3.3 The Conditional

"If you build it, he will come."
—The Voice in the 1990 movie *Field of Dreams*

Ray Kinsella, an Iowa farmer in the movie *Field of Dreams,* heard a voice from the sky. Ray interpreted it as a promise that if he would build a baseball field in his cornfield, then the ghost of Shoeless Joe Jackson (a baseball star in the early days of the twentieth century) would come to play on it. The promise came in the form of a conditional statement. A **conditional** statement is a compound statement that uses the connective *if . . . then.* For example, here are a few conditional statements.

If I read for too long, *then* I get a headache.

If looks could kill, *then* I would be dead.

If he doesn't get back soon, *then* you should go look for him.

In each of these conditional statements, the component coming after the word *if* gives a condition (but not necessarily the only condition) under which the statement coming after *then* will be true. For example. "If it is over 90°, then I'll go to the mountains" tells one possible condition under which I will go to the mountains—if the temperature is over 90°.

The conditional is written with an arrow, so that "if p, then q" is symbolized as

$$p \rightarrow q.$$

We read $p \rightarrow q$ as "p implies q" or "if p, then q." In the conditional $p \rightarrow q$, the statement p is the **antecedent,** while q is the **consequent.**

The conditional connective may not always be explicitly stated. That is, it may be "hidden" in an everyday expression. For example, the statement

Big girls don't cry

can be written in *if . . . then* form as

If you're a big girl, then you don't cry.

As another example, the statement

It is difficult to study when you are distracted

can be written

If you are distracted, then it is difficult to study.

As seen in the quote from the movie *Field of Dreams* earlier, the word "then" is sometimes not stated but understood to be there from the context of the statement. In that statement, "you build it" is the antecedent and "he will come" is the consequent.

The conditional truth table is a little harder to define than were the tables in the previous section. To see how to define the conditional truth table, let us analyze a statement made by a politician, Senator Julie Davis:

If I am elected, then taxes will go down.

As before, there are four possible combinations of truth values for the two component statements. Let p represent "I am elected," and let q represent "Taxes will go down."

As we analyze the four possibilities, it is helpful to think in terms of the following: "Did Senator Davis lie?" If she lied, then the conditional statement is considered false; if she did not lie, then the conditional statement is considered true.

Possibility	Elected?	Taxes Go Down?	
1	Yes	Yes	p is T, q is T
2	Yes	No	p is T, q is F
3	No	Yes	p is F, q is T
4	No	No	p is F, q is F

The four possibilities are as follows:

1. In the first case assume that the senator was elected and taxes did go down (p is T, q is T). The senator told the truth, so place T in the first row of the truth table. (We do not claim that taxes went down *because* she was elected; it is possible that she had nothing to do with it at all.)
2. In the second case assume that the senator was elected and taxes did not go down (p is T, q is F). Then the senator did not tell the truth (that is, she lied). So we put F in the second row of the truth table.
3. In the third case assume that the senator was defeated, but taxes went down anyway (p is F, q is T). Senator Davis did not lie; she only promised a tax reduction if she were elected. She said nothing about what would happen if she were not elected. In fact, her campaign promise gives no information about what would happen if she lost. Since we cannot say that the senator lied, place T in the third row of the truth table.
4. In the last case assume that the senator was defeated but taxes did not go down (p is F, q is F). We cannot blame her, since she only promised to reduce taxes if elected. Thus, T goes in the last row of the truth table.

The completed truth table for the conditional is defined as follows.

Truth Table for the Conditional If *p*, then *q*

If p, then q.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

It must be emphasized that the use of the conditional connective in no way implies a cause-and-effect relationship. Any two statements may have an arrow placed between them to create a compound statement. For example,

If I pass mathematics, then the sun will rise the next day

is true, since the consequent is true. (See the box after Example 1.) There is, however, no cause-and-effect connection between my passing mathematics and the sun's rising. The sun will rise no matter what grade I get in a course.

EXAMPLE 3 Construct a truth table for each statement.

(a) $(\sim p \to \sim q) \to (\sim p \wedge q)$

First insert the truth values of $\sim p$ and of $\sim q$. Then find the truth value of $\sim p \to \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Next use $\sim p$ and q to find the truth values of $\sim p \wedge q$.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim p \wedge q$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	F

Now use the conditional truth table to find the truth values of $(\sim p \to \sim q) \to (\sim p \wedge q)$.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim p \wedge q$	$(\sim p \to \sim q) \to (\sim p \wedge q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

(b) $(p \to q) \to (\sim p \vee q)$

Go through steps similar to the ones above.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \to (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As the truth table in Example 3(b) shows, the statement $(p \to q) \to (\sim p \vee q)$ is always true, no matter what the truth values of the components. Such a statement is called a **tautology**. Other examples of tautologies (as can be checked by forming truth tables) include $p \vee \sim p$, $p \to p$, $(\sim p \vee \sim q) \to \sim(q \wedge p)$, and so on. By the way, the truth tables in Example 3 also could have been found by the alternate method shown in the previous section.

3.3 EXERCISES

In Exercises 1–8, decide whether each statement is true *or* false.

1. If the antecedent of a conditional statement is false, the conditional statement is true.
2. If the consequent of a conditional statement is true, the conditional statement is true.
3. If q is true, then $(p \wedge q) \rightarrow q$ is true.
4. If p is true, then $\sim p \rightarrow (q \vee r)$ is true.
5. The negation of "If pigs fly, I'll believe it" is "If pigs don't fly, I won't believe it."
6. The statements "If it flies, then it's a bird" and "It does not fly or it's a bird" are logically equivalent.
7. Given that $\sim p$ is true and q is false, the conditional $p \rightarrow q$ is true.
8. Given that $\sim p$ is false and q is false, the conditional $p \rightarrow q$ is true.
9. In a few sentences, explain how we determine the truth value of a conditional statement.
10. Explain why the statement "If $3 = 5$, then $4 = 6$" is true.

Rewrite each statement using the if . . . then *connective. Rearrange the wording or add words as necessary.*

11. You can believe it if it's in *USA Today*.
12. It must be dead if it doesn't move.
13. Kathi Callahan's area code is 708.
14. Kara Gourley goes to Hawaii every summer.
15. All soldiers maintain their weapons.
16. Every dog has its day.
17. No koalas live in Mississippi.
18. No guinea pigs are scholars.
19. An alligator cannot live in these waters.
20. Romeo loves Juliet.

Let s represent "I study in the library," *let p represent the statement* "I pass my psychology course," *and let m represent* "I major in mathematics." *Express each compound statement in words.*

27. $\sim m \rightarrow p$
28. $p \rightarrow \sim m$
29. $s \rightarrow (m \wedge p)$
30. $(s \wedge p) \rightarrow m$
31. $\sim p \rightarrow (\sim m \vee s)$
32. $(\sim s \vee \sim m) \rightarrow \sim p$

Let d represent "I drive my car," *let s represent* "it snows," *and let c represent* "classes are cancelled." *Write each compound statement in symbols.*

33. If it snows, then I drive my car.
34. If I drive my car, then classes are cancelled.
35. If I do not drive my car, then it does not snow.
36. If classes are cancelled, then it does not snow.
37. I drive my car, or if classes are cancelled then it snows.
38. Classes are cancelled, and if it snows then I do not drive my car.
39. I'll drive my car if it doesn't snow.
40. It snows if classes are cancelled.

Find the truth value of each statement. Assume that p and r are false, and q is true.

41. $\sim r \rightarrow q$
42. $\sim p \rightarrow \sim r$
43. $q \rightarrow p$
44. $\sim r \rightarrow p$
45. $p \rightarrow q$
46. $\sim q \rightarrow r$
47. $\sim p \rightarrow (q \wedge r)$
48. $(\sim r \vee p) \rightarrow p$
49. $\sim q \rightarrow (p \wedge r)$
50. $(\sim p \wedge \sim q) \rightarrow (p \wedge \sim r)$
51. $(p \rightarrow \sim q) \rightarrow (\sim p \wedge \sim r)$
52. $(p \rightarrow \sim q) \wedge (p \rightarrow r)$

Construct a truth table for each statement. Identify any tautologies.

55. $\sim q \rightarrow p$
56. $p \rightarrow \sim q$
57. $(\sim p \rightarrow q) \rightarrow p$
58. $(\sim q \rightarrow \sim p) \rightarrow \sim q$
59. $(p \vee q) \rightarrow (q \vee p)$
60. $(p \wedge q) \rightarrow (p \vee q)$
61. $(\sim p \rightarrow \sim q) \rightarrow (p \wedge q)$
62. $r \rightarrow (p \wedge \sim q)$
63. $[(r \vee p) \wedge \sim q] \rightarrow p$
64. $(\sim r \rightarrow s) \vee (p \rightarrow \sim q)$
65. $(\sim p \wedge \sim q) \rightarrow (\sim r \rightarrow \sim s)$

Negation of $p \to q$

Suppose that someone makes the conditional statement

"If it rains, then I take my umbrella."

When will the person have lied to you? The only case in which you would have been misled is when it rains *and* the person does *not* take the umbrella. Letting p represent "it rains" and q represent "I take my umbrella," you might suspect that the symbolic statement

$$p \wedge \sim q$$

is a candidate for the negation of $p \to q$. That is,

$$\sim(p \to q) \equiv p \wedge \sim q.$$

It happens that this is indeed the case, as the next truth table indicates.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

$\uparrow$ ——— $\equiv$ ——— $\uparrow$

Negation of $p \to q$

The negation of $p \to q$ is $p \wedge \sim q$.

Since

$$\sim(p \to q) \equiv p \wedge \sim q,$$

by negating each expression we have

$$\sim[\sim(p \to q] \equiv \sim(p \wedge \sim q).$$

The left side of the above equivalence is $p \to q$, and one of De Morgan's laws can be applied to the right side.

$$p \to q \equiv \sim p \vee \sim(\sim q)$$

$$p \to q \equiv \sim p \vee q.$$

This final row indicates that a conditional may be written as a disjunction.

Write the negation of each statement. Remember that the negation of $p \to q$ is $p \wedge \sim q$.

67. If you give your plants tender, loving care, they flourish.
68. If the check is in the mail, I'll be surprised.
69. If she doesn't, he will.
70. If I say yes, she says no.
71. All residents of Boise are residents of Idaho.
72. All men were once boys.

Writing a Conditional as an "or" Statement

$p \rightarrow q$ is equivalent to $\sim p \vee q$.

EXAMPLE 4 Write the negation of each statement.

(a) If you build it, he will come.

If b represents "you build it" and q represents "he will come," then the given statement can be symbolized by $b \rightarrow q$. The negation of $b \rightarrow q$, as shown above, is $b \wedge \sim q$, so the negation of the statement is

You build it and he will not come.

(b) All dogs have fleas.

First, we must restate the given statement in *if . . . then* form:

If it is a dog, then it has fleas.

Based on our earlier discussion, the negation is

It is a dog and it does not have fleas.

A common error occurs when students try to write the negation of a conditional statement as another conditional statement. As seen in Example 4, the negation of a conditional statement is written as a conjunction.

EXAMPLE 5 Write each conditional as an equivalent statement without using *if . . . then.*

(a) If the Cubs win the pennant, then Gwen will be happy.

Since the conditional $p \rightarrow q$ is equivalent to $\sim p \vee q$, let p represent "The Cubs win the pennant" and q represent "Gwen will be happy." Restate the conditional as

The Cubs do not win the pennant or Gwen will be happy.

(b) If it's Borden's, it's got to be good.

If p represents "it's Borden's" and if q represents "it's got to be good," the conditional may be restated as

It's not Borden's or it's got to be good.

Write each statement as an equivalent statement that does not use the if . . . then *connective. Remember that $p \rightarrow q$ is equivalent to $\sim p \vee q$.*

73. If you give your plants tender, loving care, they flourish.

74. If the check is in the mail, I'll be surprised.

75. If she doesn't, he will.

76. If I say yes, she says no.

77. All residents of Boise are residents of Idaho.

78. All men were once boys.

Use truth tables to decide which of the pairs of statements are equivalent.

79. $p \rightarrow q$; $\sim p \vee q$

80. $\sim(p \rightarrow q)$; $p \wedge \sim q$

81. $p \rightarrow q$; $\sim q \rightarrow \sim p$

82. $q \rightarrow p$; $\sim p \rightarrow \sim q$

83. $\sim(\sim p)$; p

84. $p \rightarrow q$; $q \rightarrow p$

85. $p \wedge \sim q$; $\sim q \rightarrow \sim p$

86. $\sim p \wedge q$; $\sim p \rightarrow q$

3.4 More on the Conditional

The conditional statement, introduced in the previous section, is one of the most important of all compound statements. Many mathematical properties and theorems are stated in *if . . . then* form. Because of their usefulness, we need to study conditional statements that are related to a statement of the form $p \rightarrow q$.

Converse, Inverse, and Contrapositive

Any conditional statement is made up of an antecedent and a consequent. If they are interchanged, negated, or both, a new conditional statement is formed. Suppose that we begin with the direct statement

If you stay, then I go,

and interchange the antecedent ("you stay") and the consequent ("I go"). We obtain the new conditional statement

If I go, then you stay.

This new conditional is called the **converse** of the given statement.

By negating both the antecedent and the consequent, we obtain the **inverse** of the given statement:

If you do not stay, then I do not go.

If the antecedent and the consequent are both interchanged *and* negated, the **contrapositive** of the given statement is formed:

If I do not go, then you do not stay.

These three related statements for the conditional $p \rightarrow q$ are summarized below. (Notice that the inverse is the contrapositive of the converse.)

Related Conditional Statements

Direct Statement	$p \rightarrow q$	(If p, then q.)
Converse	$q \rightarrow p$	(If q, then p.)
Inverse	$\sim p \rightarrow \sim q$	(If not p, then not q.)
Contrapositive	$\sim q \rightarrow \sim p$	(If not q, then not p.)

Equivalent (Direct and Contrapositive)

Equivalent (Converse and Inverse)

		Direct	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

The direct statement and the contrapositive are equivalent, and the converse and the inverse are equivalent.

EXAMPLE 1 Given the direct statement

If I live in Tampa, then I live in Florida,

write each of the following.

(a) the converse

Let p represent "I live in Tampa" and q represent "I live in Florida." Then the direct statement may be written $p \rightarrow q$. The converse, $q \rightarrow p$, is

If I live in Florida, then I live in Tampa.

Notice that for this statement, the converse is not necessarily true, even though the direct statement is.

(b) the inverse

The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$. For the given statement, the inverse is

If I don't live in Tampa, then I don't live in Florida.

which is again not necessarily true.

(c) the contrapositive

The contrapositive, $\sim q \rightarrow \sim p$, is

If I don't live in Florida, then I don't live in Tampa.

The contrapositive, like the direct statement, is true.

3.4 EXERCISES

For each given direct statement, write (a) *the converse,* (b) *the inverse, and* (c) *the contrapositive in* if . . . then *form. In Exercises 3–10, it may be helpful to restate the direct statement in* if . . . then *form.*

1. If you lead, then I will follow.
2. If beauty were a minute, then you would be an hour.
3. If I had a nickel for each time that happened, I would be rich.
4. If it ain't broke, don't fix it.
5. Milk contains calcium.
6. Walking in front of a moving car is dangerous to your health.
7. A rolling stone gathers no moss.
8. Birds of a feather flock together.
9. Where there's smoke, there's fire.
10. If you build it, he will come.

Alternate Forms of "if p, then q"

The conditional statement "if p, then q" can be stated in several other ways in English. For example,

If you go to the shopping center, then you will find a place to park

can also be written

Going to the shopping center is *sufficient* for finding a place to park.

According to this statement, going to the shopping center is enough to guarantee finding a place to park. Going to other places, such as schools or office buildings, *might* also guarantee a place to park, but at least we *know* that going to the shopping center does. Thus, $p \rightarrow q$ can be written "p is sufficient for q." Knowing that p has occurred is sufficient to guarantee that q will also occur. On the other hand,

Turning on the set is necessary for watching television (*)

has a different meaning. Here, we are saying that one condition that is necessary for watching television is that you turn on the set. This may not be enough; the set might be broken, for example. The statement labeled (*) could be written as

If you watch television, then you turned on the set.

As this example suggests, $p \rightarrow q$ is the same as "q is necessary for p." In other words, if q doesn't happen, then neither will p. Notice how this idea is closely related to the idea of equivalence between the direct statement and its contrapositive.

Some common translations of $p \rightarrow q$ are summarized in the following box.

Common Translations of $p \rightarrow q$

The conditional $p \rightarrow q$ can be translated in any of the following ways.

If p, then q.	p is sufficient for q.
If p, q.	q is necessary for p.
p implies q.	All p's are q's.
p only if q.	q if p.

The translation of $p \rightarrow q$ into these various word forms does not in any way depend on the truth or falsity of $p \rightarrow q$.

EXAMPLE 3 The statement

If you are 18, then you can vote

can be written in any of the following ways.

You can vote if you are 18.

You are 18 only if you can vote.

Being able to vote is necessary for you to be 18.

Being 18 is sufficient for being able to vote.

All 18-year-olds can vote.

Being 18 implies that you can vote.

Biconditionals

The compound statement p *if and only if* q (often abbreviated p *iff* q) is called a **biconditional.** It is symbolized $p \leftrightarrow q$, and is interpreted as the conjunction of the two conditionals $p \rightarrow q$ and $q \rightarrow p$. Using symbols, this conjunction is written

$$(q \rightarrow p) \wedge (p \rightarrow q)$$

so that, by definition,

$$p \leftrightarrow q \equiv (q \rightarrow p) \wedge (p \rightarrow q).$$

Using this definition, the truth table for the biconditional $p \leftrightarrow q$ can be determined.

Truth Table for the Biconditional *p* if and only if *q*

p if and only if q.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 6 Tell whether each biconditional statement is *true* or *false*.

(a) $6 + 9 = 15$ if and only if $12 + 4 = 16$

Both $6 + 9 = 15$ and $12 + 4 = 16$ are true. By the truth table for the biconditional, this biconditional is true.

(b) $5 + 2 = 10$ if and only if $17 + 19 = 36$

Since the first component $(5 + 2 = 10)$ is false, and the second is true, the entire biconditional statement is false.

(c) $6 = 5$ if and only if $12 \neq 12$

Both component statements are false, so by the last line of the truth table for the biconditional, the entire statement is true. (Understanding this might take some extra thought!)

In this section and in the previous two sections, truth tables have been derived for several important types of compound statements. The summary that follows describes how these truth tables may be remembered.

Summary of Basic Truth Tables

1. $\sim p$, the **negation** of p, has truth value opposite that of p.
2. $p \wedge q$, the **conjunction,** is true only when both p and q are true.
3. $p \vee q$, the **disjunction,** is false only when both p and q are false.
4. $p \rightarrow q$, the **conditional,** is false only when p is true and q is false.
5. $p \leftrightarrow q$, the **biconditional,** is true only when p and q have the same truth value.

EXAMPLE 4 Write each statement in the form "if p, then q."

(a) You'll be sorry if I go.
If I go, then you'll be sorry.

(b) Today is Friday only if yesterday was Thursday.
If today is Friday, then yesterday was Thursday.

(c) All nurses wear white shoes.
If you are a nurse, then you wear white shoes.

Write each of the following statements in the form "if p, then q."

19. If I finish studying, I'll go to the party.
20. If it is muddy, I'll wear my galoshes.
21. "Today is Wednesday" implies that yesterday was Tuesday.
22. "17 is positive" implies that 17 + 1 is positive.

23. All whole numbers are integers.
24. All integers are rational numbers.
25. Being in Fort Lauderdale is sufficient for being in Florida.
26. Doing crossword puzzles is sufficient for driving me crazy.
27. Being an environmentalist is necessary for being elected.
28. A day's growth of beard is necessary for Greg Odjakjian to shave.
29. The principal will hire more teachers only if the school board approves.
30. I can go from Park Place to Baltic Avenue only if I pass GO.
31. No integers are irrational numbers.
32. No whole numbers are not integers.
33. Newt will be a liberal when pigs fly.
34. The Indians will win the pennant when their pitching improves.

Two statements that can both be true about the same object are **consistent.** *For example,* "It is green" *and* "It is small" *are consistent statements. Statements that cannot both be true about the same object are called* **contrary;** "It is a Ford" *and* "It is a Chevrolet" *are contrary.*

Label the following pairs of statements as either contrary *or* consistent.

51. Elvis is alive. Elvis is dead.
52. Bill Clinton is a Democrat. Bill Clinton is a Republican.
53. That animal has four legs. That animal is a dog.
54. That book is nonfiction. That book costs more than $40.
55. This number is an integer. This number is irrational.
56. This number is positive. This number is a natural number.
57. Make up two statements that are consistent.
58. Make up two statements that are contrary.