10 Modelling rates of change: exponential and logarithmic functions

Concepts

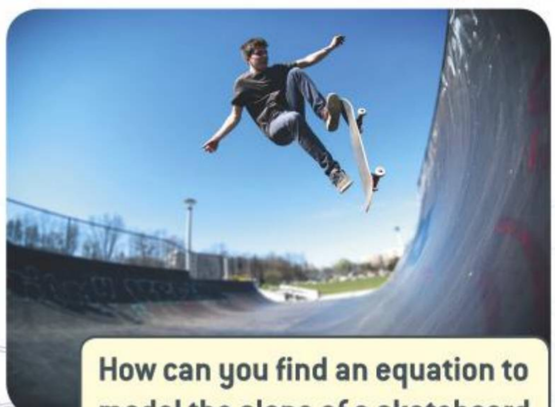
- Change
- Modelling

Microconcepts

- Exponential functions
- Horizontal asymptotes
- Geometrical sequences and series
- Common ratio
- Compound interest
- Annual depreciation
- Annuity and amortization
- Growth and decay
- Logarithms

The reduction in temperature of a hot drink, the slope of a skateboard track, compound interest, depreciation of the value of a car, even the half-life of radioactive decay all give rise to curves. This chapter looks at ways of modelling these curves with equations in order to predict various outcomes.

How long will it take you to become a millionaire if you receive US\$1 the first month, US\$2 the second month, US\$4 the third month, US\$8 the fourth month and so on?



How can you work out which rate gives you the most interest on your savings?

How can you find an equation to model the slope of a skateboard track? Or a ski slope?

If the temperature of a cup of tea reduces at a given rate, will it ever reach 0°C?

How can you work out the monthly payments on a loan?

Since the Olympic Games began, the height that sportsmen and women have been able to jump in the pole vault has increased. Can you predict the smallest height that no athlete will ever be able to jump over?

What information will you need?

What assumptions will you have to make?

How will you be able to construct a model?

Will this be different for men and women? If so, why?

Think about these questions, then **write down** your answers and **discuss** them with a classmate.

What further information do you think that you will need? What other things do you notice? What do you wonder?

Developing inquiry skills

Think about the questions you would need to ask to model other sorts of activity and achievements, like the shortest time for a marathon or the longest long jump.

Before you start

Click here for help with this skills check

You should know how to:	Skills check
1 Laws of exponents eg $2^3 \times 2^4 = 2^7$	**1** Find the values of: **a** $4^2 \times 4^9$ **b** $\frac{5^8}{5^2}$.
2 Percentages eg 6% of 24 $= \frac{6}{100} \times 24 = 1.44$	**2** Find the values of: **a** 3% of 24 **b** 28% of 150.
3 Sigma notation $\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$	**3** Find the values of: **a** $\sum_{i=1}^{4} 2^i$ **b** $\sum_{k=1}^{6} (k+3)$.

10.1 Geometric sequences and series

The game of chess was invented in India by a man named Sissa ibn Dahir. The king, Shihram, was so pleased with the game that he offered Sissa any reward that he wanted. Sissa said that he would take this reward: the king should put one grain of wheat on the first square of a chessboard, two grains of wheat on the second square, four grains on the third square, eight grains on the fourth square, and so on, doubling the number of grains of wheat with each square.

How would you be able to find out the number of grains of wheat on the last square of the chessboard?

How would you be able to find the total number of grains of wheat on the chessboard?

This is one example of a geometric sequence.

A tennis competition involves 256 players. Each game has two players and there is only one winner of each game. In the first round all 256 players will take part, in the second round only the winners from the first round will take part and so on. How can you find out the total number of rounds that take place before there is a winner? How can you find out the total number of games that are played in the competition?

This is also an example of a geometric sequence.

A sequence of numbers in which each term can be found by multiplying the preceding term by a **common ratio** is called a geometric sequence.

A geometric series is the sum of the terms of a geometric sequence.

International-mindedness

The Elements, Euclid's book from 300BC, contained geometric sequences and series.

Investigation 1

For each of the following sequences, find the next term and describe the rule for finding the next term.

a $2, 4, 6, 8, \ldots$ **b** $1, 3, 9, 27, \ldots$ **c** $1, 1, 2, 3, 5, \ldots$

d $-3, -1.5, 0, 1.5, \ldots$ **e** $1, 4, 9, 16, \ldots$ **f** $5, 15, 45, 135, \ldots$

Conceptual How can you tell whether a sequence is geometric or not? Which of these sequences are geometric?

Investigation 2

1 Complete the following geometric sequences:

a $2, 4, 8, 16, \_, \_, \_, \_$ b $-1, 3, -9, 27, \_, \_, \_, \_$

c $\_, \_, -6, -3, -\frac{3}{2}, -\frac{3}{4}, \_, \_$ d $\_, \_, \_, \_, 1, 5, 25, 125$

2 Complete this table for these sequences.

	$\frac{\text{term 2}}{\text{term 1}}$	$\frac{\text{term 3}}{\text{term 2}}$	$\frac{\text{term 4}}{\text{term 3}}$
a	$\frac{4}{2}=2$	$\frac{8}{4}=2$	$\frac{16}{8}=2$
b			
c			
d			

3 What do you notice about the ratio of consecutive terms for each sequence?

You have found the common ratio of each geometric sequence.

Each term of a geometric sequence can be written in terms of its first term and common ratio.

Sequence **a** can be written in terms of its first term, 2, and common ratio, 2:

$2, \quad 2(2), \quad 2(2)^2, \quad 2(2)^3, \quad 2(2)^4, \quad 2(2)^5, \ldots$

4 Find an expression for the nth term of sequence **a**.

Repeat for the other three sequences.

5 **Conceptual** Can you always write a geometric sequence in terms of its first term and common ratio?

The first term of a geometric sequence is called u_1, the common ratio is called r, and the nth term is called u_n.

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots$$

6 Complete the following:

$u_1 = u_1$

$u_2 = u_1 \times r = u_1 r$

$u_3 = u_2 \times r = u_1 r \times r = u_1 r^2$

$u_4 = u_3 \times r = u_1 r^2 \times r = u_1 r^3$

$u_5 =$

$u_6 =$

$\vdots$

$u_n =$

7 **Factual** What is the formula for the nth term of a geometric sequence?

8 **Conceptual** How can you tell if a sequence is geometric?

TOK

Why is proof important in mathematics?

To check whether a sequence is geometric, find the ratio of pairs of consecutive terms. If this ratio is constant, it is the common ratio and the sequence is geometric.

The nth term in a geometric sequence is given by the formula $u_n = u_1 r^{n-1}$.

Example 1

1 For each of these geometric sequences, write down:

i the first term, u_1 **ii** the common ratio, r **iii** u_{10}.

a 2, 6, 18, 54, … **b** −3, 6, −12, 24, … **c** 16, 8, 4, 2, …

2 Carolien is starting a new job. She earns €48 000 in her first year, and her salary increases by 5% each year. Show that Carolien's annual salary follows a geometric sequence, and state the common ratio. Calculate how much Carolien will earn in her fifth year at work.

3 The third term of a positive geometric sequence is 63 and the fifth term is 567. Find the common ratio and the first term.

1 a i $u_1 = 2$

ii $r = \frac{6}{2} = 3$

iii $u_{10} = 2(3)^9 = 39\,366$

b i $u_1 = -3$

ii $\frac{6}{-3} = -2$

iii $u_{10} = -3(-2)^9 = 1536$

c i $u_1 = 16$

ii $r = \frac{8}{16} = 0.5$

iii $u_{10} = 16(0.5)^9 = 0.031\,25$

2 $u_1 = 48\,000$

$u_2 = 48\,000 + 5\% \times 48\,000$

$= 48\,000(1 + 5\%)$

$= 48\,000(1 + 0.05)$

$= 48\,000 \times 1.05$

So, $r = 1.05$

Therefore,

$u_5 = u_1 r^4 = 48\,000(1.05)^4 = €58\,344.30$

3 $u_3 = 63$

Remember that $u_3 = u_1 r^2$.

So, $u_1 r^2 = 63$

$u_5 = 567$

$u_5 = u_1 r^4$

So, $u_1 r^4 = 567$

Then you can write: $\frac{u_5}{u_3} = \frac{u_1 r^4}{u_1 r^2} = \frac{567}{63} = 9$ $r^2 = 9$ $r = \pm 3$ You are told that all the terms are positive, so $r = 3$ Substituting back into the first equation: $u_1 = \frac{63}{9} = 7$	Here you can cancel the u_1 and r^2 to get r^2 on the left-hand side.

Exercise 10A

1 The first three terms of a geometric sequence are 3, 6, 12.

a Write down the common ratio.

b Calculate the value of the 15th term.

2 A marrow plant is 0.5 m long. Every week it grows by 20%.

a Find the value of the common ratio.

b Calculate how long the plant is after 12 weeks.

c Comment on the predicted length of the marrow after one year.

3 The first four terms of a geometric sequence are 27, x, 3, 1.

a Find the value of x.

b Write down the common ratio.

c Calculate the value of the eighth term.

4 There is a flu epidemic in Cozytown. On the first day, 2 people have the flu. On the second day, 10 people have the flu. On the third day, 50 people have the flu.

a Show that the number of people with the flu forms a geometric sequence.

b Hence, assuming the conjecture is true, calculate how many people have the flu after one week (seven days).

c Hence calculate how many people will have the flu after one year. Comment on your answer.

5 The second term of a geometric sequence is −32 and the fourth term is −2. Given that all the terms are negative, find the common ratio and the first term.

Example 2

Mr Farmer buys machinery for US$25 000. The machinery depreciates at a rate of 8% per annum.

a Find the value of the machinery after five years.

b Find how many years it is before Mr Farmer's machinery is worth half its original value.

Continued on next page

a If the machinery depreciates at 8% each year, then after one year it will be worth only $(100-8)\% = 92\% = 0.92$ of its original value.

So, after five years it will be worth US$$25\,000 \times 0.92^5 =$ US$16 477.04.

b Half of the original value is US$12 500

So, $25\,000 \times 0.92^n = 12\,500$

Using your Solver, $n = 8.31$

So, Mr Farmer's machinery is worth half its original value after 8.31 years.

Exercise 10B

1 At the end of 2016 the population of a city was 200 000. At the end of 2018 the population was 264 500.

- **a** Assuming that these end of year figures follow a geometric sequence, find the population at the end of 2017.
- **b** Calculate the population at the end of 2020.
- **c** Comment on whether this increase will continue. Give a reason for your answer.

2 One kilogram of tomatoes costs US$2.20 at the end of 2015. Prices rise at 2.65% per year. Find the cost of a kilogram of tomatoes at the end of 2019.

3 Petra buys a camper van for €45 000. Each year the camper van decreases in value by 5%. Find the value of the camper van at the end of six years.

4 Beau spends €15 000 buying computer materials for his office. Each year the material depreciates by 12%.

- **a** Find the value of the material after three years.
- **b** Find how many years it takes for the materials to be worth €5 000.

Developing inquiry skills

Return to the chess problem on p. 463. You should now be able to find out how many grains of wheat are on the last square.

But, how can you find the total number of grains of wheat on all the squares?

You should also be able to find the number of rounds of tennis played until the final.

But, how can you find the total number of games played?

Investigation 3

The following are geometric sequences:

a 1, 5, 25, 125, 625 **b** 6, 12, 24, 48, 96 **c** 2, −6, 18, −54, 162

Complete the following table for each sequence:

	u_1	r	r^2	r^2-1	$r-1$	Sum of first 2 terms
a						
b						
c						
	u_1	r	r^3	r^3-1	$r-1$	Sum of first 3 terms
a						
b						
c						
	u_1	r	r^4	r^4-1	$r-1$	Sum of first 4 terms
a						
b						
c						
	u_1	r	r^5	r^5-1	$r-1$	Sum of first 5 terms
a						
b						
c						

See if you can find a connection between:

- the sum of the first 2 terms, r^2-1, $r-1$ and u_1
- the sum of the first 3 terms, r^3-1, $r-1$ and u_1
- the sum of the first 4 terms, r^4-1, $r-1$ and u_1
- the sum of the first 5 terms, r^5-1, $r-1$ and u_1.

Using the connections you discovered, suggest a formula for the sum of the first n terms of a geometric series in terms of u_1, r and n. Are there any values that r cannot take?

The sum to n terms of a geometric series is written as S_n. The formula for S_n is

$$S_n = \sum_{i=1}^{n} u_i = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r},\ r \neq 1$$

You can use either form.

To see how we might get that formula, consider the series 1, 2, 4, 8, 16, 32.

Here, $u_1 = 1$ and $r = 2$. If we take the sum of this series we have

$$S_6 = 1 + 2 + 4 + 8 + 16 + 32 \qquad (1)$$

Multiply this by 2:

$2S_6 = 2 + 4 + 8 + 16 + 32 + 64 \qquad (2)$

Now do (1) − (2). You'll see that a lot of the terms cancel out:

$S_6 - 2S_6 = 1 - 64 \qquad (3)$

Solving this for S_6, you get $S_6 = 63$.

Check this against the formula:

$$S_6 = \frac{1(1-2^6)}{1-2} = \frac{(1-64)}{-1} = 63$$

If you look at (3) and compare it with the formula you will see that on the left-hand side we have $(1-2)S_6$, which is $(1-r)S_6$, and on the right-hand side we have $1 - 2^6$, which is $1 - r^6$.

So, $S_6 = \dfrac{u_1(1-r^6)}{(1-r)}$.

Example 3

1 Find the sum to eight terms of the geometric series 7, 28, 112, 448, …

2 When Kenzo starts Bright Academy, the school fees are JYN 2 500 000 (Japanese yen). The fees increase by 2% each year. Kenzo attends the school for a total of six years.

- **a** Write down the common ratio.
- **b** Calculate the school fees in year six.
- **c** Calculate the total fees paid for the six years that Kenzo attends Bright Academy.

3 The second term of a geometric sequence is 6 and the fourth term is 24. All the terms are positive.

- **a** Find the first term and the common ratio.
- **b** Find the sum of the first 10 terms.

1 $u_1 = 7$ $r = \frac{28}{7} = 4$ So, $S_8 = \frac{7(4^8-1)}{4-1} = 152\,915$	
2 a $r = 1.02$	$1 + 2\% = 1 + 0.02 = 1.02$
b $u_6 = u_1r^5 = 2\,500\,000 \times 1.02^5$ $= \text{JYN } 2\,760\,202$	Remember, $u_n = u_1r^{n-1}$
c $S_6 = \frac{u_1(r^6-1)}{(r-1)} = \frac{2\,500\,000(1.02^6-1)}{(1.02-1)}$ $= \text{JYN } 15\,770\,302$	

3 a $u_2 = u_1 r = 6$

$u_4 = u_1 r^3 = 24$

$\frac{u_4}{u_2} = \frac{u_1 r^3}{u_1 r} = \frac{24}{6} = 4$

$r^2 = 4$

$r = 2$ — You do not need $r = -2$ because all the terms are positive.

$u_1 r = 6$

$u_1(2) = 6$

$u_1 = 3$

b $S_{10} = \frac{3(2^{10} - 1)}{(2 - 1)} = 3069$

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Exercise 10C

1 A geometric series is $2 + 6 + 18 + \ldots$

a Find the common ratio.

b Find the sum of the first 10 terms.

2 Aziza receives an allowance of EGP 2500 (Egyptian pounds) each month. Every month this amount is increased by 2%.

a Find how much Aziza receives in month six.

b Calculate the total amount that Aziza receives in one year.

3 The population of a small island is 12 000. The population is expected to grow by 1.2% each year.

a Find the population at the end of seven years.

Each year, every inhabitant of the island receives a gift from the government.

b Find the total number of gifts that the government gives to the inhabitants during these seven years.

4 A geometric sequence has third term 8 and fifth term 128.

a Find the common ratio and the first term.

b Find the eighth term.

c Find the sum of the first eight terms.

5 When Chen starts at a private school, the school fees are CNY 270 000 (Chinese yuan) a year. Every year the fees increase by 4%.

a Calculate the school fees for the second and third years.

b Find the total cost of the school fees, given that Chen stays at the school for six years.

6 Kyle joins a sports club. The fee for the first year is €75.

a Given that the fee is increased by 2% every year, find the cost for the second year.

b Given that Kyle is a member of the sports club for five years, calculate how much he pays in total over that period.

7 Find the sum to nine terms of this geometric series:

$24 + 36 + 54 + 81 + \ldots$

8 Alexa wins a prize in a lottery. She receives CAD 8000 (Canadian dollars) the first month, CAD 6000 the second month, CAD 4500 the third month and so on for a total of six months.

a Calculate how much she receives in month six.

b Calculate the total prize money for the six months.

9 Find the sum of this geometric series: $1 + 3 + 9 + \ldots + 19683$.

10 Giovanna works on a farm and feeds the chickens each week. At the beginning the farm has 500 chickens. The number of chickens increases each week by 1%.

a Find the number of chickens after 15 weeks.

b Find the total number of feeds in 15 weeks.

Developing inquiry skills

Now you can find the total number of grains of wheat from the chess-board problem.

You can also find the total number of games of tennis that are played.

10.2 Compound interest, annuities, amortization

On her 10th birthday, Yun Lu inherits some money from her grandmother. She can either have US$10 each month now to invest in a bank which offers 2% interest compounded annually until she is 21, or she can have US$1400 on her 21st birthday.

How can you work out which option Yun Lu should choose?

When you invest money in a bank or other financial institution, you will be given interest at regular intervals such as annually, half-yearly, quarterly or monthly.

If the interest paid is **simple interest**, then the interest remains the same for each year that you have your money in the bank.

If the interest paid is **compound interest**, then the interest is added to the original amount and the new value is used to calculate the interest for the next period.

Investigation 4

For each of the following amounts deposited in a bank, calculate how much interest would be paid at the end of three years.

Amount	Simple interest at 5%	Compound interest at 5% compounded annually
US$100	Year 1: 5% of 100 = US$5 Year 2: also US$5 Year 3: also US$5 So, total is US$15	Year 1: 5% of 100 = US$5 Year 2: 5% of 105 = US$5.25 Year 3: 5% of 110.25 = US$5.5125 So, total is US$15.76

US\$500	Year 1: Year 2: Year 3: Total =	
US\$2000	Year 1: Year 2: Year 3: Total =	

Factual Which pays out more interest: simple or compound? Explain why.

Conceptual What is the connection between compound interest and geometric sequences?

If you invest US\$100 in a bank that offers 5% interest compounded annually, then:

- At the end of year 1 you have $100 + 5\% = 100(1 + 0.05) = 100\left(1+\frac{5}{100}\right)$
- At the end of year 2 you have $100\left(1+\frac{5}{100}\right) + 5\%$ of $100\left(1+\frac{5}{100}\right) = 100\left(1+\frac{5}{100}\right)^2$
- At the end of year 3 you have $100\left(1+\frac{5}{100}\right)^2 + 5\%$ of $100\left(1+\frac{5}{100}\right)^2 = 100\left(1+\frac{5}{100}\right)^3$

And so on. At the end of year n you would have $100\left(1+\frac{5}{100}\right)^n$.

In general, if you invest a **present** value of PV, the rate is r% compounded annually, and the number of years is n, then the **future** value (FV) is found by the following formula:

$$FV = PV\left(1+\frac{r}{100}\right)^n$$

Example 4

Peter invests AUD 2000 (Australian dollars) in a bank that offers simple interest at a rate of 3% per annum.

Paul invests AUD 2000 in another bank that offers 2.9% interest compounded annually.

a Calculate how much money they each have in the bank after 10 years.

b Determine after how many years they will each have at least AUD 5000 in their accounts.

a Peter: 3% of 2000 = AUD 60 $60 \times 10 = 600$	

Continued on next page

So at the end of 10 years, Peter has AUD 600 + his original AUD 2000 = AUD 2600

Paul:

Using the formula, Paul has

$$FV = 2000\left(1+\frac{2.9}{100}\right)^{10}$$

FV = AUD 2661.85

b Peter:

3% of 2000 = AUD 60

So, Peter earns AUD 60 interest each year.

In order to have AUD 5000, Peter will have to make 5000 − 2000 = AUD 3000 interest.

So, it will take $\frac{3000}{60} = 50$ years before he has AUD 5000 in the bank.

Paul:

Using the formula:

$$5000 = 2000\left(1+\frac{2.9}{100}\right)^{n}$$

So $n = 32.05$

It will take Paul just over 32 years to have AUD 5000 in the bank.

Exercise 10D

1 Merel and Misty each have €5000 to invest. Merel invests her money in a bank that pays 4.5% simple interest each year. Misty invests her money in a bank that pays 4.4% interest compounded annually.

Calculate who has more money at the end of 15 years.

2 Visay invests LAK 500 000 000 (Lao Kip) in a bank that pays 3.2% interest compounded annually.

- **a** Calculate how much Visay has in the bank after eight years.
- **b** Determine how many years it will take for his money to double.

3 Silvia invests UK£4500 in a bank that pays r% interest compounded annually. After five years, she has UK£5066.55 in the bank.

- **a** Find the interest rate.
- **b** Calculate how many years it will take for Silvia to have UK£8000 in the bank.

4 Sal wants to buy a scooter that costs US$1500. He deposits US$1000 in a bank that pays 7.5% interest compounded annually.

Calculate how long it will take before he can buy the scooter.

5 Huub invests €5000 in a bank that pays 1.2% interest compounded annually.

- **a** Find how much money he has in the bank after five years.

He then removes the money from this bank and puts it all into another bank that pays a higher interest rate compounded annually.

After a further five years, Huub now has €5675.33 in the bank.

- **b** Find the interest rate for the second bank.

Investigation 5

A bank pays interest at 6 $\%$ per annum compounded every half-year.

1 How many times a year does it pay interest?

2 What percentage interest does it pay each time?

3 Complete the following table:

6% interest per annum	**Number of times interest is paid**	**Percentage each time**
Compounded yearly Compounded half-yearly Compounded quarterly Compounded monthly	1 2	6% 3%
$r\%$ interest per annum		
Compounded yearly Compounded half-yearly Compounded quarterly Compounded monthly	1	$r\%$

4 Find the formula for the (future value) FV for a (present value) PV invested for n years at $r\%$ interest compounded half-yearly.

5 Find the formula for the FV for a PV invested for n years at $r\%$ interest compounded quarterly.

6 Find the formula for the FV for a PV invested for n years at $r\%$ interest compounded monthly.

7 **Conceptual** In terms of time periods and the principal how is the compound interest calculated?

The **compounding period** is the time period between interest payments. For example, for interest compounded quarterly, the compounding period is three months.

If PV is the present value, FV the future value, r the interest rate, n the number of years and k the compounding frequency, or number of times interest is paid in a year (ie $k = 1$ for yearly, $k = 2$ for half-yearly, $k = 4$ for quarterly and $k = 12$ for monthly), then the general formula for finding the future value is

$$FV = PV\left(1 + \frac{r}{k \times 100}\right)^{k \times n}$$

You can also use the Finance app on your GDC.

Example 5

1 Rafael invests BRL 5000 (Brazilian real) in a bank offering 2.5% interest compounded annually.

a Calculate the amount of money he has after five years.

After the five years, Rafael withdraws all his money and puts it in another bank that offers 2.5% interest per annum compounded monthly.

b Calculate the amount of money that he has in the bank after three more years.

2 Alexis invests RUB 80 000 (Russian ruble) in a bank that offers interest at 3% per annum compounded quarterly.

a Calculate how much money Alexis has in the bank after six years.

b Calculate how long it takes for his original amount of money to double.

1 a $FV = 5000\left(1+\frac{2.5}{1\times 100}\right)^{1\times 5}$ $= \text{BRL } 5657.04$	$PV = 5000,\ r = 2.5,\ k = 1,\ n = 5$
b $FV = 5657.04\left(1+\frac{2.5}{12\times 100}\right)^{12\times 3}$ $= \text{BRL } 6097.16$	$PV = 5657.04,\ r = 2.5,\ k = 12,\ n = 3$
Or, using the Finance app on your GDC:	
a N = 5 I% = 2.5 PV = −5000 PMT = 0 FV = PpY = 1 CpY = 1 Move the cursor back to FV and press enter to get the answer.	PV is usually negative because you have given it to the bank. PMT is periodic money transfers (also called annuity payment) and you do not need it here. PpY is periods in the year; when you are dealing with years, this is always 1. CpY is compounding periods: 1, 2, 4 or 12 depending on how often interest is paid.
b N = 3 I% = 2.5 PV = −5657.04 PMT = 0 FV = PpY = 1 CpY = 12 Move the cursor to FV and press enter.	

2 **a** $FV = 80\,000\left(1+\dfrac{3}{4\times 100}\right)^{4\times 6}$ $= \text{RUB } 95\,713.08$	N = 6 I% = 3 PV = −80 000 PMT = 0 FV = PpY = 1 CpY = 4 Move the cursor back to FV and press enter.
b $160\,000 = 80\,000\left(1+\dfrac{3}{4\times 100}\right)^{4\times n}$ Using the Finance app: $n = 23.19$ So it would take 23 years for his money to double.	Double the original amount is RUB 160 000 N = I% = 3 PV = −80 000 PMT = 0 FV = 160 000 PpY = 1 CpY = 4 Move the cursor back to N and press enter.

Exercise 10E

1 Ambiga invests MYR 8000 (Malaysian ringgits) in a bank offering interest at a rate of 4.6% per annum compounded monthly.

a Calculate the amount of money Ambiga has in the bank after seven years.

b Calculate how long it will take for her money to double.

2 A bank is offering a rate of 3.4% per annum compounded quarterly. Mrs Safe invests €3500 in this bank.

a Calculate the amount of money she has after six years.

Mr Secure invests €x in this bank. After six years, the amount in his bank is €4000.

b Calculate the value of x, correct to the nearest euro.

c Calculate the number of years it would take for Mr Secure's money to double.

3 Rik invests SGD 40 000 (Singaporean dollars) in an account that pays 5% interest per year, compounded half-yearly.

a Calculate how much he has in the bank after four years.

The bank then changes the interest rate to 4.9% per annum, compounded monthly.

b Calculate how much Rik has in the bank after another four years.

4 Mr Chen invests CNY 20 000 (Chinese yuan) in a bank that offers interest at a rate of 3.8% per annum compounded quarterly. Mrs Chang also invests CNY 20 000 in a bank that offers interest at a rate of 3.9% per annum compounded yearly.

Calculate who has earned more interest after five years.

5 Peter invests UK£400 in a bank that offers interest at a rate of 4% per annum compounded monthly.

  a Calculate how much money he has in the bank after 10 years.

  b Calculate how long it takes for his money to double.

6 Yvie invests US$1200 in a bank that offers interest at a rate of r% per annum compounded monthly. Her money doubles in 10 years. Find the value for r.

7 Colin, Ryan and Kyle each have €1500 to invest. Colin invests his money in a bank that offers 2.6% interest compounded quarterly. Ryan invests his money in a bank that offers 2.55% interest compounded monthly. Kyle invests his money in a bank that offers 2.75% interest compounded annually.

  a Calculate who has the most money in their account after six years.

  b Find how long it will take before Ryan has €2500.

  c Find how long it will take for Kyle to double his money.

Inflation measures the rate that prices for goods increase over time and, as a result, how much less your money can buy.

An **inflation adjustment** is the change in the price of an article that is a direct result of inflation.

Example 6

The inflation rate of a country was calculated as 4.48% per annum.

a Given that the same rate continues for the next five years, work out the percentage increase due to inflation at the end of the five years.

b A computer game costs US$35 today. Calculate what you would expect it to cost next year due to an inflation adjustment.

a Over five years, US$1 will become $\left(1+\frac{4.48}{100}\right)^5 = 1.245$, so the percentage increase is 24.5%. b The computer game will cost $35 \times 1.0448 = 36.568$, or US$36.57.	

What happens if you want to make payments at **regular intervals**?

An **annuity** is a fixed sum payable at specified intervals, usually annually, over a period, such as the recipient's life, in return for a premium paid either in instalments or in a single payment.

Investigation 6

Abraham wants to save up so that he can retire early. He decides to save US$5000 every year in an annuity that pays 4% interest compounded annually. If he wants to retire in 20 years' time, determine how you can work out how much he will have in the annuity at the end of the 20 years.

Complete the following spreadsheet:

Year	Add 5000	+ 4% interest
1	5000	5200
2	10 200	10 608
3	15 608	16 232.32
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		

How much more is this than 20 × 5000?
Discuss the benefits of saving money regularly.

The formula for working out an annuity is

$$FV = A\frac{(1+r)^n - 1}{r}$$

where FV is the future value, A is the amount invested each year, r is the interest rate and n is the number of years.

You can rearrange this formula if you want to calculate any of the other unknowns.

Fortunately, your GDC works all this out for you!

You can use the PMT button on the finance icon of your GDC.

TOK

"Debt certainly isn't always a bad thing. A mortgage can help you afford a home. Student loans can be a necessity in getting a good job. Both are investments worth making, and both come with fairly low interest rates"—Jean Chatzky

Do all societies view investment and interest in the same way?

What is your stance?

Example 7

Anmol decides to save for a yacht. He would like to have TRY 1 000 000 (Turkish lira) at the end of 10 years. He saves every year in an annuity that pays 4% interest. Calculate how much he has to save each year.

N = 10 I% = 4 PV = 0 PMT = FV = 1 000 000 PpY = 1 CpY = 1 Move the cursor to PMT and press Enter (or ALPHA ENTER) This gives a PMT of TRY 83 290.94. So, he will have to save TRY 83 290.94 every year for the next 10 years.	$P = \frac{FV(r)}{(1+r)^n - 1}$

The formula for working out monthly payments is $P = \frac{rPV}{1-(1+r)^{-n}}$ where P is the payment, r is the rate, PV is the present value and n is the number of months.

Fortunately, you do not need to remember this formula because you can use your calculator in the exams.

Example 8

Trintje has been left an annuity of €5000 in a will. The annuity is for five years at 8% per annum to be paid out monthly.

Find the monthly payments.

N = 60 I% = 8/12 PV = 5000 PMT = FV = 0 PpY = 1 CpY = 1 Move the cursor to PMT and press Enter (or ALPHA ENTER) This gives PMT = €101.38. So, Trintje receives €101.38 each month for five years.	Because the money is paid each month, you have to multiply the number of years by 12. The interest rate per month is 8/12 = 0.666…%

These are examples of annuities.

Reflect Discuss the pros and cons of an annuity.

Exercise 10F

1 Sarah-Jane starts saving for her pension. She puts UK£1500 into an annuity each year. Given that the annuity pays 3.5% interest, calculate how much she will have saved after 30 years. Is this a good investment?

2 Pedro puts MXN 2500 (Mexican pesos) into an annuity that pays 2.8% interest per annum every month for five years. Calculate how much he has saved at the end of the five years.

3 Stijn has been left an annuity of €25 000 in a will. The annuity is for 10 years at 6% per annum to be paid out annually. Work out the amount that Stijn receives each year.

4 Fiona has won an annuity of US$6000. It is for five years at 4.8% per annum to be paid out monthly. Find the monthly payments. Contrast the benefits of the full US$6000 in one payment or the annuity.

5 Mikey wants to receive an annuity of UK£500 a month for five years. The monthly interest rate is 0.8%. Determine the present value of the annuity.

6 You have US$20 000 and want to get a monthly income for 10 years. If the monthly interest rate is 0.9%, find how much you would receive each month. Comment on why you think that you would do this.

There are also similar examples but in the following cases the value of the item decreases each term. This is called **amortization**.

Investigation 7

Pim borrows US$1000 from a bank that charges 4% interest compounded annually. He wants to pay the loan back in six months in monthly instalments. The bank informs Pim that he must pay US$168.62 back each month.

1 Work out how much he has to pay in total.

If the interest is 4% compounded annually, then that would be 2% for half a year.

2 Work out 2% of US$1000.

3 Explain why the amount that Pim pays back in total is less than US$1000 + 2% of US$1000.

Pim sets up a spreadsheet to monitor his payments.

He pays US$168.62 each month, but the bank also charges interest each month on the amount owed.

4 If the interest rate is 4% per annum, work out what it is per month.

5 Calculate the interest for the first month.

6 Describe how you worked out how much is paid off the loan each month.

Continued on next page

7 The spreadsheet shows the first two lines of Pim's payments. Fill in the next few lines.

Amount owed	Payment	Interest	Payment – interest	Remaining loan
1000	168.62	3.33	165.29	834.71
834.71	168.62	2.78	165.84	668.87

After six months, the remaining loan should be US$0.

The formula to find the payments is

$$A = PV\frac{r(1+r)^n}{(1+r)^n - 1}$$

where A is the amount, PV is the present value, r is the rate and n is the number of periods.

Fortunately, you do not need to remember this formula because you can use your calculator in the exams.

You can use your GDC to work out payments, and so on.

You can also use any of the online websites such as www.amortization-calc.com or www.bankrate.com to calculate payments, and so on. You can adjust the length of time for the loan or the interest rate to see what effect this has.

Example 9

Tejas takes out a loan of €35 000 for a car. The loan is for 10 years at 1% interest per month. Find how much he has to repay each month.

N = 120

I% = 1

PV = 35 000

PMT =

FV = 0

PpY = 1

CpY = 1

Using your GDC, this gives PMT = €502.15.

So, Tejas has to pay €502.15 each month for 10 years.

Exercise 10G

1 Sami takes out a loan of €150 000 to buy a boat. The loan is for 20 years at 5% per annum.

  a Find how much he must pay each month.

  b Another loan was for 10 years at 5% per annum. Determine how much Sami would pay each month.

  c For each of these calculations find out the difference in the monthly payments and the amount of interest paid.

2 Mr and Mrs Jones take out a mortgage of UK£350 000 for a house. The mortgage is for 30 years at 2.3% per annum. Find their monthly repayments and discuss whether this is a reasonable amount.

3 Zak takes out a loan of €2000 at 4%. He repays the loan in five end-of-year instalments. Find his yearly payment.

4 A car costs US$28 000. Benji takes out a loan at 10% per annum for five years. Calculate the amount that Benji must pay each month and comment on your answer.

Developing inquiry skills

You should now be able to work out the answer to the original problem at the beginning of this section. What do you think that Yun Lu should do?

TOK

"A government's ability to raise and lower short-term interest rates is its primary control over the economy"—Alex Berenson

How can knowledge of mathematics result in individuals being exploited or protected from extortion?

10.3 Exponential models

The temperature, T, of a cup of tea is modelled by the function $T(x) = 21 + 55(1.9)^{-x}$, where T is measured in °C and x in minutes.

How long will the tea stay warm?

What information do you need to know?

Will the temperature of the tea ever reach 0°C?

Can you find out the temperature at different times?