Logic

3.1 Statements and Quantifiers

This section introduces the study of *symbolic logic,* which uses letters to represent statements, and symbols for words such as *and, or, not.* One of the main applications of logic is in the study of the *truth value* (that is, the truth or falsity) of statements with many parts. The truth value of these statements depends on the components that comprise them.

Many kinds of sentences occur in ordinary language, including factual statements, opinions, commands, and questions. Symbolic logic discusses only the first type of sentence, the kind that involves facts.

A **statement** is defined as a declarative sentence that is either true or false, but not both simultaneously. For example, both of the following are statements:

Radio provides a means of communication.

$2 + 1 = 6$

Each one is either true or false. However, the following sentences are not statements based on this definition:

Paint the wall.

How do you spell relief?

Shaquille O'Neal is a better basketball player than Anfernee Hardaway.

This sentence is false.

These sentences cannot be identified as being either true or false. The first sentence is a command, and the second is a question. The third is an opinion. "This sentence is false" is a paradox; if we assume it is true, then it is false, and if we assume it is false, then it is true.

A **compound statement** may be formed by combining two or more statements. The statements making up a compound statement are called **component statements.** Various **logical connectives,** or simply **connectives,** can be used in forming compound statements. Words like *and, or, not,* and *if . . . then* are examples of connectives. (While a statement such as "Today is not Tuesday" does not consist of two component statements, for convenience it is considered compound, since its truth value is determined by noting the truth value of a different statement, "Today is Tuesday.")

EXAMPLE 1 Decide whether or not each statement is compound.

(a) Trey Yuen restaurant serves Peking duck and Pat O'Brien's serves drinks called Hurricanes.

This statement is compound, since it is made up of the component statements "Trey Yuen restaurant serves Peking duck" and "Pat O'Brien's serves drinks called Hurricanes." The connective is *and.*

(b) You can pay me now or you can pay me later.

The connective here is *or.* The statement is compound.

(c) If he said it, then it must be true.

The connective here is *if . . . then,* discussed in more detail in a later section. The statement is compound.

Negations

The sentence "Tom Jones has a red car" is a statement; the **negation** of this statement is "Tom Jones does not have a red car." The negation of a true statement is false, and the negation of a false statement is true.

EXAMPLE 2 Form the negation of each statement.

(a) That state has a governor.
To negate this statement, we introduce *not* into the sentence: "That state does not have a governor."

(b) The sun is not a star.
Negation: "The sun is a star."

Symbols

To simplify work with logic, symbols are used. Statements are represented with letters, such as p, q, or r, while several symbols for connectives are shown in the following table. The table also gives the type of compound statement having the given connective.

Connective	Symbol	Type of Statement
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
not	$\sim$	Negation

The symbol $\sim$ represents the connective *not*. If p represents the statement "Millard Fillmore was president in 1850" then $\sim p$ represents "Millard Fillmore was not president in 1850."

EXAMPLE 4 Let p represent "It is 80° today," and let q represent "It is Tuesday." Write each symbolic statement in words.

(a) $p \vee q$
From the table, $\vee$ symbolizes *or;* thus, $p \vee q$ represents

It is 80° today or it is Tuesday.

(b) $\sim p \wedge q$

It is not 80° today and it is Tuesday.

(c) $\sim(p \vee q)$

It is not the case that it is 80° today or it is Tuesday.

(d) $\sim(p \wedge q)$

It is not the case that it is 80° today and it is Tuesday.

The statement in part (c) of Example 4 is usually translated in English as "Neither p nor q."

3.1 EXERCISES

Decide whether or not each of the following is a statement.

1. A ZIP code for New Orleans is 70115.
2. October 12, 1949 was a Wednesday.
3. Have a nice day.
4. Stand up and be counted.
5. $8 + 15 = 23$
6. $9 - 4 = 5$ and $2 + 1 = 5$
7. Chester A. Arthur was president in 1882.
8. Not all numbers are positive.
9. Dancing is enjoyable.
10. Since 1950, more people have died in automobile accidents than of cancer.
11. Toyotas are better cars than Dodges.

Decide whether each of the following statements is compound.

15. My sister got married in Paris.
16. I read novels and I read newspapers.
17. Denise Clark is younger than 40 years of age, and so is Rodger Klas.
18. Yesterday was Friday.
19. $4 + 2 \neq 8$
20. $5 \neq 4 + 2$
21. If Buddy is a politician, then Eddie is a crook.
22. If Earl Karn sells his quota, then Kerry Freeman will be happy.

Write a negation for each of the following statements.

23. The flowers are to be watered.
24. Her aunt's name is Lucia.
25. No rain fell in southern California today.
26. Every dog has its day.
27. All students present will get another chance.
28. Some books are longer than this book.
29. Some people have all the luck.
30. No computer repairman can play blackjack.
31. Nobody doesn't like Sara Lee.
32. Everybody loves somebody sometime.

Let p represent the statement "She has blue eyes" *and let q represent the statement* "He is 43 years old." *Translate each symbolic compound statement into words.*

39. $\sim p$
40. $\sim q$
41. $p \wedge q$
42. $p \vee q$
43. $\sim p \vee q$
44. $p \wedge \sim q$
45. $\sim p \vee \sim q$
46. $\sim p \wedge \sim q$
47. $\sim(\sim p \wedge q)$
48. $\sim(p \vee \sim q)$

Let p represent the statement "Chris collects videotapes" *and let q represent the statement* "Jack is a shortstop." *Convert each of the following compound statements into symbols.*

49. Chris collects videotapes and Jack is not a shortstop.
50. Chris does not collect videotapes or Jack is not a shortstop.
51. Chris does not collect videotapes or Jack is a shortstop.
52. Jack is a shortstop and Chris does not collect videotapes.
53. Neither Chris collects videotapes nor Jack is a shortstop.
54. Either Jack is a shortstop or Chris collects videotapes, and it is not the case that both Jack is a shortstop and Chris collects videotapes.
55. Incorrect use of quantifiers is often heard in everyday language. Suppose you hear that a local electronics chain is having a 30% off sale, and the radio advertisement states "All items are not available in all stores." Do you think that, literally translated, the ad really means what it says? What do you think is really meant? Explain your answer.
56. Repeat Exercise 55 for the following: "All people don't have the time to devote to maintaining their cars properly."

Quantifiers

The words *all, each, every,* and *no(ne)* are called **universal quantifiers,** while words and phrases like *some, there exists,* and *(for) at least one* are called **existential quantifiers.** Quantifiers are used extensively in mathematics to indicate *how many* cases of a particular situation exist. Be careful when forming the negation of a statement involving quantifiers.

The negation of a statement must be false if the given statement is true and must be true if the given statement is false, in all possible cases. Consider the statement

All girls in the group are named Mary.

Many people would write the negation of this statement as "No girls in the group are named Mary" or "All girls in the group are not named Mary." But this would not be correct. To see why, look at the three groups below.

Group I:	Mary Jones, Mary Smith, Mary Jackson
Group II:	Mary Johnson, Betty Parker, Margaret Boyle
Group III:	Shannon d'Hemecourt, Annie Ross, Patricia Gainey

These groups contain all possibilities that need to be considered. In Group I, *all* girls are named Mary; in Group II, *some* girls are named Mary (and some are not); and in Group III, *no* girls are named Mary. Look at the truth values in the chart on the next page and keep in mind that "some" means "at least one (and possibly all)."

Truth Value as Applied to:

	Group I	Group II	Group III	
(1) All girls in the group are named Mary. **(Given)**	T	F	F ←	Negation
(2) No girls in the group are named Mary. **(Possible negation)**	F	F	T	
(3) All girls in the group are not named Mary. **(Possible negation)**	F	F	T	
(4) Some girls in the group are not named Mary. **(Possible negation)**	F	T	T ←	

The negation of the given statement (1) must have opposite truth values in *all* cases. It can be seen that statements (2) and (3) do not satisfy this condition (for Group II), but statement (4) does. It may be concluded that the correct negation for "All girls in the group are named Mary" is "Some girls in the group are not named Mary." Other ways of stating the negation are:

Not all girls in the group are named Mary.

It is not the case that all girls in the group are named Mary.

At least one girl in the group is not named Mary.

The following table can be used to generalize the method of finding the negation of a statement involving quantifiers.

Negations of Quantified Statements

Statement	Negation
All do.	Some do not. (Equivalently: Not all do.)
Some do.	None do. (Equivalently: All do not.)

The negation of the negation of a statement is simply the statement itself. For instance, the negations of the statements in the Negation column are simply the corresponding original statements in the Statement column. As an example, the negation of "Some do not" is "All do."

EXAMPLE 5 Write the negation of each statement.

(a) Some dogs have fleas.

Since *some* means "at least one," the statement "Some dogs have fleas" is really the same as "At least one dog has fleas." The negation of this is "No dog has fleas."

(b) Some dogs do not have fleas.

This statement claims that at least one dog, somewhere, does not have fleas. The negation of this is "All dogs have fleas."

(c) No dogs have fleas.

The negation is "Some dogs have fleas."

Sets of Real Numbers

Natural or counting numbers $\{1, 2, 3, 4, \ldots\}$
Whole numbers $\{0, 1, 2, 3, 4, \ldots\}$
Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers $\{p/q \mid p \text{ and } q \text{ are integers, and } q \neq 0\}$
(Some examples of rational numbers are $3/5$, $-7/9$, 5, and 0. Any rational number may be written as a terminating decimal number, like 0.25 or a repeating decimal number, like 0.666. . . .)
Real numbers $\{x \mid x \text{ is a number which may be written as a decimal}\}$
Irrational numbers $\{x \mid x \text{ is a real number and } x \text{ cannot be written as a quotient of integers}\}$
(Some examples of irrational numbers are $\sqrt{2}$, $\sqrt[3]{4}$, and π. A characteristic of irrational numbers is that their decimal representations never terminate and never repeat, that is, they never reach a point where a given pattern of digits repeats from that point on.)

EXAMPLE 6 Decide whether each of the following statements about sets of numbers involving a quantifier is *true* or *false*.

(a) There exists a whole number that is not a natural number.

Because there is such a whole number (it is 0), this statement is true.

(b) Every integer is a natural number.

This statement is false, because we can find at least one integer that is not a natural number. For example, -1 is an integer but is not a natural number. (There are infinitely many other choices we could have made.)

(c) Every natural number is a rational number.

Since every natural number can be written as a fraction with denominator 1, this statement is true.

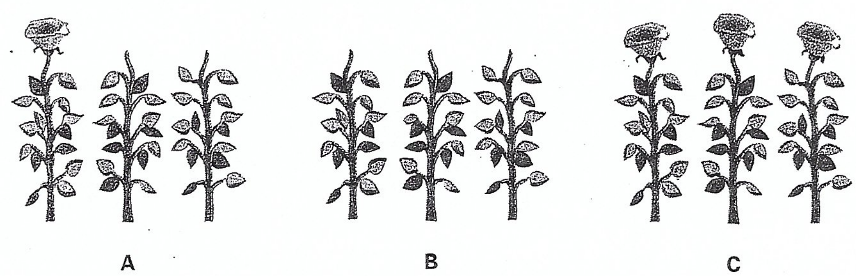
(d) There exists an irrational number that is not real.

In order to be an irrational number, a number must first be real (see the box). Therefore, since we cannot give an irrational number that is not real, this statement is false. (Had we been able to find at least one, the statement would have then been true.)

Decide whether each statement involving a quantifier is true *or* false.

57. Every natural number is an integer.
58. Every whole number is an integer.
59. There exists an integer that is not a natural number.
60. There exists a rational number that is not an integer.
61. All irrational numbers are real numbers.
62. All rational numbers are real numbers.
63. Some whole numbers are not rational numbers.
64. Some rational numbers are not integers.
65. Each rational number is a positive number.
66. Each whole number is a positive number.

Refer to the sketches labeled A, B, and C, and identify the sketch (or sketches) that is (are) satisfied by the given statement involving a quantifier.



67. All plants have a flower.
68. At least one plant has a flower.
69. No plant has a flower.
70. All plants do not have a flower.
71. At least one plant does not have a flower.
72. No plant does not have a flower.
73. Not every plant has a flower.
74. Not every plant does not have a flower.
75. Explain the difference between the following statements:

All students did not pass the test.
Not all students passed the test.

76. Write the following statement using "every": There is no one here who has not done that at one time or another.

3.2 Truth Tables

In this section, the truth values of component statements are used to find the truth values of compound statements. To begin, let us decide on the truth values of the **conjunction** *p and q*, symbolized $p \wedge q$. In everyday language, the connective *and* implies the idea of "both." The statement

> Monday immediately follows Sunday and March immediately follows February

is true, since each component statement is true. On the other hand, the statement

> Monday immediately follows Sunday and March immediately follows January

is false, even though part of the statement (Monday immediately follows Sunday) is true. For the conjunction $p \wedge q$ to be true, both p and q must be true. This

result is summarized by a table, called a **truth table,** which shows all four of the possible combinations of truth values for the conjunction *p and q*. The truth table for *conjunction* is shown here.

Truth Table for the Conjunction *p* and *q*

p and *q*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

In some cases, the logical connective *but* is used in compound statements. For example, consider the statement

> He wants to go to the mountains but she wants to go to the beach.

Here, *but* is used in place of *and* to give a different sort of emphasis to the statement. In such a case, we consider the statement as we would consider the conjunction using the word *and.* The truth table for the conjunction, given above, would apply.

In ordinary language, the word *or* can be ambiguous. The expression "this or that" can mean either "this or that or both," or "this or that but not both." For example, the statement

I will paint the wall or I will paint the ceiling

probably has the following meaning: "I will paint the wall or I will paint the ceiling or I will paint both." On the other hand, the statement

I will drive the Ford or the Datsun to the store

probably means "I will drive the Ford, or I will drive the Datsun, but I shall not drive both."

The symbol $\vee$ normally represents the first *or* described. That is, $p \vee q$ means "p or q or both." With this meaning of *or*, $p \vee q$ is called the *inclusive disjunction,* or just the **disjunction** of p and q.

In everyday language, the disjunction implies the idea of "either." For example, the disjunction

I have a quarter or I have a dime

is true whenever I have either a quarter, a dime or both. The only way this disjunction could be false would be if I had neither coin. A disjunction is false only if both component statements are false. The truth table for *disjunction* follows.

Truth Table for the Disjunction *p* or *q*

p or q

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The **negation** of a statement p, symbolized $\sim p$, must have the opposite truth value from the statement p itself. This leads to the truth table for the negation, shown here.

Truth Table for the Negation not *p*

not p

p	$\sim p$
T	F
F	T

EXAMPLE 3 Suppose p is false, q is true, and r is false. What is the truth value of the compound statement $\sim p \wedge (q \vee \sim r)$?

Here parentheses are used to group q and $\sim r$ together. Work first inside the parentheses. Since r is false, $\sim r$ will be true. Since $\sim r$ is true and q is true, find the truth value of $q \vee \sim r$ by looking in the first row of the *or* truth table. This row gives the result T. Since p is false, $\sim p$ is true, and the final truth value of $\sim p \wedge (q \vee \sim r)$ is found in the top row of the *and* truth table. From the *and* truth table, when $\sim p$ is true, and $q \vee \sim r$ is true, the statement $\sim p \wedge (q \vee \sim r)$ is true.

The paragraph above may be interpreted using a short-cut symbolic method, letting T represent a true statement and F represent a false statement:

$\sim p \wedge (q \vee \sim r)$	
$\sim F \wedge (T \vee \sim F)$	
$T \wedge (T \vee T)$	$\sim$F gives T.
$T \wedge T$	T $\vee$ T gives T.
T.	T $\wedge$ T gives T.

The T in the final row indicates that the compound statement is true.

The next two examples show the use of truth tables to determine the truth values of algebraic statements.

Use the concepts introduced in this section to answer Exercises 1–6.

1. If we know that p is true, what do we know about the truth value of $p \vee q$ even if we are not given the truth value of q?
2. If we know that p is false, what do we know about the truth value of $p \wedge q$ even if we are not given the truth value of q?
3. If p is false, what is the truth value of $p \wedge (q \vee \sim r)$?
4. If p is true, what is the truth value of $p \vee (q \vee \sim r)$?
5. Explain in your own words the condition that must exist for a conjunction of two component statements to be true.
6. Explain in your own words the condition that must exist for a disjunction of two component statements to be false.

Let p represent a false statement and let q represent a true statement. Find the truth value of the given compound statement.

7. $\sim p$ **8.** $\sim q$ **9.** $p \vee q$ **10.** $p \wedge q$

11. $p \vee \sim q$ **12.** $\sim p \wedge q$ **13.** $\sim p \vee \sim q$ **14.** $p \wedge \sim q$

15. $\sim(p \wedge \sim q)$ **16.** $\sim(\sim p \vee \sim q)$

17. $\sim[\sim p \wedge (\sim q \vee p)]$ **18.** $\sim[(\sim p \wedge \sim q) \vee \sim q]$

Let p represent a true statement, and q and r represent false statements. Find the truth value of the given compound statement.

21. $(q \vee \sim r) \wedge p$ **22.** $(p \wedge r) \vee \sim q$

23. $(\sim p \wedge q) \vee \sim r$ **24.** $p \wedge (q \vee r)$

25. $(\sim r \wedge \sim q) \vee (\sim r \wedge q)$ **26.** $\sim(p \wedge q) \wedge (r \vee \sim q)$

27. $\sim[r \vee (\sim q \wedge \sim p)]$ **28.** $\sim[(\sim p \wedge q) \vee r]$

Truth Tables

In the examples above, the truth value for a given statement was found by going back to the basic truth tables. In the long run, it is easier to first create a complete truth table for the given statement itself. Then final truth values can be read directly from this table. The procedure for making new truth tables is shown in the next few examples.

In this book we will use the following standard format for listing the possible truth values in compound statements involving two statements.

p	q	**Compound Statement**
T	T	
T	F	
F	T	
F	F	

EXAMPLE 6

(a) Construct a truth table for $(\sim p \wedge q) \vee \sim q$.

Begin by listing all possible combinations of truth values for p and q, as above. Then find the truth values of $\sim p \wedge q$. Start by listing the truth values of $\sim p$, which are the opposite of those of p.

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

Use only the "$\sim p$" column and the "q" column, along with the *and* truth table, to find the truth values of $\sim p \wedge q$. List them in a separate column.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Next include a column for $\sim q$.

p	q	$\sim p$	$\sim p \wedge q$	$\sim q$
T	T	F	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Finally, make a column for the entire compound statement. To find the truth values, use *or* to combine $\sim p \wedge q$ with $\sim q$.

p	q	$\sim p$	$\sim p \wedge q$	$\sim q$	$(\sim p \wedge q) \vee \sim q$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

(b) Suppose both p and q are true. Find the truth value of $(\sim p \wedge q) \vee \sim q$.

Look in the first row of the final truth table above, where both p and q have truth value T. Read across the row to find that the compound statement is false.

EXAMPLE 8

(a) Construct a truth table for $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$.

This statement has three component statements, p, q, and r. The truth table thus requires eight rows to list all possible combinations of truth values of p, q, and r. The final truth table, however, can be found in much the same way as the ones above.

p	q	r	$\sim p$	$\sim p \wedge r$	$\sim q$	$\sim q \wedge \sim p$	$(\sim p \wedge r) \vee (\sim q \wedge \sim p)$
T	T	T	F	F	F	F	F
T	T	F	F	F	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	T	F	F
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	T	T	T

(b) Suppose p is true, q is false, and r is true. Find the truth value of $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$.

By the third row of the truth table in part (a), the compound statement is false.

Let p represent the statement $1 > 9$, *let q represent the statement* $7 \not> 5$, *and let r represent the statement* $13 \geq 13$. *Find the truth value of the given compound statement.*

29. $p \wedge r$ **30.** $p \vee \sim q$ **31.** $\sim q \vee \sim r$ **32.** $\sim p \wedge \sim r$

33. $(p \wedge q) \vee r$ **34.** $\sim p \vee (\sim r \vee \sim q)$ **35.** $(\sim r \wedge q) \vee \sim p$ **36.** $\sim(p \vee \sim q) \vee \sim r$

Give the number of rows in the truth table for each of the following compound statements.

37. $p \vee \sim r$ **38.** $p \wedge (r \wedge \sim s)$

39. $(\sim p \wedge q) \vee (\sim r \vee \sim s) \wedge r$ **40.** $[(p \vee q) \wedge (r \wedge s)] \wedge (t \vee \sim p)$

41. $[(\sim p \wedge \sim q) \wedge (\sim r \wedge s \wedge \sim t)] \wedge (\sim u \vee \sim v)$

42. $[(\sim p \wedge \sim q) \vee (\sim r \vee \sim s)] \vee [(\sim m \wedge \sim n) \wedge (u \wedge \sim v)]$

Construct a truth table for each compound statement.

45. $\sim p \wedge q$ **46.** $\sim p \vee \sim q$ **47.** $\sim(p \wedge q)$

48. $p \vee \sim q$ **49.** $(q \vee \sim p) \vee \sim q$ **50.** $(p \wedge \sim q) \wedge p$

51. $\sim q \wedge (\sim p \vee q)$ **52.** $\sim p \vee (\sim q \wedge \sim p)$ **53.** $(p \vee \sim q) \wedge (p \wedge q)$

54. $(\sim p \wedge \sim q) \vee (\sim p \vee q)$ **55.** $(\sim p \wedge q) \wedge r$ **56.** $r \vee (p \wedge \sim q)$

Alternate Method for Constructing Truth Tables

After making a reasonable number of truth tables, some people prefer the shortcut method shown in Example 10, which repeats Examples 6 and 8 above.

EXAMPLE 10 Construct the truth table for each statement.

(a) $(\sim p \wedge q) \vee \sim q$

Start by inserting truth values for $\sim p$ and for q.

p	q	$(\sim p$	$\wedge$	$q)$	$\vee$	$\sim q$
T	T	F		T		
T	F	F		F		
F	T	T		T		
F	F	T		F		

Next, use the *and* truth table to obtain the truth values of $\sim p \wedge q$.

p	q	$(\sim p$	$\wedge$	$q)$	$\vee$	$\sim q$
T	T	F	F	T		
T	F	F	F	F		
F	T	T	T	T		
F	F	T	F	F		

Now disregard the two preliminary columns of truth values for $\sim p$ and for q, and insert truth values for $\sim q$.

p	q	$(\sim p$	$\wedge$	$q)$	$\vee$	$\sim q$
T	T		F			F
T	F		F			T
F	T		T			F
F	F		F			T

Finally, use the *or* truth table.

p	q	$(\sim p$	$\wedge$	$q)$	$\vee$	$\sim q$
T	T		F		F	F
T	F		F		T	T
F	T		T		T	F
F	F		F		T	T

These steps can be summarized as follows.

p	q	$(\sim p$	$\wedge$	$q)$	$\vee$	$\sim q$
T	T	F	F	T	F	F
T	F	F	F	F	T	T
F	T	T	T	T	T	F
F	F	T	F	F	T	T
		①	②	①	④	③

The circled numbers indicate the order in which the various columns of the truth table were found.

(b) $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$

Work as follows.

p	q	r	$(\sim p$	$\wedge$	$r)$	$\vee$	$(\sim q$	$\wedge$	$\sim p)$
T	T	T	F	F	T	F	F	F	F
T	T	F	F	F	F	F	F	F	F
T	F	T	F	F	T	F	T	F	F
T	F	F	F	F	F	F	T	F	F
F	T	T	T	T	T	T	F	F	T
F	T	F	T	F	F	F	F	F	T
F	F	T	T	T	T	T	T	T	T
F	F	F	T	F	F	T	T	T	T
			①	②	①	⑤	③	④	③

58. $(\sim r \vee \sim p) \wedge (\sim p \vee \sim q)$
60. $(\sim r \vee s) \wedge (\sim p \wedge q)$

57. $(\sim p \wedge \sim q) \vee (\sim r \vee \sim p)$
59. $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim s)$

Equivalent Statements

One application of truth tables is illustrated by showing that two statements are equivalent; by definition, two statements are **equivalent** if they have the same truth value in *every* possible situation. The columns of each truth table that were the last to be completed will be exactly the same for equivalent statements.

EXAMPLE 11 Are the statements

$$\sim p \wedge \sim q \text{ and } \sim(p \vee q)$$

equivalent?

To find out, make a truth table for each statement, with the following results.

p	q	$\sim p \wedge \sim q$
T	T	F
T	F	F
F	T	F
F	F	T

p	q	$\sim(p \vee q)$
T	T	F
T	F	F
F	T	F
F	F	T

Since the truth values are the same in all cases, as shown in the columns in color, the statements $\sim p \wedge \sim q$ and $\sim(p \vee q)$ are equivalent. Equivalence is written with a three-bar symbol, $\equiv$. Using this symbol, $\sim p \wedge \sim q \equiv \sim(p \vee q)$.

In the same way, the statements $\sim p \vee \sim q$ and $\sim(p \wedge q)$ are equivalent. We call these equivalences De Morgan's laws.

De Morgan's Laws

For any statements p and q,

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$
$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

EXAMPLE 12 Find a negation of each statement by applying De Morgan's laws.

(a) I got an A or I got a B.

If p represents "I got an A" and q represents "I got a B," then the compound statement is symbolized $p \vee q$. The negation of $p \vee q$ is $\sim(p \vee q)$; by one of De Morgan's laws, this is equivalent to

$$\sim p \wedge \sim q,$$

or, in words,

I didn't get an A and I didn't get a B.

This negation is reasonable—the original statement says that I got either an A or a B; the negation says that I didn't get *either* grade.

(b) She won't try and he will succeed.

From one of De Morgan's laws, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, so the negation becomes

She will try or he won't succeed.

(c) $\sim p \vee (q \wedge \sim p)$

Negate both component statements and change $\vee$ to $\wedge$.

$$\sim[\sim p \vee (q \wedge \sim p)] \equiv p \wedge \sim(q \wedge \sim p)$$

Now apply De Morgan's law again.

$$p \wedge \sim(q \wedge \sim p) \equiv p \wedge (\sim q \vee \sim(\sim p))$$
$$\equiv p \wedge (\sim q \vee p)$$

A truth table will show that the statements

$$\sim p \vee (q \wedge \sim p) \quad \text{and} \quad p \wedge (\sim q \vee p)$$

are negations.

Use one of De Morgan's laws to write the negation of each statement.

61. You can pay me now or you can pay me later.

62. I am not going or she is going.

63. It is summer and there is no snow.

64. 1/2 is a positive number and -12 is less than zero.

65. I said yes but she said no.

66. Kelly Bell tried to sell the book, but she was unable to do so.

67. $5 - 1 = 4$ and $9 + 12 \neq 7$

68. $3 < 10$ or $7 \neq 2$

69. Dasher or Dancer will lead Santa's sleigh next Christmas.

70. The lawyer and the client appeared in court.

Identify each of the following statements as true *or* false.

71. For every real number y, $y < 12$ or $y > 4$.

72. For every real number t, $t > 3$ or $t < 3$.

73. For some integer p, $p \geq 5$ and $p \leq 5$.

74. There exists an integer n such that $n > 0$ and $n < 0$.

75. Complete the truth table for *exclusive disjunction.* The symbol $\veebar$ represents "one or the other is true, but not both."

p	q	$p \veebar q$
T	T	
T	F	
F	T	
F	F	

Exclusive disjunction

76. Lawyers sometimes use the phrase "and/or." This phrase corresponds to which usage of the word *or*: inclusive or exclusive?