5

Euler Paths and Circuits

United Parcel Service (UPS) is the largest package delivery company in the world. On a typical day UPS delivers roughly 15 million packages to over 6 million customers worldwide; on a busy day much more than that (the week before Christmas 2011, UPS delivered more than 120 million packages). Such remarkable feats of logistics require tremendous resources, superb organization, and (surprise!) a good dose of mathematics. In this chapter we will discuss some of the mathematical ideas that make this possible.

THE BRIDGES OF MADISON COUNTY

Madison County is a quaint old place, famous for its beautiful bridges. The Madison River runs through the county, and there are four islands (A, B, C, and D) and 11 bridges joining the islands to both banks of the river (R and L) and one another (Fig. 5-4). A famous photographer is hired to take pictures of each of the 11 bridges for a national magazine. The photographer needs to drive across each bridge once for the photo shoot. The problem is that there is a $25 toll (the locals call it a "maintenance tax") every time an out-of-town visitor drives across a bridge, and the photographer wants to minimize the total cost of the trip. The street-routing problem here is to find a route that passes through each bridge at least once and recrosses as few bridges as possible. Moreover, the photographer can start the route on either bank of the river and, likewise, end it on either bank of the river.

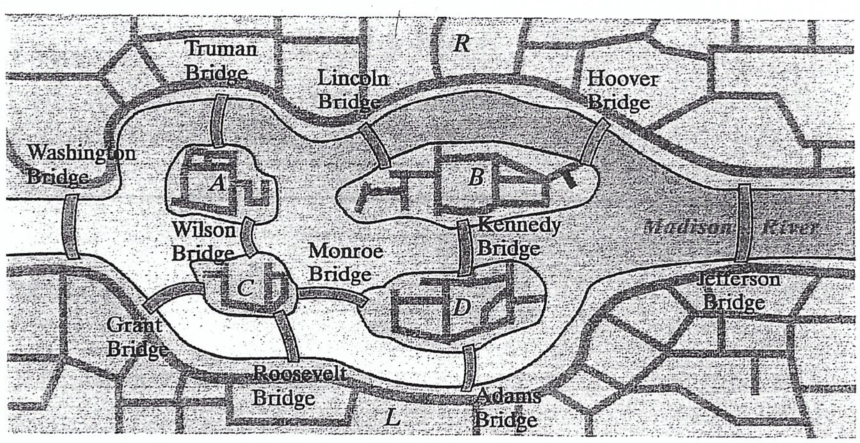


FIGURE 5-4 Bridges on the Madison River.

EXAMPLE 5.1 THE SECURITY GUARD PROBLEM

After a rash of burglaries, a private security guard is hired to patrol the streets of the Sunnyside neighborhood shown in Fig. 5-1. The security guard's assignment is to make an exhaustive patrol, on foot, through the entire neighborhood. Obviously, he doesn't want to walk any more than what is necessary. His starting point is the corner of Elm and J streets across from the school (*S* in Fig. 5-1)—that's where he usually parks his car. (This is relevant because at the end of his patrol he needs to come back to *S* to pick up his car.) Being a practical person, the security guard would like the answers to the following questions:

1. Is it possible to start and end at *S*, cover every block of the neighborhood, and pass through each block *just once*?
2. If some of the blocks will have to be covered more than once, what is an *optimal* route that covers the entire neighborhood? ("Optimal" here means "with the minimal amount of walking.")
3. Can a better route (i.e., less walking) be found by choosing a different starting and ending point? We will answer all of these questions in Section 5.4.

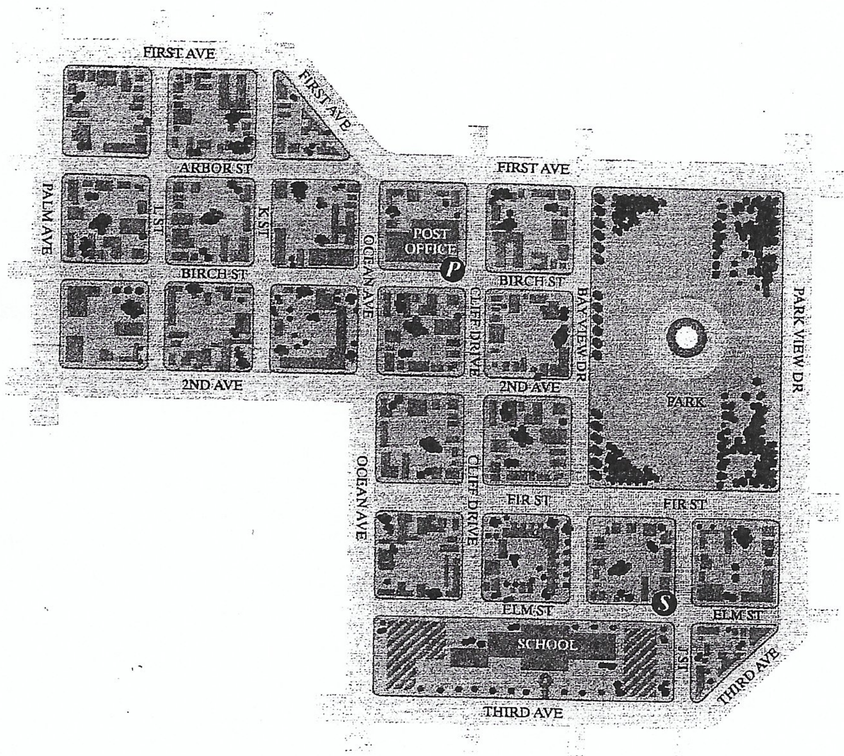


FIGURE 5-1 The Sunnyside neighborhood.

EXAMPLE 5.2 THE MAIL CARRIER PROBLEM

A mail carrier has to deliver mail in the same Sunnyside neighborhood (Fig. 5-1). The difference between the mail carrier's route and the security guard's route is that the mail carrier must make *two* passes through blocks with buildings on both sides of the street and only one pass through blocks with buildings on only one side of the street (and where there are no buildings on either side of the street, the mail carrier does not have to walk at all). In addition, the mail carrier has no choice as to her starting and ending points—she has to start and end her route at the local post office (*P* in Fig. 5-1). Much like the security guard, the mail carrier wants to find the optimal route that would allow her to cover the neighborhood with the least amount of walking. (Put yourself in her shoes and you would do the same—good weather or bad, she walks this route 300 days a year!)

5.2 An Introduction to Graphs

A note of warning: The graphs we will be discussing here have no relation to the graphs of functions you may have studied in algebra or calculus.

The key tool we will use to tackle the street-routing problems introduced in Section 5.1 is the notion of a **graph**. The most common way to describe a *graph* is by means of a picture. The basic elements of such a picture are a set of "dots" called the **vertices** of the graph and a collection of "lines" called the **edges** of the graph. (Unfortunately, this terminology is not universal. In some applications the word "nodes" is used for the vertices and the word "links" is used for the edges. We will stick to vertices and edges as much as possible.) On the surface, that's all there is to it—edges connecting vertices. Below the surface there is a surprisingly rich theory. Let's look at a few examples first.

EXAMPLE 5.6 BASIC GRAPH CONCEPTS

Figure 5-5 shows several examples of graphs. We will discuss each separately.

- Figure 5-5(a) shows a graph with six vertices labeled *A, B, C, D, E,* and *F* (it is customary to use capital letters to label the vertices of a graph). For convenience we refer to the set of vertices of a graph as the **vertex set.** In this graph, the vertex set is $\{A, B, C, D, E, F\}$. The graph has 11 edges (described by listing, in any order, the two vertices that are connected by the edge): *AB, AD, BC*, etc.
  - When two vertices are connected by an edge we say that they are **adjacent vertices.** Thus, *A* and *B* are adjacent vertices, but *A* and *E* are not adjacent. The edge connecting *B* with itself is written as *BB* and is called a **loop.** Vertices *C* and *D* are connected twice (i.e., by two separate edges), so when we list the edges we include *CD* twice. Similarly, vertices *E* and *F* are connected by three edges, so we list *EF* three times. We refer to edges that appear more than once as **multiple edges.**
  - The complete list of edges of the graph, the **edge list,** is *AB, AD, BB, BC, BE, CD, CD, DE, EF, EF, EF.*
  - The number of edges that meet at each vertex is called the **degree** of the vertex and is denoted by $\deg(X)$. In this graph we have $\deg(A) = 2$, $\deg(B) = 5$ (please note that the loop contributes 2 to the degree of the vertex), $\deg(C) = 3$, $\deg(D) = 4$, $\deg(E) = 5$, and $\deg(F) = 3$. It will be important in the next section to distinguish between vertices depending on whether their degree is an odd or an even number. We will refer to vertices like *B, C, E,* and *F* with an odd degree as **odd vertices** and to vertices with an even degree like *A* and *D* as **even vertices.**

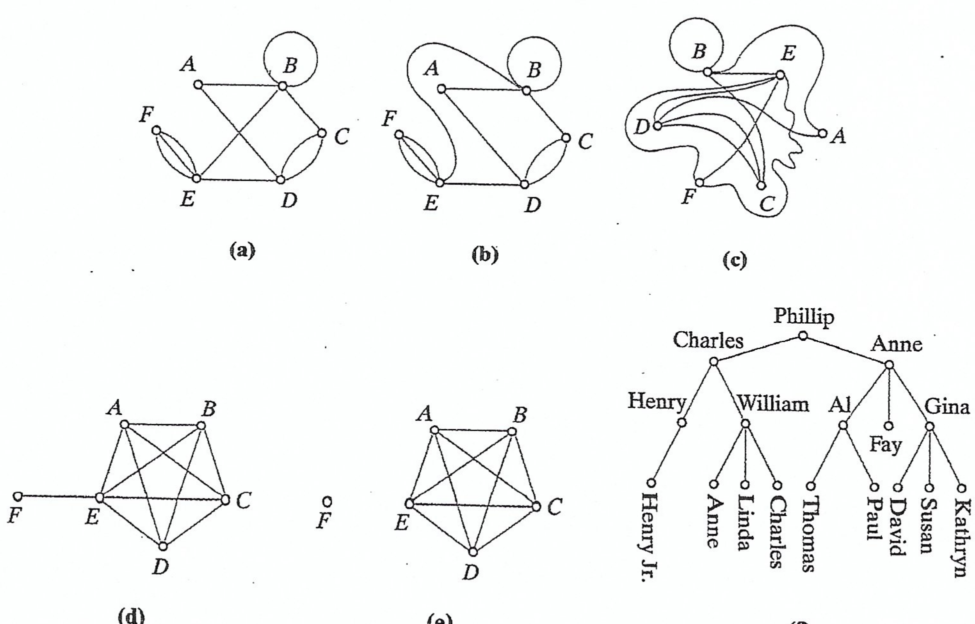


FIGURE 5-5 (a), (b), and (c) are all pictures of the same graph; (d) is a simple, connected graph; (e) is a simple, disconnected graph; and (f) is a graph labeled with names instead of letters.

- Figure 5-5(b) is very similar to Fig. 5-5(a)—the only difference is the way the edge BE is drawn. In Fig. 5-5(a) edges AD and BE cross each other, but the crossing point is not a vertex of the graph—it's just an irrelevant crossing point. Fig. 5-5(b) gets around the crossing by drawing the edge in a more convoluted way, but the way we draw an edge is itself irrelevant. The key point here is that as graphs, Figs. 5-5(a) and 5-5(b) are the same. Both have exactly the same vertices and exactly the same edge list.
- Figure 5-5(c) take the idea one step further—it is in fact, another rendering of the graph shown in Figs. 5-5(a) and (b). The vertices have been moved around and put in different positions, and the edges are funky—no other way to describe it. Despite all the funkiness, this graph conveys exactly the same information that the graph in Fig. 5-5(a) does. You can check it out— same set of vertices and same edge list. The moral here is that while graphs are indeed pictures connecting "dots" with "lines," it is not the specific picture that matters but the story that the picture tells: which dots are connected to each other and which aren't. We can move the vertices around, and we can draw the edges any funky way we want (straight line, curved line, wavy line, etc.)—none of that matters. The only thing that matters is the set of vertices and the list of edges.
- Figure 5-5(d) shows a graph with six vertices. Vertices A, B, C, D, and E form what is known as a **clique**—each vertex is connected to each of the other four. Vertex F, on the other hand, is connected to only one other vertex. This graph has no loops or multiple edges. Graphs without loops or multiple edges are called **simple graphs**. There are many applications of graphs where loops and multiple edges cannot occur, and we have to deal only with simple graphs. (In Examples 5.7 and 5.8 we will see two applications where only simple graphs occur.)
- Figure 5-5(e) shows a graph very similar to the one in Fig. 5-5(d). The only difference between the two is the absence of the edge EF. In this graph there are no edges connecting F to any other vertex. For obvious reasons, F is called an **isolated** vertex. This graph is made up of two separate and disconnected "pieces"—the clique formed by the vertices A, B, C, D, and E and the isolated vertex F. Because the graph is not made of a single "piece," we say that the graph is **disconnected**, and the separate pieces that make up the graph are called the *components* of the graph.
- Figure 5-5(f) shows a connected simple graph. The vertices of this graph are names (there is no rule about what the labels of a vertex can be). Can you guess what this graph might possibly represent?

EXAMPLE 5.9 PATHS AND CIRCUITS

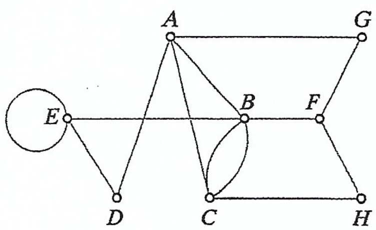


FIGURE 5-8 Graph for Example 5.9.

We say that two edges are **adjacent edges** when they share a common vertex. In Fig. 5-8 for example, *AB* is adjacent to *AC* and *AD* (they share vertex *A*), as well as to *BC*, *BF*, and *BE* (they share vertex *B*). A sequence of *distinct* edges each adjacent to the next is called a **path**, and the number of edges in the path is called the **length** of the path. For example in Fig. 5-8, the edges *AB*, *BF*, and *FG* form a path of length 3. A good way to think of a path is as a real-world path—a way to "hike" along the edges of the graph, traveling along the first edge, then the next, and so on. To shorten the notation, we describe the path by just listing the vertices in sequence separated by commas. For example *A*, *B*, *F*, *G* describes the path formed by the edges *AB*, *BF*, and *FG*.

Here are a few more examples of paths in Fig. 5-8:

- *A*, *B* is a path of length 1. Any edge can be thought of as a path of length 1—not very interesting, but it allows us to apply the concept of a path even to single edges.
- *A*, *B*, *C*, *A*, *D*, *E* is a path of length 5 starting at *A* and ending at *E*. The path goes through vertex *A* a second time, but that's OK. It is permissible for a path to revisit some of the vertices. On the other hand, *A*, *C*, *B*, *A*, *C* does not meet the definition of a path because the edges of the path cannot be revisited and here *AC* is traveled twice. So, in a path it's OK to revisit some of the vertices but not OK to revisit any edges.
- *A*, *B*, *C*, *A*, *D*, *E*, *E*, *B* is a path of length 7. Notice that this "trip" is possible because of the loop at *E*.
- *A*, *B*, *C*, *A*, *D*, *E*, *B*, *C* is also a legal path of length 7. Here we can use the edge *BC* twice because there are in fact two distinct edges connecting *B* and *C*.

When a trip along the edges of the graph closes back on itself (i.e., starts and ends with the same vertex) we specifically call it a **circuit** rather than a path. Thus, we will restrict the term *path* to open-ended trips and the word *circuit* to closed trips.

Here are a few examples of circuits in Fig. 5-8:

- *A*, *D*, *E*, *B*, *A* is a circuit of length 4. Even though it appears like the circuit designates *A* as the starting (and ending) vertex, a circuit is independent of where we designate the start. In other words, the same circuit can be written as *D*, *E*, *B*, *A*, *D* or *E*, *B*, *A*, *D*, *E*, etc. They are all the same circuit, but we have to choose one (arbitrary) vertex to start the list.
- *B*, *C*, *B* is a circuit of length 2. This is possible because of the double edge *BC*. On the other hand, *B*, *A*, *B* is not a circuit because the edge *AB* is being traveled twice. (Just as in a path, the edges of a circuit have to be distinct.)
- *E*, *E* is a circuit of length 1. A loop is the only way to have a circuit of length 1.

In Example 5.9 we saw several examples of paths (and circuits) that are part of the graph in Fig. 5-8, but the important idea we will discuss next in this: Can the path (or circuit) be the entire graph, not just a part of it? In other words, we want to consider the possibility of a path (or a circuit) that *exhausts* all the edges of the graph.

An **Euler path** (named after Leonhard Euler) is a path that covers *all* the edges of the graph. Likewise, an **Euler circuit** is a circuit that covers all the edges of the graph. In other words, we have an Euler path (or circuit) when the entire graph can be written as a path (or circuit).

EXAMPLE 5.10 EULER PATHS AND EULER CIRCUITS

Figures 5-9, 5-10, and 5-11 illustrate the three possibilities that can occur:

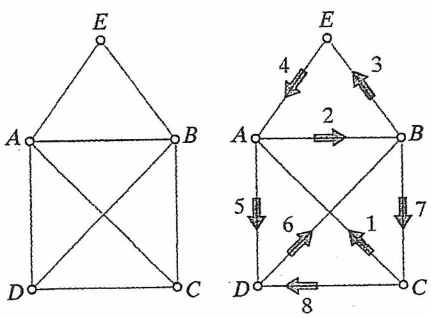


FIGURE 5-9 An Euler path starting at *C* and ending at *D*.

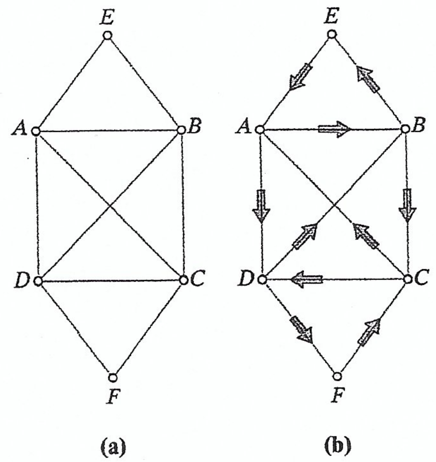


FIGURE 5-10 An Euler circuit.

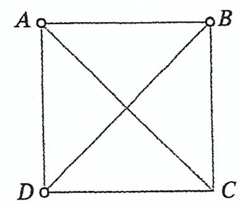


FIGURE 5-11 No Euler path or circuit.

- The graph in Fig. 5-9(a) has an Euler path—in fact, it has several. One of the possible Euler paths is shown in Fig. 5-9(b). The path starts at C and ends at D—just follow the arrows and you will be able to "trace" the edges of the graph without retracing any (just like in elementary school).
- The graph in Fig. 5-10(a) has many possible Euler circuits. One of them is shown in Fig. 5-10(b). Just follow the arrows. Unlike the Euler path in Fig. 5-9(b), the arrows are not numbered. You can start this circuit at any vertex of the graph, follow the arrows, and you will return to the starting vertex having covered all the edges once.
- The graph in Fig. 5-11 has neither an Euler path nor an Euler circuit. That's the way it goes sometimes—some graphs just don't have it!

We introduced the idea of a *connected* or *disconnected* graph in Example 5.6. Formally, we say that a graph is **connected** if you can get from any vertex to any other vertex along some path of the graph. Informally, this says that you can get from any point to any other point by "hiking" along the edges of the graph. Even more informally, it means that the graph is made of one "piece." A graph that is not connected is called **disconnected** and consists of at least two (maybe more) separate "pieces" we call the **components** of the graph.

EXAMPLE 5.11 BRIDGES

Figure 5-12 shows three different graphs. The graph in Fig. 5-12(a) is connected; the graph in Fig. 5-12(b) is disconnected and has two components; the graph in Fig. 5-12(c) is disconnected and has three components (the isolated vertex *G* is a component—that's as small a component as you can get!).

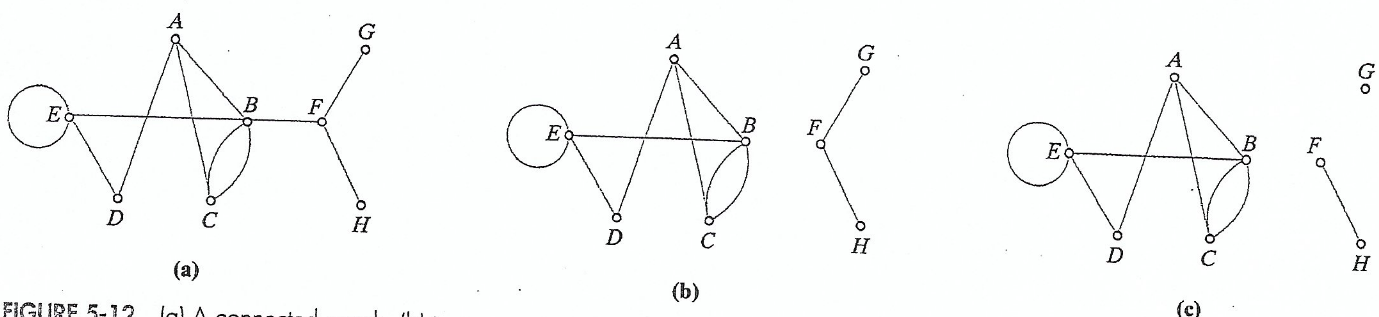


FIGURE 5-12 (a) A connected graph, (b) two components, and (c) three components.

Notice that the only difference between the *disconnected* graph in Fig. 5-12(b) and the *connected* graph in Fig. 5-12(a) is the edge BF. Think of *BF* as a "bridge" that connects the two components of the graph in Fig. 5-12(b). Not surprisingly, we call such an edge a *bridge*. A **bridge** in a connected graph is an edge that keeps the graph connected—if the bridge were not there, the graph would be disconnected. The graph in Fig. 5-12(a) has three bridges: *BF*, *FG*, and *FH*.

Vertices

- **adjacent:** any two vertices connected by an edge
- **vertex set:** the set of vertices in a graph
- **degree:** number of edges meeting at the vertex
- **odd (even):** degree is an odd (even) number
- **isolated:** no edges connecting the vertex (i.e., degree is 0)

Edges

- **adjacent:** two edges that share a vertex
- **loop:** an edge that connects a vertex with itself
- **multiple edges:** more than one edge connecting the same two vertices
- **edge list:** a list of all the edges in a graph
- **bridge:** an edge in a connected graph without which the graph would be disconnected

Paths and circuits

- **path:** a sequence of edges each adjacent to the next, with no edge included more than once, and starting and ending at different vertices
- **circuit:** same as a path, but starting and ending at the same vertex
- **Euler path:** a path that covers all the edges of the graph
- **Euler circuit:** a circuit that covers all the edges of the graph
- **length:** number of edges in a path or a circuit

Graphs

- **simple:** a graph with no loops or multiple edges
- **connected:** there is a path going from any vertex to any other vertex
- **disconnected:** not connected; consisting of two or more components
- **clique:** a set of completely interconnected vertices in the graph (every vertex is connected to every other vertex by an edge)

EXAMPLE 5.13 MODELING THE SECURITY GUARD PROBLEM

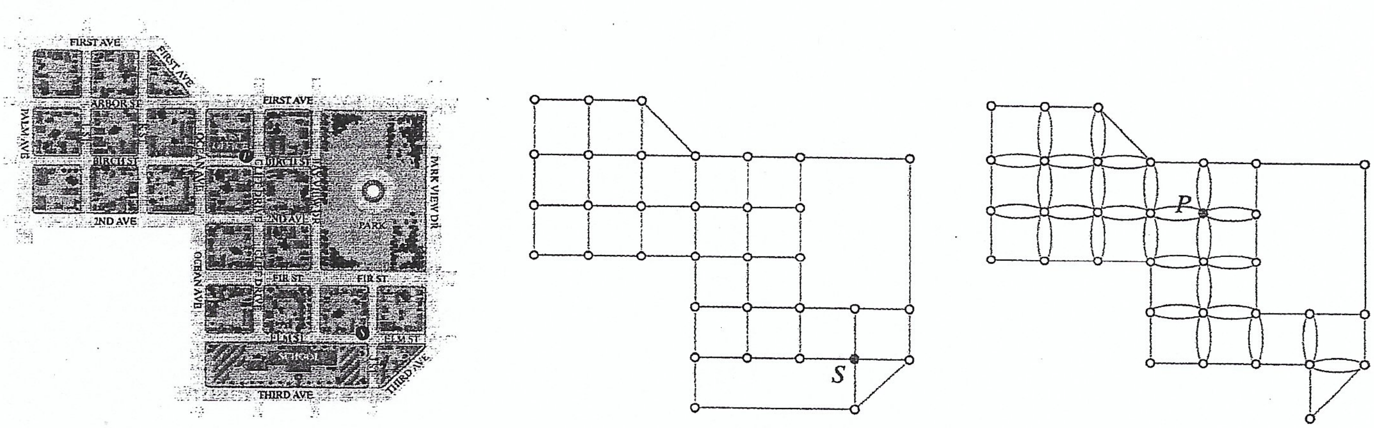
In Example 5.1 we were introduced to the problem of the security guard who needs to walk the streets of the Sunnyside neighborhood [Fig. 5-14(a)]. The graph in Fig. 5-14(b)—where each edge represents a block of the neighborhood and each vertex an intersection—is a graph model of this problem. The questions raised in Example 5.1 can now be formulated in the language of graphs.

1. Does the graph in Fig. 5-14(b) have an Euler circuit that starts and ends at S?
2. What is the fewest number of edges that have to be added to the graph so that there is an Euler circuit?

We will learn how to answer such questions in the next couple of sections.

EXAMPLE 5.14 MODELING THE MAIL CARRIER PROBLEM

Unlike the security guard, the mail carrier in Example 5.2 must make two passes through every block that has homes on both sides of the street (she has to physically place the mail in the mailboxes), must make one pass through blocks that have homes on only one side of the street, and does not have to walk along blocks where there are no houses. In this situation an appropriate graph model requires two edges on the blocks that have homes on both sides of the street, one edge for the blocks that have homes on only one side of the street, and no edges for blocks having no homes on either side of the street. The graph that models this situation is shown in Fig. 5-14(c).



(a) (b) (c)

FIGURE 5-14 (a) The Sunnyside neighborhood. (b) A graph model for the security guard. (c) A graph model for the mail carrier.

5.3 Euler's Theorems and Fleury's Algorithm

In this section we are going to develop the basic theory that will allow us to determine if a graph has an Euler circuit, an Euler path, or neither. This is important because, as we saw in the previous section, what are Euler circuit or Euler path questions in theory are real-life street-routing questions in practice. The three theorems we are going to see next (all due to Euler) are surprisingly simple and yet tremendously useful.

EULER'S CIRCUIT THEOREM

- If a graph is *connected* and *every vertex is even*, then it has an Euler circuit (at least one, usually more).
- If a graph has *any odd vertices*, then it does not have an Euler circuit.

If we want to know if a graph has an Euler circuit or not, here is how we can use Euler's circuit theorem. First we make sure the graph is connected. (If it isn't, then no matter what else, an Euler circuit is impossible.) If the graph is connected, then we start checking the degrees of the vertices, one by one. As soon as we hit an odd vertex, we know that an Euler circuit is out of the question. If there are no odd vertices, then we know that the answer is yes—the graph does have an Euler circuit! (The theorem doesn't tell us how to find it—that will come soon.) Figure 5-16 illustrates the three possible scenarios. The graph in Fig. 5-16(a) cannot have an Euler circuit for the simple reason that it is disconnected. The graph in Fig. 5-16(b) is connected, but we can quickly spot odd vertices (*C* is one of them; there are others). This graph has no Euler circuits either. But the graph in Fig. 5-16(c) is connected and all the vertices are even. This graph does have Euler circuits.

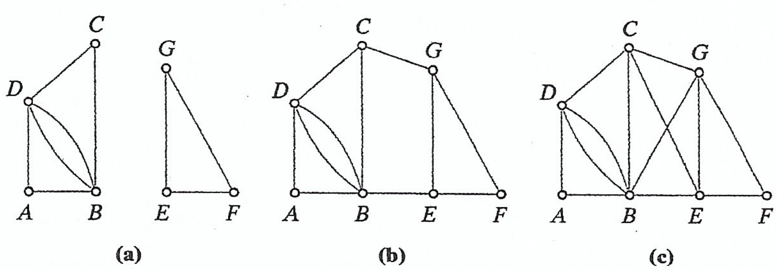


FIGURE 5-16 (a) Not connected; (b) some vertices are odd; (c) all vertices are even.

EULER'S PATH THEOREM

- If a graph is *connected* and has *exactly two odd vertices*, then it has an Euler path (at least one, usually more). Any such path must start at one of the odd vertices and end at the other one.
- If a graph has *more than two* odd vertices, then it cannot have an Euler path.

EULER'S SUM OF DEGREES THEOREM

- The sum of the degrees of all the vertices of a graph equals twice the number of edges (and, therefore, is an even number).
- A graph always has an even number of *odd* vertices.

Euler's sum of degrees theorem is based on the following basic observation: Take any edge—let's call it XY. The edge contributes once to the degree of vertex X and once to the degree of vertex Y, so, in all, that edge makes a total contribution of 2 to the sum of the degrees. Thus, when the degrees of all the vertices of a graph are added, the total is twice the number of edges. Since the total sum is an even number, it is impossible to have just one odd vertex, or three odd vertices, or five odd vertices, and so on. To put it in a slightly different way, *the odd vertices of a graph always come in twos.*

Table 5-2 is a summary of Euler's three theorems. It shows the relationship between the number of odd vertices in a connected graph G and the existence of Euler paths or Euler circuits. (The assumption that G is connected is essential—a disconnected graph cannot have Euler paths or circuits regardless of what else is going on.)

Number of odd vertices	Conclusion
0	G has Euler circuit
2	G has Euler path
4, 6, 8, . . .	G has neither
1, 3, 5, . . .	This is impossible!

TABLE 5-2 Euler's theorems (summary)

Euler's theorems help us answer the following existence question: Does the graph have an Euler circuit, an Euler path, or neither? But when the graph has an Euler circuit or path, how do we find it? For small graphs, simple trial-and-error usually works fine, but real-life applications sometimes involve graphs with hundreds, or even thousands, of vertices. In these cases a trial-and-error approach is out of the question, and what is needed is a systematic strategy that tells us how to create an Euler circuit or path. In other words, we need an *algorithm*.

Fleury's algorithm is based on a simple principle: To find an Euler circuit or an Euler path, *bridges are the last edges you want to cross.* Only do it if you have no choice! Simple enough, but there is a rub: The graph whose bridges we are supposed to avoid is not necessarily the original graph of the problem. Instead, it is that part of the original graph that has yet to be traveled. The point is this: Once we travel along an edge, we are done with it! We will never cross it again, so from that point on, as far as we are concerned, it is as if that edge never existed. Our only concern is how we are going to get around in the *yet-to-be-traveled* part of the graph. Thus, when we talk about bridges that we want to leave as a last resort, we are really referring to *bridges of the to-be-traveled part of the graph.*

FLEURY'S ALGORITHM FOR FINDING AN EULER CIRCUIT (PATH)

- **Preliminaries.** Make sure that the graph is connected and either (1) has no odd vertices (circuit) or (2) has just two odd vertices (path).
- **Start.** Choose a starting vertex. [In case (1) this can be any vertex; in case (2) it must be one of the two *odd* vertices.]
- **Intermediate steps.** At each step, if you have a choice, *don't choose a bridge of the yet-to-be-traveled part* of the graph. However, if you have only one choice, take it.
- **End.** When you can't travel any more, the circuit (path) is complete. [In case (1) you will be back at the starting vertex; in case (2) you will end at the other odd vertex.]

The only complicated aspect of Fleury's algorithm is the bookkeeping. With each new step, the untraveled part of the graph changes and there may be new bridges formed. Thus, in implementing Fleury's algorithm it is critical to separate the *past* (the part of the graph that has already been traveled) from the *future* (the part of the graph that still needs to be traveled). While there are many different ways to accomplish this (you are certainly encouraged to come up with one of your own), a fairly reliable way goes like this: Start with *two* copies of the graph. Copy 1 is to keep track of the "future"; copy 2 is to keep track of the "past." Every time you travel along an edge, *erase* the edge from copy 1, but mark it (say in red) and label it with the appropriate number on copy 2. As you move forward, copy 1 gets smaller and copy 2 gets redder. At the end, copy 1 has disappeared; copy 2 shows the actual Euler circuit or path.

EXAMPLE 5.17 IMPLEMENTING FLEURY'S ALGORITHM

The graph in Fig. 5-19(a) is a very simple graph—it would be easier to find an Euler circuit just by trial-and-error than by using Fleury's algorithm. Nonetheless, we will do it using Fleury's algorithm. The real purpose of the example is to see the algorithm at work. Each step of the algorithm is explained in Figs. 5-19(b) through (h).

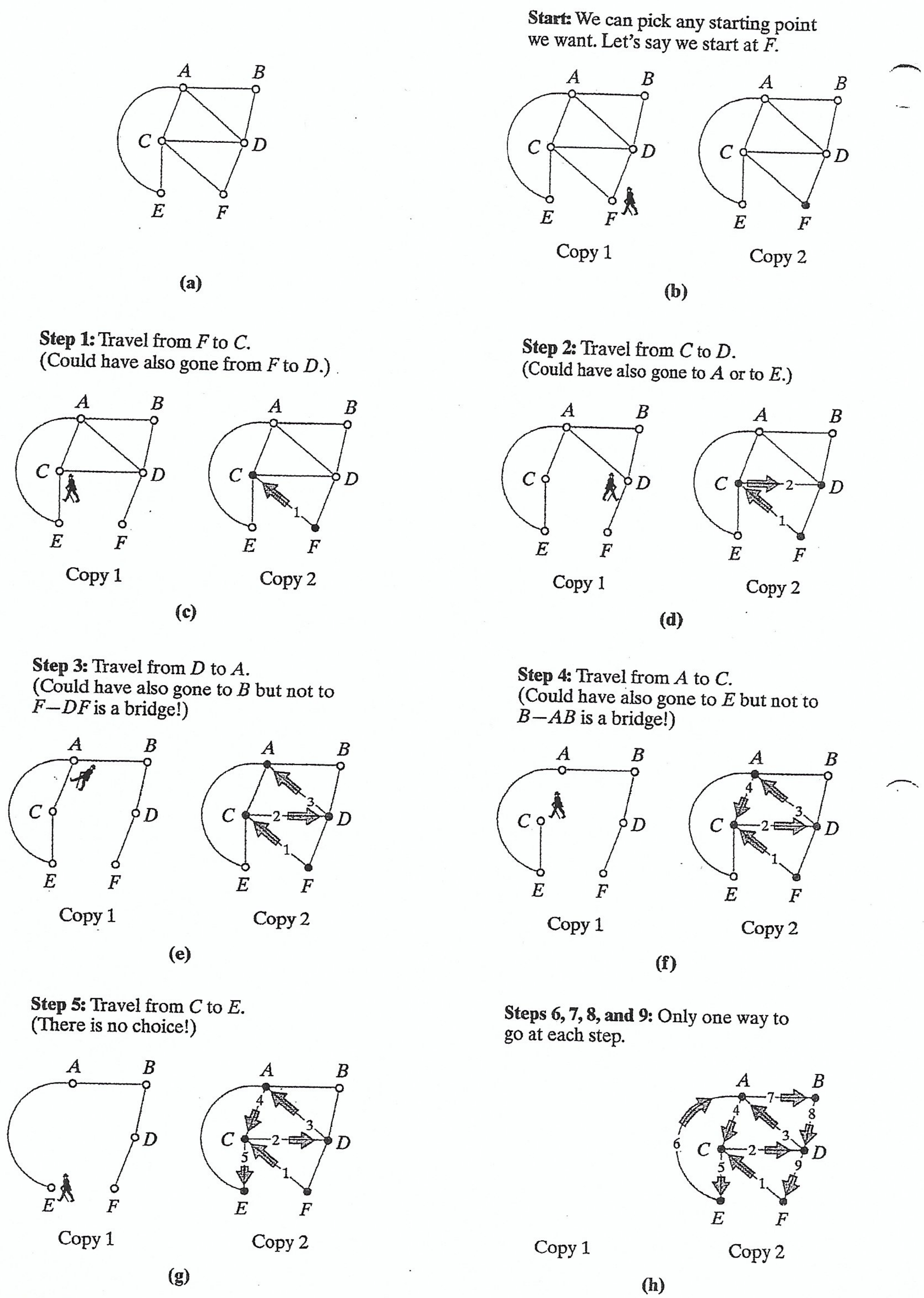


FIGURE 5-19 Fleury's algorithm at work.

EXAMPLE 5.18 FLEURY'S ALGORITHM FOR EULER PATHS

We will apply Fleury's algorithm to the graph in Figure 5-20. Since it would be a little impractical to show each step of the algorithm with a separate picture as we did in Example 5.17, you are going to have to do some of the work. Start by making two copies of the graph. (If you haven't already done so, get some paper, a pencil, and an eraser.)

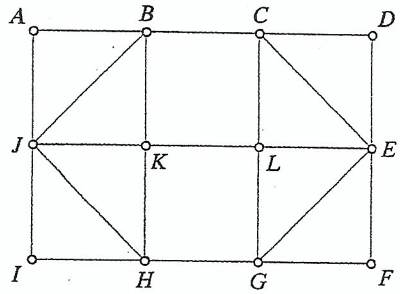
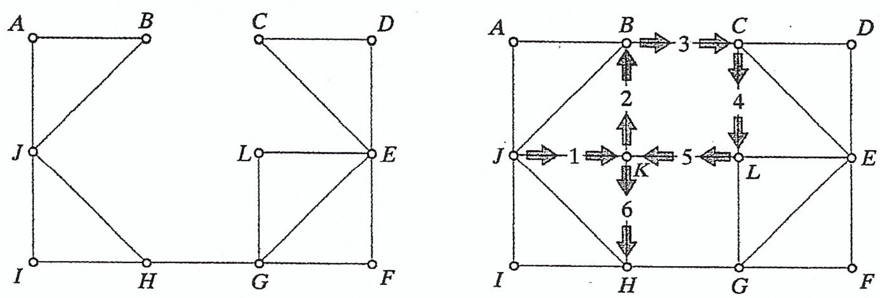


FIGURE 5-20

- **Start.** This graph has two odd vertices, *E* and *J*. We can pick either one as the starting vertex. Let's start at *J*.
- **Step 1.** From *J* we have five choices, all of which are OK. We'll randomly pick *K*. (Erase *JK* on copy 1, and mark and label *JK* with a 1 on copy 2.)
- **Step 2.** From *K* we have three choices (*B*, *L*, or *H*). Any of these choices is OK. Say we choose *B*. (Now erase *KB* from copy 1 and mark and label *KB* with a 2 on copy 2.)
- **Step 3.** From *B* we have three choices (*A*, *C*, or *J*). Any of these choices is OK. Say we choose *C*. (Now erase *BC* from copy 1 and mark and label *BC* with a 3 on copy 2.)
- **Step 4.** From *C* we have three choices (*D*, *E*, or *L*). Any of these choices is OK. Say we choose *L*. (EML—that's shorthand for erase, mark, and label.)
- **Step 5.** From *L* we have three choices (*E*, *G*, or *K*). Any of these choices is OK. Say we choose *K*. (EML.)
- **Step 6.** From *K* we have only one choice—to *H*. Without further ado, we choose *H*. (EML.)
- **Step 7.** From *H* we have three choices (*G*, *I*, or *J*). But for the first time, one of the choices is a bad choice. We should not choose *G*, as *HG* is a bridge of the yet-to-be-traveled part of the graph (Fig. 5-21). Either of the other two choices is OK. Say we choose *J*. (EML.)



Copy 1 at Step 7

Copy 2 at Step 7

FIGURE 5-21

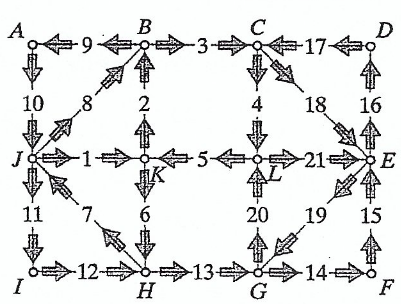


FIGURE 5-22

- **Step 8.** From *J* we have three choices (*A*, *B*, or *I*), but we should not choose *I*, as *JI* has just become a bridge. Either of the other two choices is OK. Say we choose *B*. (EML)
- **Steps 9 through 13.** Each time we have only one choice. From *B* we have to go to *A*, then to *J*, *I*, *H*, and *G*.
- **Steps 14 through 21.** Not to belabor the point, let's just cut to the chase. The rest of the path is given by *G*, *F*, *E*, *D*, *C*, *E*, *G*, *L*, *E*. There are many possible endings, and you should find a different one by yourself.

The completed Euler path (one of hundreds of possible ones) is shown in Fig. 5-22. ■

5.4 Eulerizing and Semi-Eulerizing Graphs

In this section we will finally answer some of the street-routing problems introduced in Section 5.1. The common thread in all these problems is to find routes that (1) cover all the edges of the graph that models the original problem and (2) recross the fewest number of edges. The first requirement typically comes with the problem; the second requirement comes from the desire to be as efficient as possible. In many applications, each edge represents a unit of cost. The more edges along the route, the higher the cost of the route. In a street-routing problem the first pass along an edge is a requirement of the job. Any additional pass along that edge represents a wasted expense (these extra passes are often described as *deadhead* travel). Thus, an optimal route is one with the fewest number of deadhead edges. (This is only true under the assumption that each edge equals one unit of cost.)

We are now going to see how the theory developed in the preceding sections will help us design optimal street routes for graphs with many (more than two) odd vertices. The key idea is that we can turn odd vertices into even vertices by adding "duplicate" edges in strategic places.

Eulerizations

EXAMPLE 5.19 COVERING A 3-BY-3 STREET GRID

The graph in Fig. 5-23(a) models a 3-block-by-3-block street grid. The graph has 24 edges, each representing a block of the street grid. How can we find an optimal route that covers all the blocks of the street grid and ends back at the starting point?

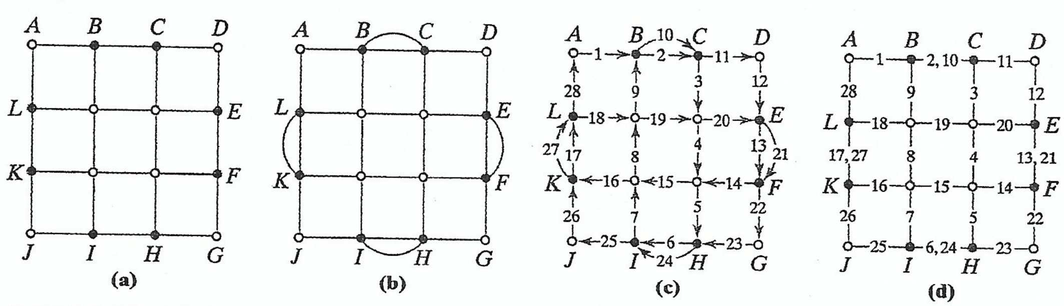


FIGURE 5-23 (a) The original graph model (odd vertices shown in red); (b) an optimal eularization; (c) an Euler circuit; (d) an optimal route on the street grid.

Our first step is to identify the odd vertices in the graph model. This graph has eight odd vertices (B, C, E, F, H, I, K, and L), shown in red. When we add a duplicate copy of edges BC, EF, HI, and KL, we get the graph in Fig. 5-23(b). This is called an **eulerization** of the original graph—in this **eulerized** version the vertices are all even, so we know this graph has an Euler circuit. Moreover, with eight odd vertices we need to add *at least* four duplicate edges, so this is the best we can do.

Figure 5-23(c) shows one of the many possible Euler circuits, with the edges numbered in the order they are traveled. The Euler circuit described in Fig. 5-23(c) represents a route that covers every block of the 3-by-3 street grid and ends back at the starting point, using only four deadhead blocks [Fig. 5-23(d)]. The total length of this route is 28 blocks (24 blocks in the grid plus 4 deadhead blocks), and this route is optimal—no matter how clever you are or how hard you try, if you want to travel along each block of the grid and start and end at the same point, you will have to pass through a minimum of 28 blocks! (There are many other ways to do it using just 28 blocks, but none with fewer than 28.)

EXAMPLE 5.21 THE SECURITY GUARD PROBLEM SOLVED

We are now ready to solve the security guard street-routing problem introduced in Example 5.1 and subsequently modeled in Example 5.13. Let's recap the story: The security guard is required to walk each block of the Sunnyside neighborhood at least once and he wants to deadhead as few blocks as possible. He usually parks his car at *S* (there is a donut shop on that corner) and needs to end his route back at the car. Figure 5-25 shows the evolution of a solution: Fig. 5-25(a) shows the original neighborhood that the security guard must cover; Fig. 5-25(b) shows the graph model of the problem (with the 18 odd vertices of the graph highlighted in red); Fig. 5-25(c) shows an eulerization of the graph in (b), with the 9 duplicate edges shown in red. This is the fewest number of edges required to eulerize the graph in Fig. 5-25(b), so the eulerization is optimal. The eulerized graph in Fig. 5-25(c) has all even vertices, and an Euler circuit can be found. Since a circuit can be started at any vertex we will start the circuit at *S*. Figure 5-25(d) shows one of the many possible optimal routes for the security guard (just follow the numbers). Note that using a different starting vertex will not make the route shorter, so the security guard can continue parking in front of the donut shop—it will not hurt!

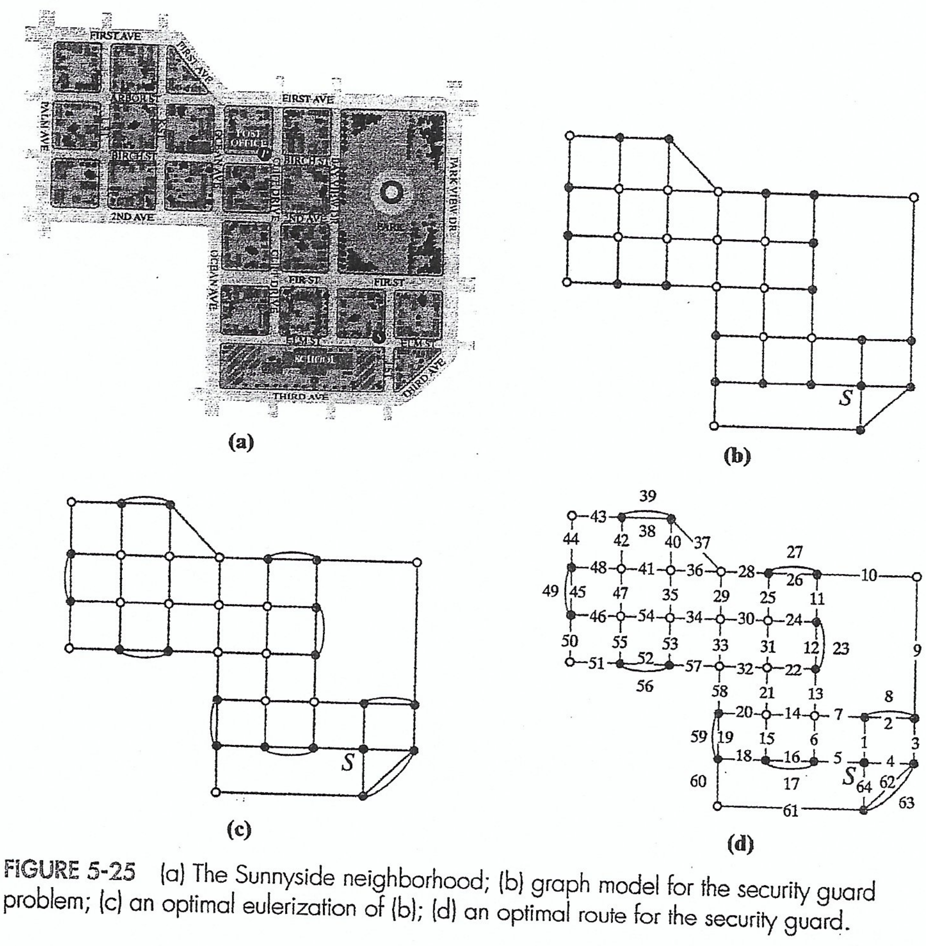


FIGURE 5-25 (a) The Sunnyside neighborhood; (b) graph model for the security guard problem; (c) an optimal eulerization of (b); (d) an optimal route for the security guard.

EXAMPLE 5.22 THE MAIL CARRIER PROBLEM SOLVED

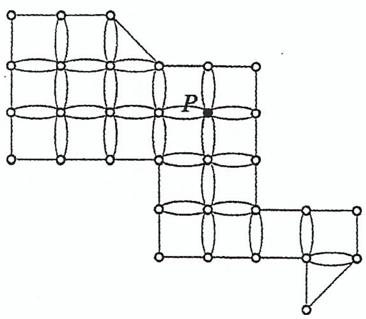


FIGURE 5-26 Graph model for the mail carrier.

The solution to the mail carrier problem follows essentially the same story line as the one for the security guard. Let's recap this story: The mail carrier needs to start and end her route at the post office (*P*), cover both sides of the street when there are buildings on both sides, and cover just one side on blocks where there are buildings on only one side, with no need to cover any streets where there are no buildings (like the back side of the park and the school). Fig. 5-26 shows the graph model for the mail carrier (see Example 5.14). The interesting thing about this graph is that every vertex is already even (there are some vertices of degree 2 in the corners, there are a couple of vertices of degree 6, and there are lots of vertices of degrees 4 and 8). This means that the graph does not have to be eulerized, as it has Euler circuits in its present form. An optimal route for the mail carrier can be found by finding an Euler circuit of the graph that starts and ends at *P*. We know how to do that and leave it as an exercise for the reader to find such a route. There are hundreds of possible routes that will work. (*Note*: Although Fleury's algorithm is a sure bet, in a case like this trial and error is almost guaranteed to work best.) ■

Semi-Eulerizations

In cases where a street route is not required to end back where it started (either because we can choose to start and end in different places or because the starting and ending points are required to be different) we are looking for Euler paths, rather than Euler circuits. In these cases we are looking for a graph that has two odd vertices and the rest even. We need the two odd vertices to give us a starting and ending point for our route. The process of adding additional edges to a graph so that all the vertices except two are even is called a **semi-eulerization**, and we say that the graph has been *semi-eulerized.*

EXAMPLE 5.23 THE BRIDGES OF MADISON COUNTY SOLVED

In Example 5.5 we introduced the problem of routing a photographer across all the bridges in Madison County, shown in Fig. 5-27(a). To recap the story: The photographer needs to cross each of the 11 bridges at least once (for her photo shoot). Each crossing of a bridge costs \$25, and the photographer is on a tight budget, so she wants to cover all the bridges once but recross as few bridges as possible. The other relevant fact is that the photographer can start her trip at either bank of the river and end the trip at either bank of the river. Figure 5-27(b) is a graph model of the Madison bridges layout. A la Euler, we let the vertices represent the land masses and the edges represent the bridges. The graph has four odd vertices (R, L, B, and D). The photographer can start the shoot at either bank and end at either bank—say she chooses to start the route at R and end it at L. Figure 5-27(c) shows a semi-eulerization of the graph in (b), with R and L left as odd vertices, and the edge BD (i.e., the Kennedy bridge) recrossed so that now B and D are even vertices. The numbers in Fig. 5-27(c) show one possible optimal route for the photographer, with a total of 12 bridge crossings and a total cost of \$300.

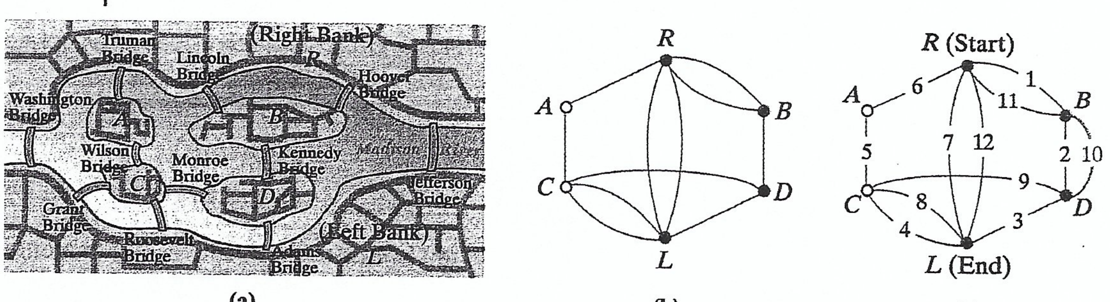


FIGURE 5-27 (a) The original layout; (b) graph model; (c) semi-eulerization of (b) and an optimal route.

EXERCISES

WALKING

5.1 Street-Routing Problems

No exercises for this section.

5.2 An Introduction to Graphs

1. For the graph shown in Fig. 5-29,

(a) give the vertex set.

(b) give the edge list.

(c) give the degree of each vertex.

(d) draw a version of the graph without crossing points.

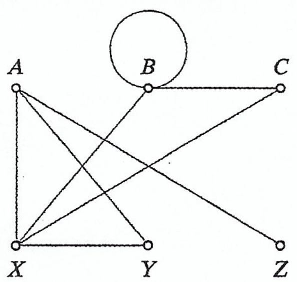


FIGURE 5-29

2. For the graph shown in Fig. 5-30,

(a) give the vertex set.

(b) give the edge list.

(c) give the degree of each vertex.

(d) sraw a version of the graph without crossing points.

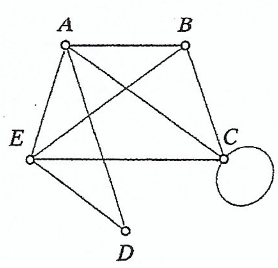


FIGURE 5-30

3. For the graph shown in Fig. 5-31,

(a) give the vertex set.

(b) give the edge list.

(c) give the degree of each vertex.

(d) give the number of components of the graph.

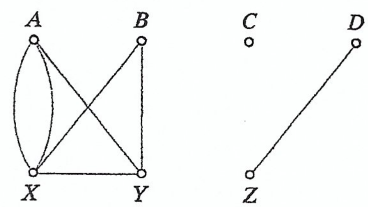


FIGURE 5-31

4. For the graph shown in Fig. 5-32,

(a) give the vertex set.

(b) give the edge list.

(c) give the degree of each vertex.

(d) give the number of components of the graph.

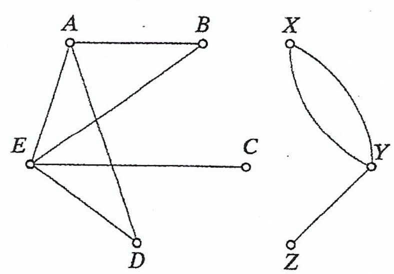


FIGURE 5-32

5. Consider the graph with vertex set $\{K, R, S, T, W\}$ and edge list RS, RT, TT, TS, SW, WW, WS. Draw two different pictures of the graph.

6. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list AC, AE, BD, BE, CA, CD, CE, DE. Draw two different pictures of the graph.

7. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list AD, AE, BC, BD, DD, DE. Without drawing a picture of the graph,

(a) list all the vertices adjacent to D.

(b) list all the edges adjacent to BD.

(c) find the degree of D.

(d) find the sum of the degrees of the vertices.

8. Consider the graph with vertex set $\{A, B, C, X, Y, Z\}$ and edge list $AX, AY, AZ, BB, CX, CY, CZ, YY$. Without drawing a picture of the graph,

(a) list all the vertices adjacent to Y.

(b) list all the edges adjacent to AY.

(c) find the degree of Y.

(d) find the sum of the degrees of the vertices.

9. **(a)** Give an example of a connected graph with six vertices such that each vertex has degree 2.

(b) Give an example of a disconnected graph with six vertices such that each vertex has degree 2.

(c) Give an example of a graph with six vertices such that each vertex has degree 1.

10. **(a)** Give an example of a connected graph with eight vertices such that each vertex has degree 3.

(b) Give an example of a disconnected graph with eight vertices such that each vertex has degree 3.

(c) Give an example of a graph with eight vertices such that each vertex has degree 1.

11. Consider the graph in Fig. 5-33.

(a) Find a path from C to F passing through vertex B but not through vertex D.

(b) Find a path from C to F passing through both vertex B and vertex D.

(c) Find a path of length 4 from C to F.

(d) Find a path of length 7 from C to F.

(e) How many paths are there from C to A?

(f) How many paths are there from H to F?

(g) How many paths are there from C to F?

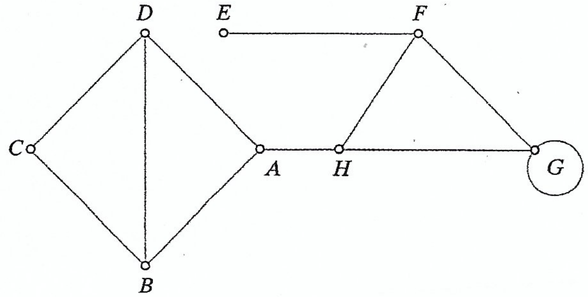


FIGURE 5-33

12. Consider the graph in Fig. 5-33.

(a) Find a path from D to E passing through vertex G only once.

(b) Find a path from D to E passing through vertex G twice.

(c) Find a path of length 4 from D to E.

(d) Find a path of length 8 from D to E.

(e) How many paths are there from D to A?

(f) How many paths are there from H to E?

(g) How many paths are there from D to E?

13. Consider the graph in Fig. 5-33.

(a) Find all circuits of length 1.

(b) Find all circuits of length 2.

(c) Find all circuits of length 3.

(d) Find all circuits of length 4.

(e) What is the total number of circuits in the graph?

14. Consider the graph in Fig. 5-34.

(a) Find all circuits of length 1.

(b) Find all circuits of length 2.

(c) Find all circuits of length 3.

(d) Find all circuits of length 4.

(e) Find all circuits of length 5.

(f) What is the total number of circuits in the graph?

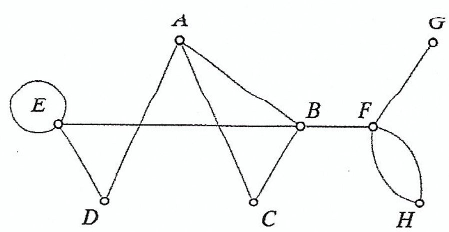


FIGURE 5-34

15. List all the bridges in each of the following graphs:

(a) the graph in Fig. 5-33.

(b) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AE, BC, CD, DE.

(c) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, BE, CD.

16. List all the bridges in each of the following graphs:

(a) the graph in Fig. 5-34.

(b) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AD, AE, BC, CE, DE.

(c) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, CD, DE.

6. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list $AC, AE, BD, BE, CA, CD, CE, DE$. Draw two different pictures of the graph.

7. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list AD, AE, BC, BD, DD, DE. Without drawing a picture of the graph,

(a) list all the vertices adjacent to D.

(b) list all the edges adjacent to BD.

(c) find the degree of D.

(d) find the sum of the degrees of the vertices.

8. Consider the graph with vertex set $\{A, B, C, X, Y, Z\}$ and edge list $AX, AY, AZ, BB, CX, CY, CZ, YY$. Without drawing a picture of the graph,

(a) list all the vertices adjacent to Y.

(b) list all the edges adjacent to AY.

(c) find the degree of Y.

(d) find the sum of the degrees of the vertices.

9. (a) Give an example of a connected graph with six vertices such that each vertex has degree 2.

(b) Give an example of a disconnected graph with six vertices such that each vertex has degree 2.

(c) Give an example of a graph with six vertices such that each vertex has degree 1.

10. (a) Give an example of a connected graph with eight vertices such that each vertex has degree 3.

(b) Give an example of a disconnected graph with eight vertices such that each vertex has degree 3.

(c) Give an example of a graph with eight vertices such that each vertex has degree 1.

11. Consider the graph in Fig. 5-33.

(a) Find a path from C to F passing through vertex B but not through vertex D.

(b) Find a path from C to F passing through both vertex B and vertex D.

(c) Find a path of length 4 from C to F.

(d) Find a path of length 7 from C to F.

(e) How many paths are there from C to A?

(f) How many paths are there from H to F?

(g) How many paths are there from C to F?

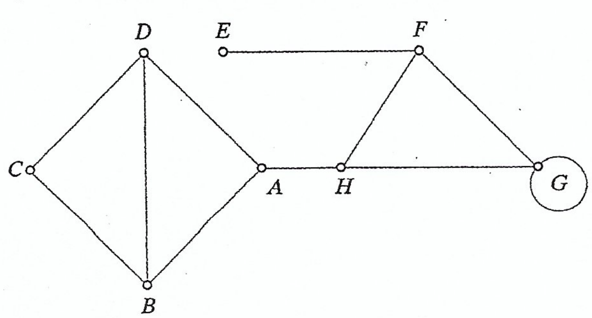


FIGURE 5-33

12. Consider the graph in Fig. 5-33.

(a) Find a path from D to E passing through vertex G only once.

(b) Find a path from D to E passing through vertex G twice.

(c) Find a path of length 4 from D to E.

(d) Find a path of length 8 from D to E.

(e) How many paths are there from D to A?

(f) How many paths are there from H to E?

(g) How many paths are there from D to E?

13. Consider the graph in Fig. 5-33.

(a) Find all circuits of length 1.

(b) Find all circuits of length 2.

(c) Find all circuits of length 3.

(d) Find all circuits of length 4.

(e) What is the total number of circuits in the graph?

14. Consider the graph in Fig. 5-34.

(a) Find all circuits of length 1.

(b) Find all circuits of length 2.

(c) Find all circuits of length 3.

(d) Find all circuits of length 4.

(e) Find all circuits of length 5.

(f) What is the total number of circuits in the graph?

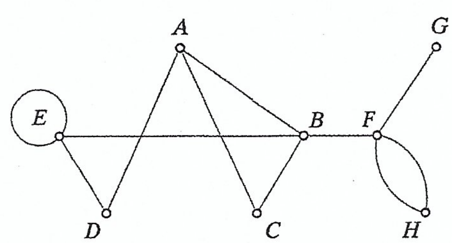


FIGURE 5-34

15. List all the bridges in each of the following graphs:

(a) the graph in Fig. 5-33.

(b) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AE, BC, CD, DE.

(c) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, BE, CD.

16. List all the bridges in each of the following graphs:

(a) the graph in Fig. 5-34.

(b) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AD, AE, BC, CE, DE.

(c) the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, CD, DE.

17. Consider the graph in Fig. 5-35.

(a) Find the largest clique in this graph.

(b) List all the bridges in this graph.

(c) If you remove *all* the bridges from the graph, how many components will the resulting graph have?

(d) Find the *shortest* path (i.e., the path with least length) from C to J. What is the length of the shortest path?

(e) Find a *longest* path from C to J. What is the length of a longest path?

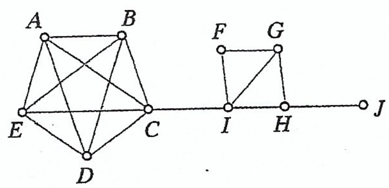


FIGURE 5-35

18. Consider the graph in Fig. 5-36.

(a) Find the largest clique in this graph.

(b) List all the bridges in this graph.

(c) If you remove *all* the bridges from the graph, how many components will the resulting graph have?

(d) Find the *shortest* path (i.e., the path with least length) from E to J. What is the length of the shortest path?

(e) What is the length of a *longest* path from E to J?

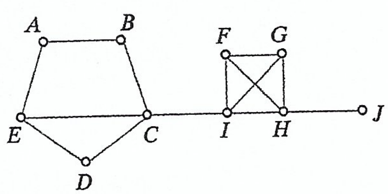


FIGURE 5-36

19. Figure 5-37 shows a map of the downtown area of the picturesque hamlet of Kingsburg, with the Kings River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by seven bridges. You have been hired by the Kingsburg Chamber of Commerce to organize the annual downtown parade. Part of your job is to plan the route for the parade. Draw a graph that models the layout of Kingsburg.

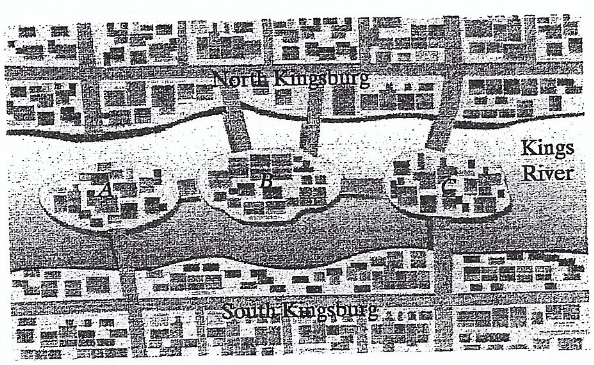


FIGURE 5-37

20. Figure 5-38 is a map of downtown Royalton, showing the Royalton River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by eight bridges. The Downtown Athletic Club wants to design the route for a marathon through the downtown area. Draw a graph that models the layout of Royalton.

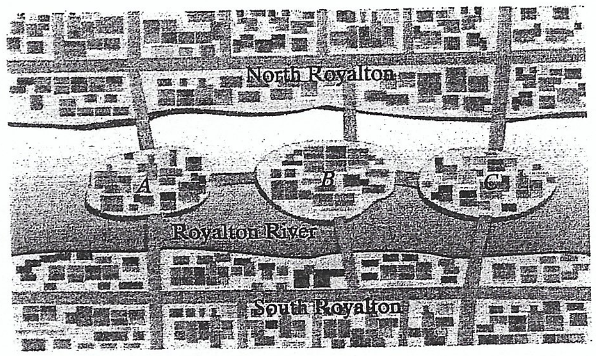


FIGURE 5-38

21. A night watchman must walk the streets of the Green Hills subdivision shown in Fig. 5-39. The night watchman needs to walk only once along each block. Draw a graph that models this street-routing problem.

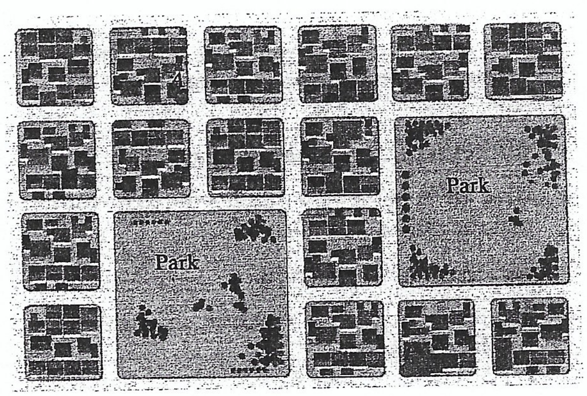


FIGURE 5-39

22. A mail carrier must deliver mail on foot along the streets of the Green Hills subdivision shown in Fig. 5-39. The mail carrier must make two passes on every block that has houses on both sides of the street (once for each side of the street), but only one pass on blocks that have houses on only one side of the street. Draw a graph that models this street-routing problem.

23. Six teams (A, B, C, D, E, and F) are entered in a softball tournament. The top two seeded teams (A and B) have to play only three games; the other teams have to play four games each. The tournament pairings are A plays against C, E, and F; B plays against C, D, and F; C plays against every team except F; D plays against every team except A; E plays against every team except B; and F plays against every team except C. Draw a graph that models the tournament.

24. The Kangaroo Lodge of Madison County has 10 members ($A, B, C, D, E, F, G, H, I,$ and J). The club has five working committees: the Rules Committee ($A, C, D, E, I,$ and J), the Public Relations Committee ($B, C, D, H, I,$ and J), the Guest Speaker Committee ($A, D, E, F,$ and H), the New Year's Eve Party Committee ($D, F, G, H,$ and I), and the Fund Raising Committee ($B, D, F, H,$ and J).

(a) Suppose we are interested in knowing which pairs of members are on the same committee. Draw a graph that models this problem. (*Hint*: Let the vertices of the graph represent the members.)

(b) Suppose we are interested in knowing which committees have members in common. Draw a graph that models this problem. (*Hint*: Let the vertices of the graph represent the committees.)

25. Table 5-3 summarizes the Facebook friendships between a group of eight individuals [an *F* indicates that the individuals (row and column) are Facebook friends]. Draw a graph that models the set of friendships in the group. (Use the first letter of the name to label the vertices.)

	Fred	Pat	Mac	Ben	Tom	Hale	Zac	Cher
Fred		F			F	F		
Pat	F				F	F		F
Mac				F			F	
Ben			F				F	
Tom	F	F				F		
Hale	F	F			F			F
Zac			F	F				
Cher		F				F		

TABLE 5-3

26. The Dean of Students' office wants to know how the seven general education courses selected by incoming freshmen are clustered. For each pair of general education courses, if 30 or more incoming freshmen register for both courses, the courses are defined as being "significantly linked." Table 5-4 shows all the significant links between general education courses (indicated by a 1). Draw a graph that models the significant links between the general education courses. (Use the first letter of each course to label the vertices of the graph.)

	Math	Chemistry	Biology	English	Physics	History	Art
Math		1	1	1	1		
Chemistry	1		1				
Biology	1	1		1		1	
English	1		1		1	1	1
Physics	1			1		1	1
History			1	1	1		1
Art				1	1	1	

TABLE 5-4

27. Figure 5-40 shows the downtown area of the small village of Kenton. The village wants to have a Fourth of July parade that passes through all the blocks of the downtown area, except for the 14 blocks highlighted in yellow, which the police department considers unsafe for the parade route. Draw a graph that models this street-routing problem.

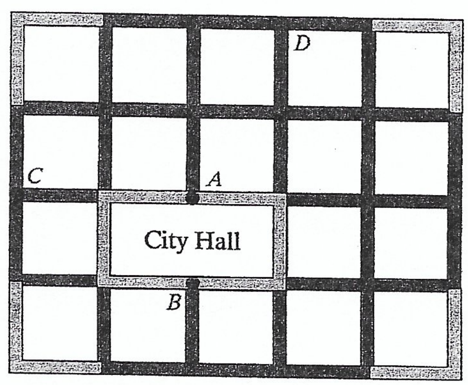


FIGURE 5-40

28. Figure 5-40 shows the downtown area of the small village of Kenton. At regular intervals at night, a police officer must patrol every downtown block at least once, and each of the six blocks along City Hall at least twice. Draw a graph that models this street-routing problem.

5.3 Euler's Theorems and Fleury's Algorithm

In Exercises 29 through 34 choose from one of the following answers and provide a short explanation for your answer.

(A) the graph has an Euler circuit.

(B) the graph has an Euler path.

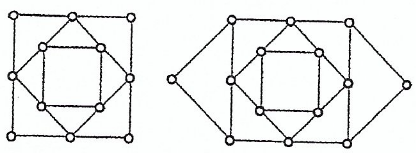
(C) the graph has neither an Euler circuit nor an Euler path.

(D) the graph may or may not have an Euler circuit.

(E) the graph may or may not have an Euler path. You do not have to show an actual path or circuit.

29. **(a)** Fig. 5-41(a) **(b)** Fig. 5-41(b)

(c) A graph with six vertices, all of degree 2

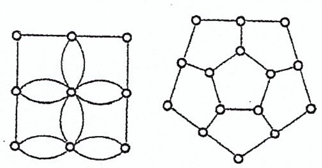


(a) (b)

FIGURE 5-41

30. **(a)** Fig. 5-42(a) **(b)** Fig. 5-42(b)

(c) A graph with eight vertices: six vertices of degree 2 and two vertices of degree 3

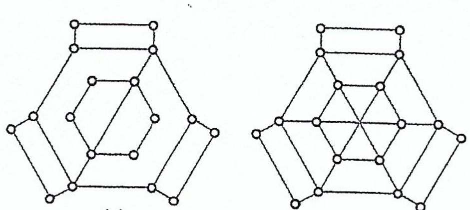


(a) (b)

FIGURE 5-42

31. **(a)** Fig. 5-43(a) **(b)** Fig. 5-43(b)

(c) A disconnected graph with six vertices, all of degree 2

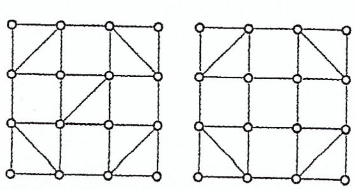


(a) (b)

FIGURE 5-43

32. **(a)** Fig. 5-44(a) **(b)** Fig. 5-44(b)

(c) A disconnected graph with eight vertices: six vertices of degree 2 and two vertices of degree 3

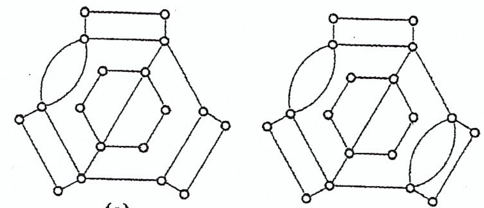


(a) (b)

FIGURE 5-44

33. **(a)** Fig. 5-45(a) **(b)** Fig. 5-45(b)

(c) A graph with six vertices, all of degree 1. [*Hint:* Try Exercise 9(c) first.]

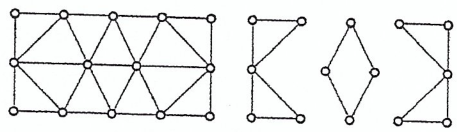


(a) (b)

FIGURE 5-45

34. **(a)** Fig. 5-46(a) **(b)** Fig. 5-46(b)

(c) A graph with eight vertices, all of degree 1.



(a) (b)

FIGURE 5-46

35. Find an Euler circuit for the graph in Fig. 5-47. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

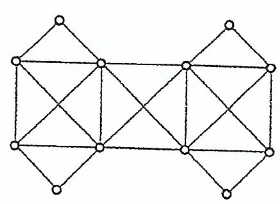


FIGURE 5-47

36. Find an Euler circuit for the graph in Fig. 5-48. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they can be traveled.

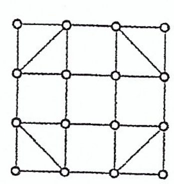


FIGURE 5-48

37. Find an Euler path for the graph in Fig. 5-49. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

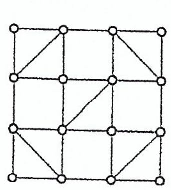


FIGURE 5-49

38. Find an Euler path for the graph in Fig. 5-50. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

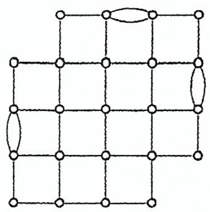


FIGURE 5-50

39. Find an Euler circuit for the graph in Fig. 5-51. Use B as the starting and ending point of the circuit. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

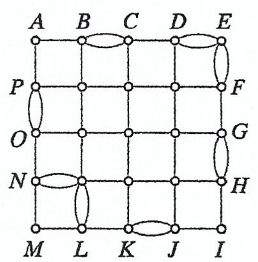


FIGURE 5-51

40. Find an Euler circuit for the graph in Fig. 5-52. Use S as the starting and ending point of the circuit. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

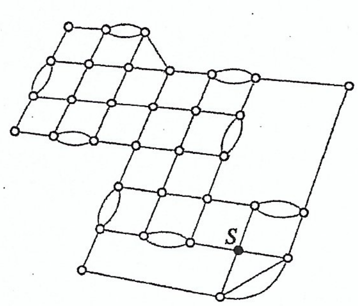


FIGURE 5-52

41. Suppose you are using Fleury's algorithm to find an Euler circuit for a graph and you are in the middle of the process. The graph in Fig. 5-53 shows both the already traveled part of the graph (the red edges) and the yet-to-be traveled part of the graph (the blue edges).

(a) Suppose you are standing at P. What edge(s) could you choose next?

(b) Suppose you are standing at B. What edge should you *not* choose next?

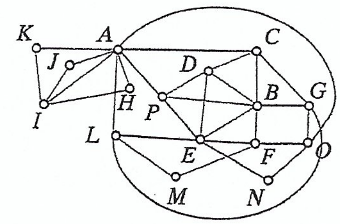


FIGURE 5-53

42. Suppose you are using Fleury's algorithm to find an Euler circuit for a graph and you are in the middle of the process. The graph in Fig. 5-53 shows both the already traveled part of the graph (the red edges) and the yet-to-be traveled part of the graph (the blue edges).

(a) Suppose you are standing at C. What edge(s) could you choose next?

(b) Suppose you are standing at A. What edge should you *not* choose next?

5.4 Eulerizing and Semi-Eulerizating Graphs

43. Find an optimal eulerization for the graph in Fig. 5-54.

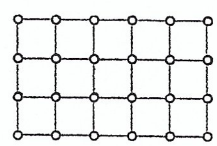


FIGURE 5-54

44. Find an optimal eulerization for the graph in Fig. 5-55.

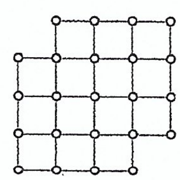


FIGURE 5-55

45. Find an optimal eulerization for the graph in Fig. 5-56.

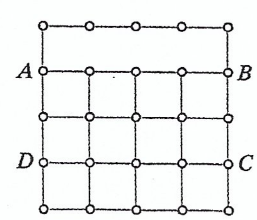


FIGURE 5-56

46. Find an optimal eulerization for the graph in Fig. 5-57.

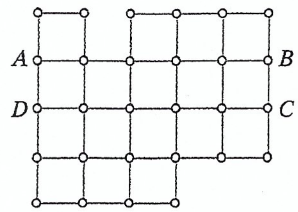


FIGURE 5-57

47. Find an optimal semi-eulerization for the graph in Fig. 5-56. You are free to choose the starting and ending vertices.

48. Find an optimal semi-eulerization for the graph in Fig. 5-57. You are free to choose the starting and ending vertices.

49. Find an optimal semi-eulerization of the graph in Figure 5-56 when A and D are required to be the starting and ending points of the route.

50. Find an optimal semi-eulerization of the graph in Figure 5-57 when A and B are required to be the starting and ending points of the route.

51. Find an optimal semi-eulerization of the graph in Figure 5-56 when B and C are required to be the starting and ending points of the route.

52. Find an optimal semi-eulerization of the graph in Fig. 5-57 when A and D are required to be the starting and ending points of the route.

53. A security guard must patrol on foot the streets of the Green Hills subdivision shown in Fig. 5-39. The security guard wants to start and end his walk at the corner labeled A, and he needs to cover each block of the subdivision at least once. Find an optimal route for the security guard. Describe the route by labeling the edges 1, 2, 3, and so on in the order in which they are traveled. (*Hint*: You should do Exercise 21 first.)

54. A mail carrier must deliver mail on foot along the streets of the Green Hills subdivision shown in Fig. 5-39. His route must start and end at the Post Office, labeled P in the figure. The mail carrier must walk along each block twice if there are houses on both sides of the street and once along blocks where there are houses on only one side of the street. Find an optimal route for the mail carrier. Describe the route by labeling the edges 1, 2, 3, and so on in the order in which they are traveled. (*Hint*: You should do Exercise 22 first.)

55. This exercise refers to the Fourth of July parade problem introduced in Exercise 27. Find an optimal route for the parade that starts at A and ends at B (see Fig. 5-40). Describe the route by labeling the edges 1, 2, 3, . . . etc. in the order they are traveled. [*Hint*: Start with the graph model for the parade route (see Exercise 27); then find an optimal semi-eulerization of the graph that leaves A and B odd; then find an Euler path in this new graph.]

56. This exercise refers to the Fourth of July parade problem introduced in Exercise 27. Find an optimal route for the parade that starts at C and ends at D (see Fig. 5-40). Describe the route by labeling the edges 1, 2, 3, . . . etc. in the order they are traveled. [*Hint*: Start with the graph model for the parade route (see Exercise 27); then find an optimal semi-eulerization of the graph that leaves C and D odd; then find an Euler path in this new graph.]

JOGGING

57. Assume you want to trace the diagram of a basketball court shown in Fig. 5-58 without retracing any lines. How many times would you have to lift your pencil to do it? Explain.

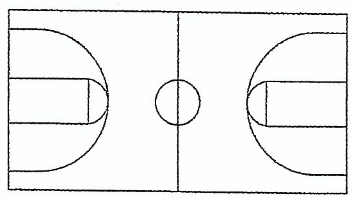


FIGURE 5-58

58. **(a)** Explain why in every graph the sum of the degrees of all the vertices equals twice the number of edges.

    (b) Explain why every graph must have an even number of odd vertices.

59. If G is a connected graph with no bridges, how many vertices of degree 1 can G have? Explain your answer.

60. **Regular graphs.** A graph is called *regular* if every vertex has the same degree. Let G be a connected regular graph with N vertices.

    (a) Explain why if N is odd, then G must have an Euler circuit.

    (b) When N is even, then G may or may not have an Euler circuit. Give examples of both situations.

61. **Complete bipartite graphs.** A complete bipartite graph is a graph having the property that the vertices of the graph can be divided into two groups A and B and each vertex in A is adjacent to each vertex in B, as shown in Fig. 5-59. Two vertices in A are never adjacent, and neither are two vertices in B. Let m and n denote the number of vertices in A and B, respectively, and assume $m \leq n$.

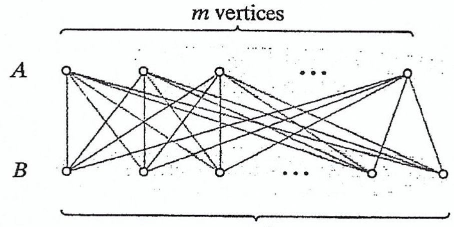


FIGURE 5-59

(a) Describe all the possible values of m and n for which the complete bipartite graph has an Euler circuit. (*Hint*: There are infinitely many values of m and n.)

(b) Describe all the possible values of m and n for which the complete bipartite graph has an Euler path.

62. Consider the following game. You are given N vertices and are required to build a graph by adding edges connecting these vertices. Each time you add an edge you must pay $1. You can stop when the graph is connected.

(a) Describe the strategy that will cost you the least amount of money.

(b) What is the minimum amount of money needed to build the graph? (Give your answer in terms of N.)

63. Consider the following game. You are given N vertices and allowed to build a graph by adding edges connecting these vertices. For each edge you can add, you make $1. You are not allowed to add loops or multiple edges, and you must stop before the graph is connected (i.e., the graph you end up with must be disconnected).

(a) Describe the strategy that will give you the greatest amount of money.

(b) What is the maximum amount of money you can make building the graph? (Give your answer in terms of N.)

64. Figure 5-60 shows a map of the downtown area of the picturesque hamlet of Kingsburg. You have been hired by the Kingsburg Chamber of Commerce to organize the annual downtown parade. Part of your job is to plan the route for the parade. An *optimal* parade route is one that keeps the bridge crossings to a minimum and yet crosses each of the seven bridges in the downtown area at least once.

(a) Find an optimal parade route if the parade is supposed to start in North Kingsburg but can end anywhere.

(b) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end in South Kingsburg.

(c) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end on island B.

(d) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end on island A.

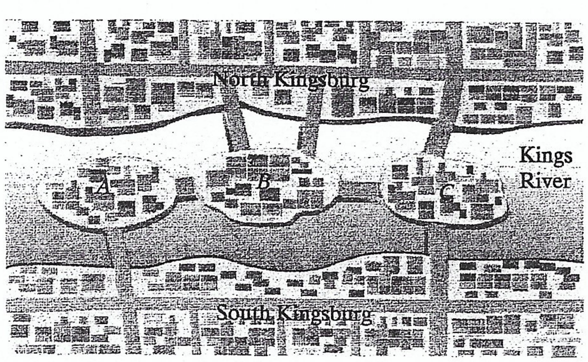


FIGURE 5-60

65. A policeman has to patrol on foot the streets of the subdivision shown in Fig. 5-61. The policeman needs to start his route at the police station, located at X, and end the route at the local coffee shop, located at Y. He needs to cover each block of the subdivision at least once, but he wants to make his route as efficient as possible and duplicate the fewest possible number of blocks.

(a) How many blocks will he have to duplicate in an optimal trip through the subdivision?

(b) Describe an optimal trip through the subdivision. Label the edges 1, 2, 3, and so on in the order the policeman would travel them.

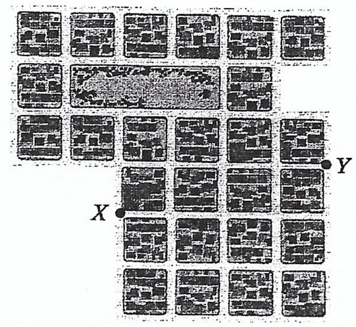


FIGURE 5-61

Exercises 66 through 68 refer to Example 5.23. In this example, the problem is to find an optimal route (i.e., a route with the fewest bridge crossings) for a photographer who needs to cross each of the 11 bridges of Madison County for a photo shoot. The layout of the 11 bridges is shown in Fig. 5-62. You may find it helpful to review Example 5.23 before trying these two exercises.

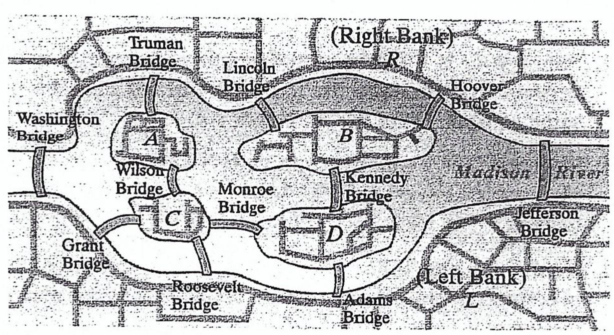


FIGURE 5-62

66. Describe an optimal route for the photographer if the route must start at B and end at L.

67. Describe an optimal route for the photographer if the route must start and end in D and the first bridge crossed must be the Adams Bridge.

68. Describe an optimal route for the photographer if the route must start and end in the same place, the first bridge crossed must be the Adams bridge, and the last bridge crossed must be the Grant Bridge.

69. This exercise comes to you courtesy of Euler himself. Here is the question in Euler's own words, accompanied by the diagram shown in Fig. 5-63.

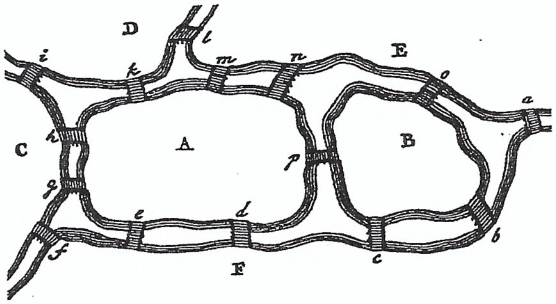


FIGURE 5-63

> *Let us take an example of two islands with four rivers forming the surrounding water. There are fifteen bridges marked a, b, c, d, etc., across the water around the islands and the adjoining rivers. The question is whether a journey can be arranged that will pass over all the bridges but not over any of them more than once.*

What is the answer to Euler's question? If the "journey" is possible, describe it. If it isn't, explain why not.

RUNNING

70. Suppose G is a connected graph with N vertices, all of even degree. Let k denote the number of bridges in G. Find the value(s) of k. Explain your answer.

71. Suppose G is a connected graph with $N-2$ even vertices and two odd vertices. Let k denote the number of bridges in G. Find all the possible values of k. Explain your answer.

72. Suppose G is a disconnected graph with exactly two odd vertices. Explain why the two odd vertices must be in the same component of the graph.

73. Suppose G is a simple graph with N vertices ($N \geq 2$). Explain why G must have at least two vertices of the same degree.

74. **Kissing circuits.** When two circuits in a graph have no edges in common but share a common vertex v, they are said to be *kissing at v*.

(a) For the graph shown in Fig. 5-64, find a circuit kissing the circuit A, D, C, A (there is only one), and find two different circuits kissing the circuit A, B, D, A.

(b) Suppose G is a connected graph and every vertex in G is even. Explain why the following statement is true: *If a circuit in G has no kissing circuits, then that circuit must be an Euler circuit.*

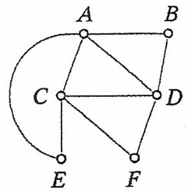


FIGURE 5-64

75. **Hierholzer's algorithm.** *Hierholzer's algorithm* is another algorithm for finding an Euler circuit in a graph. The basic idea behind Hierholzer's algorithm is to start with an arbitrary circuit and then enlarge it by patching to it a *kissing circuit*, continuing this way and making larger and larger circuits until the circuit cannot be enlarged any farther. (For the definition of kissing circuits, see Exercise 74.) More formally, Hierholzer's algorithm is as follows:

Step 1. Start with an arbitrary circuit C_0.

Step 2. Find a kissing circuit to C_0. If there are no kissing circuits to C_0, then you are finished—C_0 is itself an Euler circuit of the graph [see Exercise 74(b)]. If there is a kissing circuit to C_0, let's call it K_0, and let V denote the vertex at which the two circuits kiss. Go to Step 3.

Step 3. Let C_1 denote the circuit obtained by "patching" K_0 to C_0 at vertex V (i.e., start at V, travel along C_0 back to V, and then travel along K_0 back again to V). Now find a kissing circuit to C_1. (If there are no kissing circuits to C_1, then you are finished—C_1 is your Euler circuit.) If there is a kissing circuit to C_1, let's call it K_1, and let W denote the vertex at which the two circuits kiss. Go to Step 4.

Steps 4, 5, and so on. Continue this way until there are no more kissing circuits available.

(a) Use Hierholzer's algorithm to find an Euler circuit for the graph shown in Fig. 5-65 (this is the graph model for the mail carrier in Example 5.14).

(b) Describe a modification of Hierholzer's algorithm that allows you to find an Euler path in a connected graph having exactly two vertices of odd degree. (*Hint*: A path can also have a kissing circuit.)

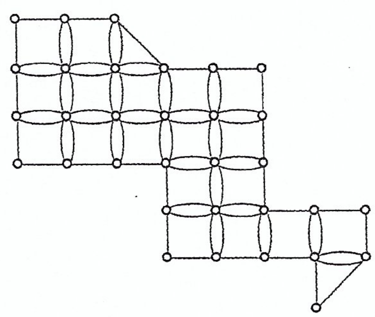


FIGURE 5-65