+3)\times 2 = 13$

Area of trapezoid 4 $= \frac{1}{2}(3+4)\times 2 = 7$

Area under the curve $= 18 + 23 + 13 + 7 = 61$



Example 9

Estimate the area under the graph of $f(x) = e^x$ over the interval $0 \le x \le 1$ using five trapezoids.

x	0	0.2	0.4	0.6	0.8	1
y	1	$e^{0.2}$	$e^{0.4}$	$e^{0.6}$	$e^{0.8}$	e

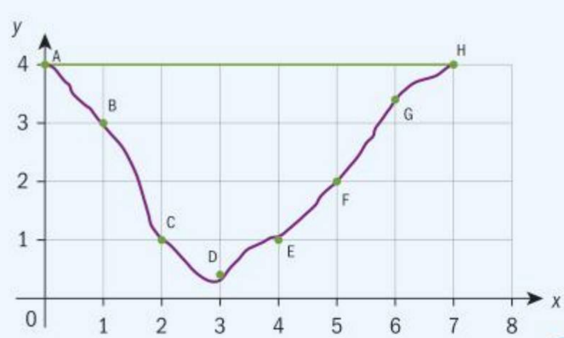
Draw a table of values.

$$\int_a^b e^x\,dx \cong \frac{1}{2}\times 0.2\times\left(1+2(e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8})+e\right)$$
$$\cong 1.7240...$$
$$\cong 1.72$$

Substitute into the trapezoid rule using $a = 0,\ b = 1,\ n = 5$, and with heights of trapezoids $= \frac{1-0}{5} = 0.2$.

Example 10

The cross-section of a river is shown here. If the water is flowing at 0.8 m/s use the trapezoidal rule, with seven trapezoids, to find an approximation for the volume of water passing this point in one minute. All lengths are in metres.



A	B	C	D	E	F	G	H
(0,4)	(1, 3)	(2, 1)	(3, 0.4)	(4, 1)	(5, 2)	(6, 3.4)	(7, 4)

Use the trapezoidal rule to find the area under the curve:	The lengths of the parallel lines are given by the y-coordinates in the table.
$A \approx \frac{1}{2} \times 1\left(4+4+2\left(3+1+0.4+1+2+3.4\right)\right)$	The trapezoidal rule is applied to these values.
$= \frac{1}{2} \times 29.6 = 14.8\ \text{m}^2$ Cross-sectional area of river is: $28 - 14.8 = 13.2\ \text{m}^2$	
Volume of water per minute: $= 13.2 \times 0.8 \times 60 \approx 634\ \text{m}^3$	Volume of water is the amount that passes per second multiplied by 60.

Exercise 13E

1 Estimate the area under a curve over the interval $1 \le x \le 9$, with the x- and y-values given in the following table.

x	1	3	5	7	9
y	5	7	6	10	4

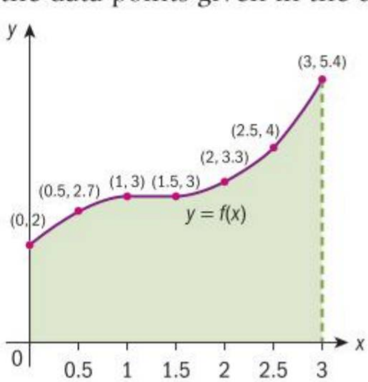
2 Estimate the area under a curve over the interval $0 \le x \le 6$, with the x- and y-values given in the following table.

x	0	1.5	3	4.5	6
y	1	4	2	5.5	0

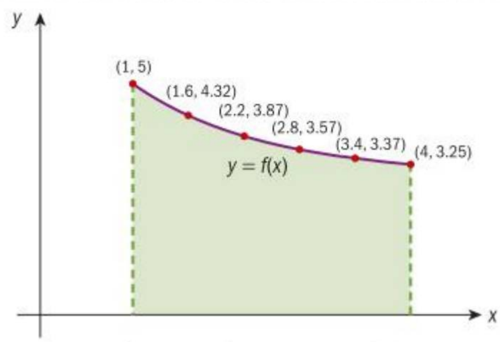
3 Estimate the area under a curve over the interval $-1 \le x \le 4$, with the x- and y-values given in the following table.

x	−1	0.25	1.5	2.75	4
y	5	7	3.5	6	8

4 Estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.



5 Estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.



6 Estimate the area between each curve and the x-axis over the given interval, using the specified number of trapezoids. Give your answers to four significant figures.

a $f(x) = \sqrt{x}$, interval $0 \le x \le 4$ with $n = 5$

b $f(x) = 2^x$, interval $-1 \le x \le 4$ with $n = 4$

c $f(x) = \dfrac{10}{x} + 1$, interval $2 \le x \le 5$ with $n = 6$

d $y = -0.5x(x-5)(x+1)$, interval $0 \le x \le 5$ with $n = 5$

7 Consider the region enclosed by the curve $y = -2(x-3)(x-6)$ and the x-axis.

a Sketch the curve and shade the region.

b **i** Write down a definite integral that represents the area of this region.

ii Find the area of this region.

c Estimate the area of this region using six trapezoids.

d Find the percentage error made with the estimation made in part **c**.

8 Consider the region enclosed by the graph of the function $f(x) = 1 + e^x$, the x-axis and the vertical lines $x = 0$ and $x = 2$.

a Sketch the function f and shade the region.

b **i** Write down a definite integral that represents the area of this region.

ii Find the area of this region. Give your answer correct to four significant figures.

c Estimate the area of this region using five trapezoids. Give your answer correct to four significant figures.

d Find the percentage error made with the estimation found in part **c**.

Developing inquiry skills

In the opening scenario for this chapter you looked at how to estimate the area of an island.

How could you improve your estimation of the area of the island using what you have studied in this section?

How close is your estimate to the claimed area? Why is your answer an estimate?

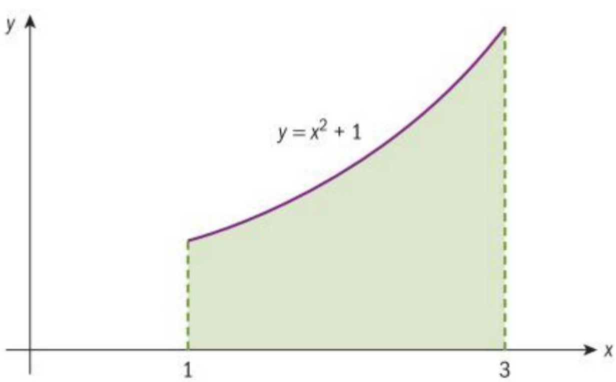
TOK

Galileo said that the universe is a grand book written in the language of mathematics.

Where does mathematics come from? Does it start in our brains or is it part of the universe?

13.2 Integration: the reverse process of differentiation

You have so far found areas under curves between two given values. For example, you have seen how to find the area enclosed between $y = x^2 + 1$, the x-axis and the vertical lines $x = 1$ and $x = 3$.



HINT

The area of the shaded region is $\int_1^3 (x^2+1)\,dx = \frac{32}{3}$.

You are now going to find expressions for areas when one of the limits is fixed and the other is variable.