Example 3

The side of a hill can be modelled by the function $f(x) = -0.02x^2 + 3.4x$ for certain values of x.

a Sketch the graph of $f(x) = -0.02x^2 + 3.4x$.

There is a monastery at the top of the hill which could be an interesting tourist attraction. So, it is decided to build a funicular railway to reach to almost the top of the hill. The funicular can be modelled by the tangent to the curve where $x = 50$.

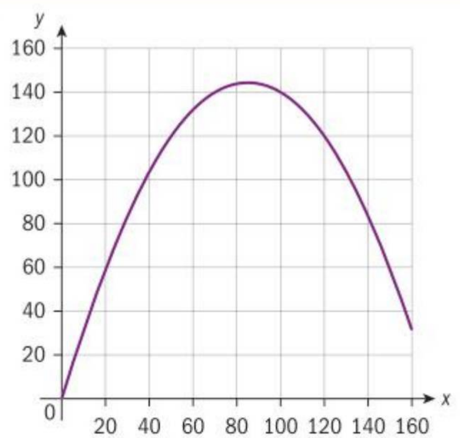
b Find the derivative of $f(x) = -0.02x^2 + 3.4x$.

c Find the gradient of the tangent at the point where $x = 50$.

d Find $f(x)$ when $x = 50$.

e Hence, find the equation of the tangent at $x = 50$ and sketch this on the graph.

a



b $f'(x) = -0.04x + 3.4$

c $f'(50) = -0.04(50) + 3.4 = 1.4$

So, 1.4 is the gradient of the tangent at $x = 50$.

Substitute 50 for x into the derivative. You can also use your GDC to find this value.

d $f(50) = -0.02(50)^2 + 3.4(50) = 120$

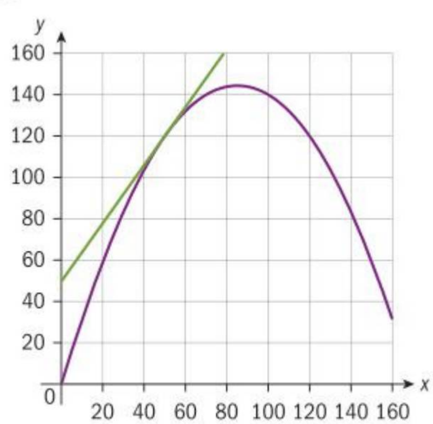
e Using $y = mx + c$, $120 = 1.4(50) + c$

So, $c = 50$.

Therefore, the equation of the tangent is

$y = 1.4x + 50$

Substitute 50 for x into the original function to find $f(50)$. Your GDC will also give you the equation of the tangent.

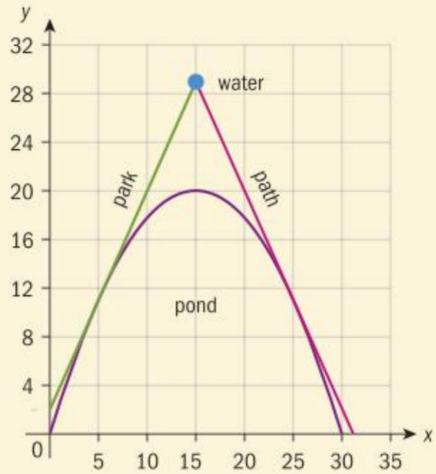


Investigation 3

Part of a pond is in the shape of a parabola which can be modelled by the function

$$f(x) = -\frac{8}{90}x^2 + \frac{8}{3}x.$$

The local authority decides to build a path on either side of the pond. The two paths will meet at the drinking fountain that is already in place at a point with coordinates (15, 29). It is decided that the most economical way to achieve this is to build paths that just touch the side of the pond at 5 m and 25 m from the edge, as shown in the diagram. These paths would then be tangents to the curve.



1 How can the local authority work out the equations for these paths?

2 How can they be certain that the paths will meet at the drinking fountain?

3 Using technology, sketch the graph of $f(x) = -\frac{8}{90}x^2 + \frac{8}{3}x$.

4 Find the derivative of $f(x) = -\frac{8}{90}x^2 + \frac{8}{3}x$.

5 What does this represent?

6 Find the gradient of the tangent to the curve when $x = 5$.

7 Find the value of $f(x)$ when $x = 5$.

8 Hence, find the equation of the tangent when $x = 5$.

9 Use technology to draw this line on your graph.

10 Repeat this process for $x = 25$.

Now that you have the equations of the two tangents, how can you find out where they meet?

Do they meet at the location of the drinking fountain?

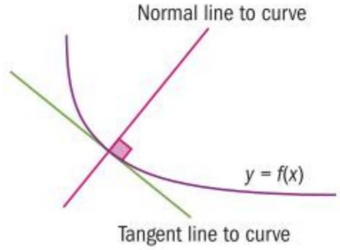
11 **Conceptual** What is the relation between the gradient of a tangent at a given point and the derivative of a function at the same point?

TOK

Who do you think should be considered the discoverer of calculus?

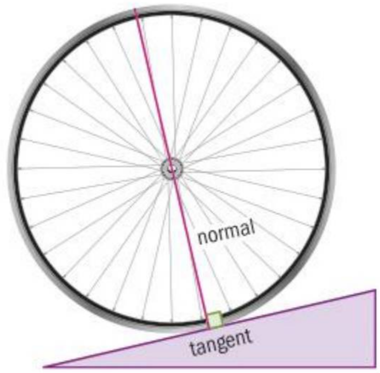
The derivative of a function at a point on its graph represents the gradient of the tangent at that point.

The **normal** line is perpendicular to the tangent line.



Calculus

For example, on a bicycle the spokes are at right angles to the tyre at the point of contact. Hence the direction of the road will be a tangent to the wheel at the point of contact and the spoke at that point will be a normal to the point of contact.



HINT

If two lines are perpendicular then the product of their gradients is -1.

If the gradient of the tangent is $\frac{a}{b}$, then the gradient of the normal is $-\frac{b}{a}$.

Reflect What is a normal to a curve?

Example 4

Find the equation of the normal to the curve with equation $f(x) = 2x^3 + 3x - 2$ at the point where $x = 1$.

$f(1) = 2(1)^3 + 3(1) - 2 = 3$	Find the y-coordinate of the point.
The point is $(1, 3)$.	
$f'(x) = 6x^2 + 3$	Next find $f'(x)$.
$f'(1) = 6(1)^2 + 3 = 9$	Find $f'(1)$ to work out the gradient of the tangent line.
The gradient of the tangent line is 9.	
Gradient of the normal line $= -\frac{1}{9}$	Gradient of normal line $= \frac{-1}{\text{tangent gradient}}$
$3 = -\frac{1}{9}(1) + c$	Substitute the point $(1, 3)$ into the equation of the normal line.
$c = 3\frac{1}{9}$	
The equation of the normal is $y = -\frac{1}{9}x + 3\frac{1}{9}$	

Example 5

The gradient of the normal to the curve with equation $f(x) = kx^3 - 2x + 1$ at the point $(1, b)$ is $-\frac{1}{4}$. Find the values of k and b.

$f'(x) = 3kx^2 - 2$	
If the gradient of the normal is $-\frac{1}{4}$, then the gradient of the tangent is 4.	

So, $f'(1) = 3k(1)^2 - 2 = 4$ $3k = 6$ $k = 2$	The derivative is the gradient of the tangent at any point. Here the point is $x = 1$. So, you put $x = 1$ into the derivative and equate it to 4, which is the gradient of the tangent at that point.
$f(1) = 2(1)^3 - 2(1) + 1 = 1$ So, $b = 1$	To find b you need to substitute 1 for x into the original equation.

Exercise 12C

1 Find the equation of the tangent to the curve with equation $f(x) = 2x^2 - 4$ at the point where $x = 3$.

2 Brian makes a seesaw for his children from part of a log and a plank of wood. The shape of the log can be modelled by the equation $f(x) = -x^2 + 2x$ for certain values of x.

The plank of wood is a tangent to the log at the point where $x = 1$.

Find the equation of the plank.

3 A cyclist is cycling up a hill. The path of the bicycle on the hill can be modelled by part of the function $f(x) = -2x^2$.

a Find the gradient of the wheel at the point where $x = 1$.

b Find the gradient of the spoke that is perpendicular to the wheel when it touches the hill at the point where $x = 1$.

4 Find the equation of the normal to the curve with equation $f(x) = 3x^2 - 4x + 5$ at the point $(1, 4)$.

5 Find the equation of the tangent and the normal to the curve with equation $y = x^4 - 6x + 3$ at the point where $x = 2$.

6 The edge of a lake can be modelled by part of the function $f(x) = x^2$.

A fountain is to be placed in the lake at the point where the equations of the normals at $x = 2$ and $x = -2$ meet.

Find the equations of the normals and the point where the fountain will be placed.

7 The perimeter of a park can be modelled by part of the function $f(x) = -0.112x^2 + 5.6x$.

a Sketch the graph of this function.

The local authority wants to place a memorial statue inside the park. They decide to place it where the two normal lines to the curve at $x = 15$ and $x = 35$ meet.

b Find the equations of the two normal lines.

c Find the point of intersection of the two normal lines.

d State whether or not this a suitable place for the memorial. Explain your answer.

8 The gradient of the tangent to the curve with equation $f(x) = ax^2 + 3x - 1$ at the point $(2, b)$ is 7. Find the values of a and b.

9 The gradient of the tangent to the function with equation $f(x) = x^2 + kx + 3$ at the point $(1, b)$ is 3. Find the values of k and b.

10 The gradient of the normal to the function with equation $y = ax^2 + bx + 1$ at the point $(1, -2)$ is 1. Find the values of a and b.

HINT

There are many websites that will give the derivative of a function as an equation and indeed some calculators do this as well. These are **not** allowed in exams.

However, in examinations, it will be expected that you use your GDC to find numerical values for the derivatives at given points and also to draw the graph of the derivative. This widens the range of functions that you might need to find the gradient for. Unless told otherwise, always use your GDC if it makes answering the question simpler.

Example 6

Consider $y = \frac{x+2}{x-1}$, $x \neq 1$.

Find the gradient of the curve at the points where $x = 2$ and $x = 3$.

-3 and -0.75	The gradient is found using the numerical derivative function on your GDC.

Exercise 12D

1 Find the gradient of the following curves at the point where $x = 3$.

a $y = \frac{x^2}{x-1}$ **b** $y = x\ln x$ **c** $f(x) = \frac{2x^2-1}{x+3}$ **d** $s = \frac{t+5}{t^3+1}$ **e** $y = xe^{2x}$ **f** $g(x) = (x^2-3)(x-1)^5$

12.3 Maximum and minimum points and optimization

Investigation 4

The number of bacteria, B, in thousands, in a culture at time, t minutes, is given by the formula:

$B(t) = 0.00314x^3 - 0.1926x^2 + 2.848x + 5$

a Sketch the graph of $B(t) = 0.00314x^3 - 0.1926x^2 + 2.848x + 5$ for $0 \leq t \leq 40$.

b Find $B'(t)$.

c Find the values of t when there are the most and the least bacteria.

d Find the gradient of the function at the two points found in part **b**.

e Calculate the gradient of the function at a point just before and just after the values of t found in part **b**.

f Use your results to explain the difference between the local maximum and the local minimum.

g Conceptual How do stationary points help solve real-life problems?

International-mindedness

Maria Agnesi, an 18th century, Italian mathematician, published a text on calculus and also studied curves of the form $y = \frac{a^2}{x^2} + a^2$.

TOK

How can you justify the rise in tax for plastic containers eg plastic bags, plastic bottles etc using optimization?