HINT

There are many websites that will give the derivative of a function as an equation and indeed some calculators do this as well. These are **not** allowed in exams.

However, in examinations, it will be expected that you use your GDC to find numerical values for the derivatives at given points and also to draw the graph of the derivative. This widens the range of functions that you might need to find the gradient for. Unless told otherwise, always use your GDC if it makes answering the question simpler.

Example 6

Consider $y = \frac{x+2}{x-1}$, $x \neq 1$.

Find the gradient of the curve at the points where $x = 2$ and $x = 3$.

-3 and -0.75	The gradient is found using the numerical derivative function on your GDC.

Exercise 12D

1 Find the gradient of the following curves at the point where $x = 3$.

a $y = \frac{x^2}{x-1}$ **b** $y = x\ln x$ **c** $f(x) = \frac{2x^2-1}{x+3}$ **d** $s = \frac{t+5}{t^3+1}$ **e** $y = xe^{2x}$ **f** $g(x) = (x^2-3)(x-1)^5$

12.3 Maximum and minimum points and optimization

Investigation 4

The number of bacteria, B, in thousands, in a culture at time, t minutes, is given by the formula:

$B(t) = 0.00314x^3 - 0.1926x^2 + 2.848x + 5$

a Sketch the graph of $B(t) = 0.00314x^3 - 0.1926x^2 + 2.848x + 5$ for $0 \leq t \leq 40$.

b Find $B'(t)$.

c Find the values of t when there are the most and the least bacteria.

d Find the gradient of the function at the two points found in part **b**.

e Calculate the gradient of the function at a point just before and just after the values of t found in part **b**.

f Use your results to explain the difference between the local maximum and the local minimum.

g **Conceptual** How do stationary points help solve real-life problems?

International-mindedness

Maria Agnesi, an 18th century, Italian mathematician, published a text on calculus and also studied curves of the form $y = \frac{a^2}{x^2} + a^2$.

TOK

How can you justify the rise in tax for plastic containers eg plastic bags, plastic bottles etc using optimization?

You can also use technology and solve $f'(x) = 0$ to find the value of stationary points.

For a function $f(x)$, if $f'(c) = 0$, then the point $(c, f(c))$ is called a **stationary point** as the instantaneous rate of change is 0 at that point.

If the signs of the gradients are different on both sides of the stationary point then it means it is either a maximum or minimum point.

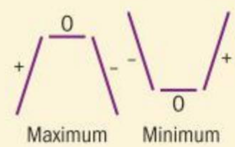
In many cases, we are interested in when a function reaches its maximum or its minimum as in real-life cases these points can be important. For example, if it is a function of profit versus the number of products produced, we would like to know where the profit is a maximum.

A stationary point can be a local **maximum** point or a local **minimum** point. At a local maximum or minimum point $f'(x) = 0$.

HINT
The local maximum or local minimum points may not be the maximum or minimum values of the graph.

If the sign of the derivative goes from negative to positive, there is a minimum point.

If the sign of the derivative goes from positive to negative, there is a maximum point.



Example 7

Consider the curve $s = 3t^3 + 2t^2 - 4t + 2$.

a Find $\frac{ds}{dt}$.

b On the same axes, sketch $s = 3t^3 + 2t^2 - 4t + 2$ and its derivative.

c Solve the equation $\frac{ds}{dt} = 0$.

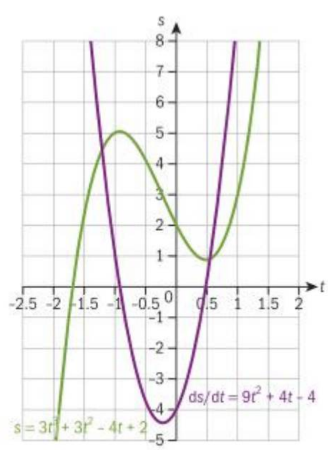
d State the feature of $s = 3t^3 + 2t^2 - 4t + 2$ indicated by these points.

e If the domain of the function is restricted to $-2 \le t \le 2$, find the actual maximum and minimum values of the function.

a $\frac{ds}{dt} = 9t^2 + 4t - 4$

Using the usual rules for differentiating but with the new notation.

b



You can either plot the curve found in part **a** or use the derivative function on the GDC to do so.

Use your GDC to find the roots of $9t^2 + 4t - 4 = 0$.

Continued on next page

Calculus

c $t = -0.925,\ 0.481$	This observation provides an alternative method for solving part **c**: using the GDC to find the maximum and minimum points on the original curve.
d Each of these values of t the curve gives a stationary point, one is a local maximum point and the other a local minimum point.	The maximum and minimum values (sometimes called the **global** maximum and minimum) are at the end points of the curve.
e 26 and −6	

Example 8

Consider the curve $y = x + \frac{1}{x}, x \neq 0$.

a Find an expression for $\frac{dy}{dx}$.

b Sketch the curve.

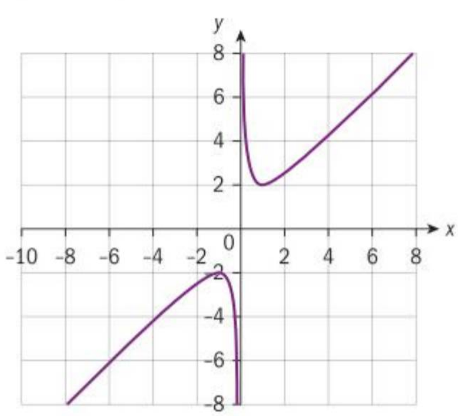
c Find the coordinates of any local maximum or minimum points.

d Find the gradient of two points, close to but on either side of the minimum point.

e Explain why this shows that the point found is a local minimum.

a $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

b



c Local maximum point is $(-1, -2)$.
Local minimum point is $(1, 2)$.

These points can be found either by solving $\frac{dy}{dx} = 0$ or by finding the points directly from the GDC.

d $x = 0.5,\ \frac{dy}{dx} = 1 - 4 = -3$

$x = 1.5,\ \frac{dy}{dx} \approx 1 - 0.444 = 0.556$

Any points will do as long as there are no discontinuities or other turning points between them and the local maximum or minimum.

e Gradient changes from negative to positive.

Exercise 12E

1 Jules throws a stone into the air. The height of the stone can be modelled by the equation $f(t) = 1 + 7.25t - 1.875t^2$ where t is the time, in seconds, that has passed since the stone was thrown, and $f(t)$ is the height of the stone, in metres.

a Find $f'(t)$.

b Find the stationary point and the time when this occurred.

c Show that this is a maximum point.

2 A company's profits, in thousands of dollars, can be modelled by the function:

$P(x) = 0.08x^3 - 1.9x^2 + 12.5x$, where x is the number of units sold (in millions) each week.

a Sketch a graph of $P(x)$ for $0 \le x \le 15$.

b Calculate the average rate of change between $x = 2$ and $x = 3$, including the units. Explain the meaning of the value you find.

c Calculate the instantaneous rate of change at $x = 3$, $x = 8$ and $x = 13$. Explain the meaning of these values.

d State the values for which the instantaneous rate of change is positive. State the values of x for which the instantaneous rate of change is negative. Explain the meaning of each of these results.

e Write down the values of x for which the instantaneous rate of change is zero. Justify your answer.

f Find the maximum and minimum values and show that these are maximum and minimum values.

3 The path of a shot-put can be modelled by the function $f(t)=1.75 + 0.75t - 0.0625t^2$, where t is the time (in seconds) and $f(t)$ is the height of the shot-put (in metres) above the ground.

Find the maximum height of the shot-put and the time when this occurred.

4 The profit $\$P$ that Yarona makes each day is dependent on the number of brownies, n, that she bakes.

Her daily profit can be modelled by the function $P(n) = -0.056n^2 + 5.6n - 20$.

Find the maximum profit that Yarona makes in a day and the number of brownies she needs to sell to make this profit.

5 A business buys engine parts from a factory. The business is currently deciding which of three purchasing strategies to use for the next stage of its development. Its researchers produce models for each of the strategies. P is the expected profit (in €10 000) and n is the number of parts it buys (in 1000s). The largest number of parts the factory can sell to the business is 5000 and there is no minimum.

a For each model find the maximum profit and the number of parts the business needs to buy to make this profit.

b State which strategy it should adopt.

i $P = 0.5n + 1.5 + \dfrac{4}{n+1}$

ii $P = \dfrac{n^3}{3} - \dfrac{5n^2}{2} + 6n - 4$

iii $P = \dfrac{n^3}{24} - \dfrac{5n^2}{8} + 3n$

6 John is a keen cyclist and is planning a route in the Alps. The profile of the route he would like to take can be modelled by the function:

$y = -0.081x^4 + 0.89x^3 - 2.87x^2 + 3x$, $0 \le x \le 6$, where y (× 100 m) is the height of the point on the route that is a horizontal distance x (× 10 km) from his starting point.

Plot the graph and find the maximum height John will climb to on this route.

International-mindedness

The Greeks' mistrust of zero meant that Archimedes' work did not lead to calculus.

Calculus

Optimization

Many practical problems involve finding maximum or minimum values. For example, we may want to maximize an area or minimize cost. Such problems are called optimization problems.

Example 9

A can of dog food contains 500 cm^3 of food. The manufacturer wants to receive the maximum profits by making sure that the surface area of the can has optimal dimensions. Let the radius of the can be r cm and the height, h cm.

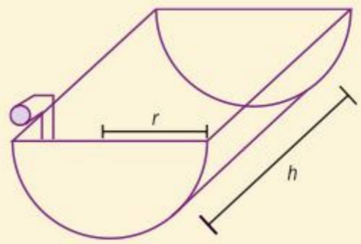
Find the dimensions of the can that will have the minimum surface area.

$V = \pi r^2 h = 500$ So, $h = \dfrac{500}{\pi r^2}$ Surface area, $S = 2\pi rh + 2\pi r^2$	In order to make maximum profits, the surface area of the can needs to be as small as possible. You are given the volume of the can and you use this to find an expression for the height in terms of the radius.
$S = 2\pi r\left(\dfrac{500}{\pi r^2}\right) + 2\pi r^2 = \dfrac{1000}{r} + 2\pi r^2$ $= 1000r^{-1} + 2\pi r^2$	This expression can be substituted into the formula for the surface area. Then you have an equation in terms of r only.
$\dfrac{dS}{dr} = -1000r^{-2} + 4\pi r$ $\dfrac{dS}{dr} = 0$ at maximum or minimum point So, $-1000r^{-2} + 4\pi r = 0$ Using your solver, $r = 4.3$ Check $\dfrac{dS}{dr}$ at $r = 4$ and $r = 5$ At $r = 4$, $\dfrac{dS}{dr} = -12.2$ and at $r = 5$, $\dfrac{dS}{dr} = 22.8$ The derivative goes from negative to positive and so it is a minimum turning point. Therefore, the best dimensions for the can are radius = 4.3 cm and height = 8.6 cm.	You can then graph the function to find the local minimum point or you can find the derivative and let $\dfrac{dS}{dr} = 0$ and solve for r.

Investigation 5

1 A semicircular water trough has surface area 10 m^2 as shown.

Explain why troughs can be designed that have the same surface area but different volumes.



2 Find the formula for the outside surface area in terms of r and h.

3 As you increase r, describe what must happen to h to keep the surface area at 10 m^2. Does the volume of the trough increase as r increases? Describe what happens to the volume.

4 Rearrange your formula from part **2** to make h the subject.

5 Find the formula for the volume of the trough. Why can you not find the maximum volume of the trough yet?

6 Substitute your expression for h from **4** to show that $V = 5r - \frac{1}{2}\pi r^3$.

7 Draw a sketch or use your GDC to plot the graph of $f(r) = 5r - \frac{1}{2}\pi r^3$.

8 How do you know that the trough has a maximum volume? How does the maximum value relate to $f(r) = 5r - \frac{1}{2}\pi r^3$?

9 Find $f'(r)$ and solve $f'(r) = 0$. What does your solution represent?

10 Find the maximum volume of the trough.

11 Explain how you can check that it is indeed the maximum value.

12 **Conceptual** How does finding the maximum or minimum values help in solving real-life problems?

HINT

Optimization is covered in Chapter 9, but this does not include differentiation.

TOK

Does the fact that Leibniz and Newton came across the calculus at similar times support the argument of Platonists over Constructivists?

If a function to be optimized has only one variable then either a GDC or differentiation can be used directly to find the maximum or minimum point. If a function to be optimized has more than one variable then a constraint must also be given. This constraint can then be written as an equation and substituted into the function to eliminate one of the variables.

Exercise 12F

1 An open cylinder has a volume of 400 cm^3. The radius of the base is r cm and the height is h cm.

- **a** Show that $\pi r^2 h = 400$.
- **b** Write down an expression for the surface area, A, of the open cylinder.
- **c** Show that this can be written as $A = \pi r^2 + \frac{800}{r}$.
- **d** Sketch the graph of $A = \pi r^2 + \frac{800}{r}$.
- **e** Find the minimum area and the value of r when this occurs.
- **f** Verify that it is a minimum point.

2 The total surface area of a closed cylinder is 5000 cm^2.

Find the dimensions of the cylinder that maximize its volume and state this maximum volume.

Verify that it is a maximum point.

3 A vegetable garden is in the shape of a rectangle. The garden is surrounded by 100 m of fencing.

a If the width of the garden is x metres, find an expression for the length.

b Show that the area of the garden is $A = x(50 - x)$ m^2.

c Find $\frac{dA}{dx}$.

d Find the maximum area and the dimensions of the garden at this point.

4 A cone has radius, r cm, and height $(18 - r)$ cm.

Find the maximum volume of the cone and the values for the radius and height that give this volume.

5 A rectangular piece of card measures 20 cm by 24 cm. Equal squares of side x cm are cut out of each corner.

The rest of the card is folded up to make a box.

a Find the volume of the box in terms of x.

b Find the value of x for which the volume is a maximum and find the maximum volume.

6 A closed cylinder has a volume of 300 cm^3.

a If the radius of the base is r cm and the height is h cm, find an expression for the volume.

b Find an expression for the total surface area in terms of r.

c Find the minimum surface area and the values for r and h that give this area.

7 The profit, \$$P$, that a firm makes per day can be modelled by the function $P(n) = -0.092n^2 + 33.3n - 313$, where n represents the number of goods sold per day.

Find the maximum profit and the value of n when this occurs.

8 A company's weekly profit, in dollars, in relation to the number of units sold each week, x, can be modelled by the function $f(x) = -0.9x^2 + 52x - 360$.

Find the maximum profit and the value of x when this occurs.

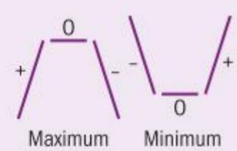
Developing your toolkit

Now do the Modelling and investigation activity on page 546.

Chapter summary

- The gradient of a tangent is the **limit** of a secant.
- The **average rate of change** between two points is the same as the gradient of the secant between the two points.
- The **instantaneous rate of change** at a point is the gradient of the tangent at that point.
- The gradient function is called the **derivative** and it is denoted by $f'(x)$.
- The **derivative** of $f(x)$ **with respect to** x is written as $f'(x)$.
- An alternative notation for $f'(x)$ is $\frac{dy}{dx}$ or, if different variables are used, $\frac{ds}{dt}$ or $\frac{dV}{dt}$ etc.
- The derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}$, $n \in \mathbb{Z}$.

- To find the derivative of a polynomial, you have to find the derivative of each term separately.
  - For example, if $f(x) = ax^n + bx^{n-1} + cx + d$, then $f'(x) = anx^{n-1} + b(n-1)\ x^{n-2} + c$.
  - Remember that the derivative of $y = mx$ is m and the derivative of $y = c$ is 0.
  - Note these last two also follow the same rule as the first if you write mx as mx^1 and c as cx^0.
- A function is increasing if $f'(x) > 0$, is stationary if $f'(x) = 0$ and is decreasing if $f'(x) < 0$.
- Usually the prime notation, $f'(x)$, is used when the question is about functions and the fractional notation, $\frac{\mathrm{d}y}{\mathrm{d}x}$, is used for curves.
- There are no set rules however. $y' = 2x - 1$ and $\frac{\mathrm{d}f}{\mathrm{d}x}$ are both correct, though less frequently seen.
- The derivative of a function at any point represents the gradient of the tangent at that point.
- The equation of the **normal** is perpendicular to the equation of the tangent.
- If two lines are perpendicular then the product of their gradients is -1.
- If the gradient of a tangent is $\frac{a}{b}$, then the gradient of the normal is $-\frac{b}{a}$.
- For function $f(x)$, if $f'(c) = 0$, then the point $(c, f(c))$ is called a **stationary point** as the instantaneous rate of change is 0 at that point.
- If the signs of the gradients are different on both sides of the stationary point then it means it is either a maximum or minimum point.
- In many cases, we are interested in when a function reaches its maximum or its minimum as in real-life cases these points can be important. For example, if it is a function of profit versus the number of products produced we would like to know where the profit is a maximum.
- A stationary point can be a local **maximum** point or a local **minimum** point.
- At a local maximum or minimum point $f'(x) = 0$.
  - The local maximum or local minimum points may not be the maximum or minimum values of the graph.
  - If the sign of the derivative goes from negative to positive, there is a minimum point.
  - If the sign of the derivative goes from positive to negative, there is a maximum point.



- If a function to be optimized has only one variable then either a GDC or differentiation can be used directly to find the maximum or minimum point.
- If a function to be optimized has more than one variable then a **constraint** must also be given. This constraint can then be written as an equation and substituted into the function to eliminate one of the variables.

Developing inquiry skills

Return to the opening problem. Has what you have learned in this chapter helped you to answer the questions?

What information did you manage to find?

What assumptions did you make?

How will you be able to construct a model?

What other things did you wonder about?