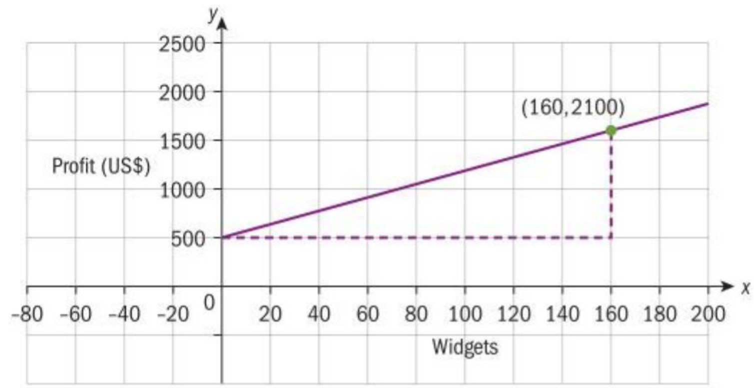
12.1 Limits and derivatives

The graph below shows how the profits of a company increase with the number of widgets it sells.



From previous work on linear functions you will know that the rate at which the profit increases for each new widget made (profit per widget) is $\frac{2100-500}{160-0}=10$ US$ per widget.

This is equivalent to the **gradient** of the curve. If the curve had a steeper gradient, the profit per widget would be higher; if the gradient had a less steep gradient, the profit per widget would be lower.

Hence the **rate of change** of one variable with respect to another is equivalent to the **gradient** of the line.

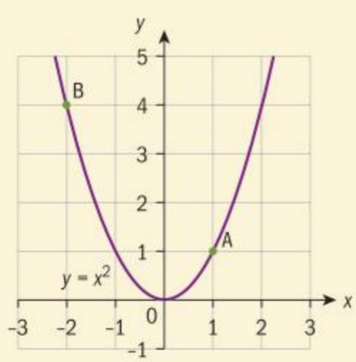
This is easy to calculate when the functions are **linear**. The following investigation will consider how the gradient of a curve at a point might be calculated.

TOK

What value does the knowledge of limits have?

Investigation 1

Consider the curve $y=x^2$ shown below.



1 a Give the range of x for which the gradient of the curve is **i** negative **ii** positive **iii** equal to zero.

b At which point is the gradient of the curve greatest, A or B?

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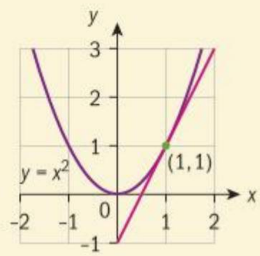
Calculus

The **tangent** to a curve at a point is the straight line that just touches, but does not cross, the curve at that point.

2 **a** On your GDC or on other software plot the curve $y=x^2$ and draw the tangent at the point $(1, 1)$, as shown below.



b Zoom in to the point $(1, 1)$. What do you notice about the gradient of the tangent and the gradient of the curve at the point of contact?

c Use your GDC or online software to draw tangents to $y=x^2$ at the points listed below and, in each case, write down the gradient of the tangents.

x	−3	−2	−1	0	1	2	3
Gradient of tangent at x					2		

d What do you think the gradient of the tangent at $x=5$ is?

e What do you think the gradient of the tangent at $x=-8$ is?

f Can you find a general expression for the gradient of the tangent to $y=x^2$ at the point on the curve with coordinates (x, x^2)?

g To justify your expression in part **f**, find the slope of the chord from point A $(1, 1)$ to a point B $(1.1, 1.21)$.

h Find the slope of the chord from point A $(1, 1)$ to a point B $(1.01, 1.0201)$.

i Find the slope of the chord from point A $(1, 1)$ to a point B $(1.001, 1.002001)$.

j What happens to the slope the closer point B gets to point A?

k Does this value fit your expression in part **f**?

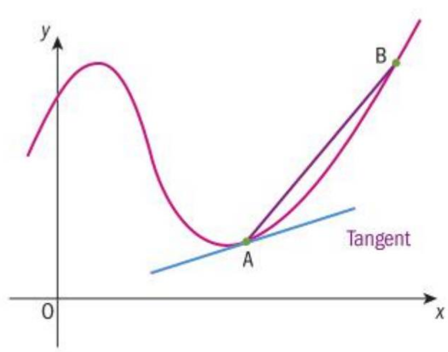
The line joining points A and B is called a **chord**.

The line through points A and B is called a **secant**.

The **average rate of change** from A to B is the same as the **gradient** of the line AB.

As B moves closer to A, the gradient of the line AB gets closer to the **gradient of the tangent** at A.

The gradient of a secant identifies the average rate of change while the gradient of a tangent identifies the instantaneous rate of change.



The **gradient** of a tangent is the **limit** of a secant.

The **average rate of change** between two points is the same as the gradient of the secant between the two points.

The **instantaneous rate of change** at a point is the gradient of the tangent at that point.

The gradient function is called the **derivative** and it is denoted by $f'(x)$.
The derivative of $f(x)$ with respect to x is defined as $f'(x)$.

An alternative notation for $f'(x)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ or, if different variables are used, $\frac{\mathrm{d}s}{\mathrm{d}t}$ or $\frac{\mathrm{d}V}{\mathrm{d}t}$ etc.

International-mindedness

The $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation was used by German mathematician, Gottfried Leibniz in the 17th century

Investigation 2

1 Write down the gradients of the following functions:

$y=3, y=-5, y=8, y=-10$

a What is the gradient of $y=c$?

b What is the derivative of $y=c$?

2 Write down the gradients of the following functions:

$y=3x, y=-7x, y=4x, y=x$

a What is the gradient of $y=mx$?

b What is the derivative of $y=mx$?

3 **a** **i** Using technology, find the gradients of the tangents to $f(x)=2x^2$ at $x=-2, -1, 0, 1, 2$.

ii Find a general expression for the derivative of $f(x)=2x^2$ at any point x.

b **i** Using technology, find the gradients of the tangents to $f(x)=3x^2$ at $x=-2, -1, 0, 1, 2$.

ii Find a general expression for the derivative of $f(x)=3x^2$ at any point x.

c State an expression for the derivative of $f(x)=ax^2$.

4 **a** **i** Using technology, find the gradients of the tangents to $f(x)=x^3$ at $x=-2, -1, 0, 1, 2$.

ii Find a general expression for the derivative of $f(x)=x^3$ at any point x.

b **i** Using technology, find the gradients of the tangents to $f(x)=2x^3$ at $x=-2, -1, 0, 1, 2$.

ii Find a general expression for the derivative of $f(x)=2x^3$ at any point x.

c State an expression for the derivative of $f(x)=ax^3$.

5 **a** State the derivative for $y=x^4$.

b Generalize the derivatives for $y=x^n$.

c Generalize the derivatives for $y=ax^n$.

6 **Conceptual** What does the derivative of a function at any point represent?

HINT

There are many interactive demos that you can use to find these answers.

The derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}$, $n \in \mathbb{Z}$.
To find the derivative of a polynomial, you have to find the derivative of each term separately.

For example, if $f(x) = ax^n + bx^{n-1} + cx + d$, then
$f'(x) = anx^{n-1} + b(n-1)x^{n-2} + c$.

Remember that the derivative of $y = mx$ is m and the derivative of $y = c$ is 0.

HINT
Note the last two also follow the same rule as the first if you write mx as mx^1 and c as cx^0.

Example 1

Find the derivatives of the following functions and find the gradient of the tangent at the point where $x = 2$:

1 $y = 3x^4$ **2** $y = 5x^3$ **3** $y = 2x^2 + 3x - 5$

4 $y = x^3 + 2x + 1$ **5** $y = \frac{2}{x} + x,\ x \neq 0$ **6** $y = \frac{3}{x^2},\ x \neq 0$

For parts **3** and **5**:

a Sketch the function and its derivative on the same axes and say whether the function is increasing or decreasing at the point where $x = 2$.

b Write down the range of values of x for which the function is increasing.

1 $\frac{dy}{dx} = 12x^3$

$3 \times 4x^{4-1}$

When $x = 2$,

$\frac{dy}{dx} = 12(2)^3 = 12(8) = 96$

To find the gradient of the tangent at $x = 2$, substitute 2 for x in the derivative. Your GDC will also give you the numerical value of the derivative at any point.

2 $\frac{dy}{dx} = 15x^2$

$5 \times 3x^{3-1}$

When $x = 2$, $\frac{dy}{dx} = 15(2)^2 = 60$

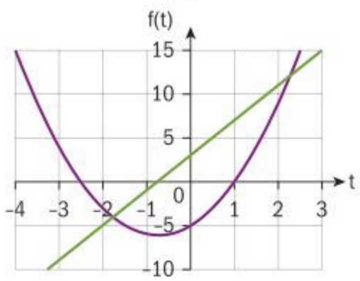
3 $\frac{dy}{dx} = 4x + 3$

$2 \times 2x^{2-1} + 3 \times 1x^{1-1} - 0$

When $x = 2$, $\frac{dy}{dx} = 11$

The gradient of $y = mx$ is m. So, the derivative of $y = 3x$ is 3.

The gradient of $y = c$ is 0. So, the derivative of $y = -5$ is 0.



a At $x = 2$, the gradient is positive and so is increasing.

b $x > -0.75$

This can be found by either finding the minimum point of $f(x) = 2x^2 + 3x - 5$ or by finding where $f'(x) > 0$.

4 $\dfrac{dy}{dx} = 3x^2 + 2$

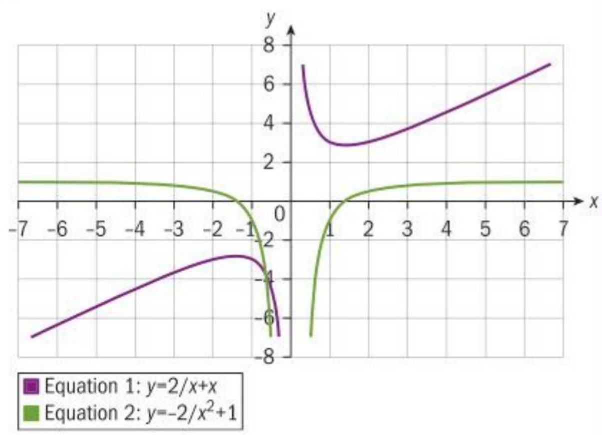
When $x = 2$, $\dfrac{dy}{dx} = 14$

5 $\dfrac{2}{x} + x = 2x^{-1} + x$, $\dfrac{dy}{dx} = -2x^{-2} + 1 = \dfrac{-2}{x^2} + 1$

When $x = 2$, $\dfrac{dy}{dx} = 0.5$

a At $x = 2$, the gradient is positive and so is increasing.

b $x > 1.41$, $x < -1.41$



6 $\dfrac{dy}{dx} = -6x^{-3} = \dfrac{-6}{x^3}$

When $x = 2$, $\dfrac{dy}{dx} = -0.75$

In order to use the rule for differentiation you need to first write the expression using negative indices.

$\dfrac{2}{x} = 2x^{-1}$, so $\dfrac{dy}{dx} = 2 \times -1x^{-1-1}$

and the derivative of x is 1.

Again the range in which the curve is increasing can be found by finding the maximum and minimum points using a GDC or by finding where $f'(x) > 0$.

$\dfrac{3}{x^2} = 3x^{-2}$, so $\dfrac{dy}{dx} = 3 \times -2x^{-2-1}$

Reflect What is the relationship between the gradient of a curve and the sign of its derivative?

A function is increasing if $f'(x) > 0$, is stationary if $f'(x) = 0$ and is decreasing if $f'(x) < 0$.

Usually the prime notation, $f'(x)$, is used when dealing with functions and the fractional notation, $\dfrac{dy}{dx}$, is used when dealing with curves.
There are no set rules, however; y' and $\dfrac{df}{dx}$ are both correct, though less frequently seen.

International-mindedness

French mathematician Joseph Lagrange, a close friend of Leonhard Euler, introduced the $f'(x)$ notation in the 18th century.

The following exercise contains a mixture of both notations. Remember that they both indicate that the function gives the gradient, or rate of change, of the variable.

Exercise 12A

i Find the derivative of each of the following functions with respect to x.

ii Find the gradient of the tangent at the point where $x = 1$ and state whether the function is increasing or decreasing at this point.

iii Write down the range of values of x for which the function is increasing.

a $y = 6$ **b** $y = 4x$

c $f(x) = 3x^2$ **d** $f(x) = 5x^2 - 3x$

e $f(x) = 3x^4 + 7x - 3$

f $f(x) = 5x^4 - 3x^2 + 2x - 6$

g $y = 2x^2 - \dfrac{3}{x}$

h $y = \dfrac{6}{x^3} + 4x - 3$

i $y = (2x - 1)(3x + 4)$

j $f(x) = 2x(x^3 - 4x - 5)$

k $y = \dfrac{7}{x^3} + 8x^4 - 6x^2 + 2$

International-mindedness

The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?

Using differentiation in everyday life problems

Example 2

1 In economics the marginal cost is the cost of producing one more unit. This can be approximated by the gradient of a cost curve.

A company produces motorcycle helmets and the daily cost function can be modelled as

$C(x) = 600 + 7x - 0.0001x^3$ for $0 \le x \le 150$ where x is the number of motorcycle helmets produced and C the cost in US dollars.

a Write down the daily cost to the company if no helmets are produced.

b Find an expression for the marginal cost, $C'(x)$, of producing the helmets.

c Find the marginal cost of producing **i** 20 helmets and **ii** 80 helmets.

d State appropriate units for the marginal cost.

2 A hammock can be modelled by the function $f(x) = 2x^2 + 3x - 4$. A mosquito, travelling in a straight line of gradient 11, touches the hammock at point A.

Find the coordinates of A.

1 a US$ 600

b $C'(x) = 7 - 0.0003x^2$

c i $C'(20) = 6.88$

ii $C'(80) = 5.08$

d US$ per helmet

2 $f'(x) = 4x + 3$	First, find the derivative.
$4x + 3 = 11$ $4x = 8$ $x = 2$	You know that the derivative is the same as the gradient of the tangent. Therefore, equate the derivative to 11 and solve for x.
$f(x) = 2(2)^2 + 3(2) - 4 = 10$	Now substitute 2 for x into the original equation.
So, the coordinates of A are $(2, 10)$.	

Exercise 12B

1 The area, A, of a circle of radius r is given by the formula $A = \pi r^2$.

a Find $\frac{dA}{dr}$.

b Find the rate of change of the area with respect to the radius when $r = 2$.

2 The profit, USP, made from selling cupcakes, c, is modelled by the function $P = -0.056c^2 + 5.6c - 20$.

a Find $\frac{dP}{dc}$.

b Find the rate of change of the profit with respect to the number of cupcakes when $c = 20$ and $c = 60$.

c Comment on your answers for part **b**.

3 The distance of a bungee jumper below his starting point can be modelled by the function $f(t) = 80t^2 - 160t$, $0 \le t \le 2$, where t is the time in seconds.

a Find $f'(t)$.

b State the quantity represented by $f'(t)$.

c Find $f'(0.5)$ and $f'(1.5)$ and comment on the values obtained.

d Find $f(2)$ and comment on the validity of the model.

4 Points A and B lie on the curve $f(x) = x^3 + x^2 + 2x$ and the gradient of the curve at A and B is 3.

Find the coordinates of points A and B.

5 The outline of a building can be modelled by the function $h = 2x - 0.1x^2$ where h is the height of the building and x the horizontal distance from the start of the building.

An observer stands at the point A. The angle of elevation from his position to the highest point he can see on the building is 45°.

Calculate the height above the ground of the highest point he can see on the building.

12.2 Equations of tangent and normal

The equation for the tangent to a curve at a given point can be easily found using the gradient to the curve and the coordinates of the point.