3 The Mathematics of Sharing

Fair-Division Games

As children, the Elghanayan brothers probably did not give a lot of thought to how to best divide their Halloween stash of candy. Fifty years later, when Henry, Fred, and Tom Elghanayan decided to break up their forty-year partnership and split up the assets, they did indeed give a great deal of thought to the question of division. After all, they were about to divide $3 billion worth of New York City real estate holdings.

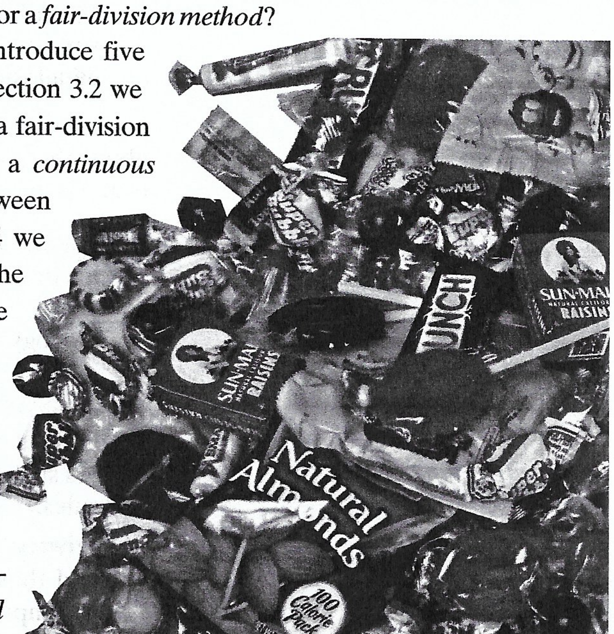
The fact that the Elghanayan brothers were able to split up so much valuable property (8000 apartments, 9 office buildings, and 9 development projects) without lawsuits or bad blood is a testament to the power of a surprising idea: When different parties—with different preferences and values—have to divide commonly owned assets, it is actually possible to carry out the division in a way that is so fair that *each party* comes out of the deal feeling that they got more than they deserved.

The problem of dividing a real estate empire worth $3 billion and the problem of dividing a stash of Halloween candy are not all that different. Other than the stakes involved they share the same basic elements: a set of assets, a group of individuals that have some equity in the assets, and a shared goal of dividing the assets in a fair way. These types of problems are part of an interesting and important class of problems known as *fair-division problems*.

Fairness is an innate human value, based on empathy, the instinct for cooperation, and, to some extent, self-interest. A theory for sharing things fairly based on cooperation, reason, and logic is one of the great achievements of social science, and, once again, we can trace the roots of this achievement to a branch of mathematics known as game theory. In this chapter we will give a basic introduction to the subject.

> “I'm happy with the assets I ended up with, and my guess is that they [the brothers] are happy also. Assets have different values to different people. There is no such thing as a fair market value; it's a fair market value in the mind of each person.”
>
> – *Henry Elghanayan*

The chapter is organized in a manner very similar to that of Chapters 1 and 2. We will start (Section 3.1) with some general background: What are the elements of a *fair-division game* and what are the requirements for a *fair-division method*? In the remaining five sections we introduce five different fair-division methods. In Section 3.2 we discuss the *divider-chooser* method, a fair-division method that applies when dividing a *continuous* (i.e., infinitely divisible) asset between two players. In Sections 3.3 and 3.4 we discuss two different extensions of the divider-chooser method that can be used to divide a continuous asset among three or more players: the *lone-divider* method and the *lone-chooser* method. In Sections 3.5 and 3.6 we discuss two different fair-division methods that can be used when dividing a set of discrete (i.e., indivisible) assets: the method of *sealed bids* and the method of *markers*.



3.1 Fair-Division Games

In this section we will introduce some of the basic concepts and terminology of fair division. Much as we did in Chapter 2 when we studied weighted voting systems, we will think of fair-division problems in terms of games—with players, goals, rules, and strategies.

Basic Elements of a Fair-Division Game

The underlying elements of every *fair-division game* are as follows:

- **The assets.** This is the formal name we will give to the "goodies" being divided. Typically, the assets are tangible physical objects such as real estate, jewelry, art, candy, cake, and so on. In some situations the assets being divided may be intangible things such as rights—water rights, drilling rights, broadcast licenses, and so on. (While the term "assets" has a connotation of positive value, sometimes what is being divided has negative value—chores, obligations, liabilities, etc. In this chapter we will focus on the division of positive assets, but in Example 3.12 we discuss a situation in which negative-valued assets have to be divided.) Regardless of the nature of the assets, we will use the symbol S throughout this chapter to denote the set of all assets being divided.
- **The players.** In every fair-division game there is a set of parties with the right (or in some cases the duty) to divide among themselves a set of jointly owned assets. They are the *players* in the game. Most of the time the players in a fair-division game are individuals, but it is worth noting that some of the most significant applications of fair division occur when the players are *institutions* (ethnic groups, political parties, states, and even nations).
- **The value systems.** The fundamental assumption we will make is that each player has the ability to give a value to any part of the assets. Specifically, this means that each player can look at the set S or any subset of S and assign to it a value—either in absolute terms ("to me, that plot of land is worth $875,000") or in relative terms ("to me, that plot of land is worth 30% of the total value of all assets").
- **A fair-division method.** These are the rules that govern the way the game is played. Much like the voting methods discussed in Chapter 1, fair-division methods are very specific and leave no room for ambiguity. We will discuss in much greater detail various fair-division methods in the remaining sections of the chapter.

Like most games, fair-division games are predicated on certain assumptions about the players. For the purposes of our discussion, we will make the following four assumptions:

- **Rationality.** Each of the players is a thinking, rational entity seeking to maximize his or her share of the assets. We will further assume that in the pursuit of this goal, a player's moves are based on reason alone (we are taking emotion, psychology, mind games, and all other nonrational elements out of the picture).
- **Cooperation.** The players are willing participants and accept the rules of the game as binding. The rules are such that after a *finite* number of moves by the players, the game terminates with a division of the assets. (There are no outsiders such as judges or referees involved in these games—just the players and the rules.)
- **Privacy.** Players have *no* useful information on the other players' value systems and thus of what kinds of moves they are going to make in the game. (This assumption does not always hold in real life, especially if the players are siblings or friends.)

- **Symmetry.** Players have *equal* rights in sharing the assets. A consequence of this assumption is that, at a minimum, each player is entitled to a *proportional* share of the assets—when there are two players, each is entitled to at least one-half of the assets, with three players each is entitled to at least one-third of the assets, and so on. (Fair-division methods can also be used in situations in which different players are entitled by right to different-sized shares of the assets. For more details on *asymmetric fair division*, see Exercise 70 and Project 2.)

Fair Shares and Fair Divisions

Given a set of players $P_1, P_2, \ldots, P_N$ and a set of assets S, the ultimate purpose of a fair-division game is to produce a *fair division* of S. But what does this mean? To formalize the precise meaning of this seemingly subjective idea, we introduce two very important definitions.

- **Fair share (to a player P).** Suppose that s denotes a share of the set of assets S and that P is one of the players in a fair-division game with N players. The share s is called a *proportional fair share* (or just simply a *fair share*) *to P*, if, in P's opinion, the value of s is at least $\frac{1}{N}$th of the value of S.
- **Fair division (of the set of assets S).** Suppose we are able to divide the set of assets S into N shares (call them $s_1, s_2, \ldots, s_N$) and assign each of these shares to one of the players. If each player considers the share he or she received to be a fair share (i.e., worth at least $\frac{1}{N}$th of the value of S), then we have achieved a *fair division* of the set of assets S.

EXAMPLE 3.1 FAIR DIVISIONS

Three brothers—say Henry, Tom, and Fred—are splitting up their partnership and dividing a bunch of assets of unspecified value. The set of assets S is divided (at this point we don't care how) into three shares: s_1, s_2, and s_3. In Henry's opinion, the values of the three shares (expressed as a percentage of the value of S) are, respectively, 32%, 31% and 37%. This information is displayed in the first row of Table 3-1. Likewise, the second and third rows of Table 3-1 show the values that each of the other two brothers assigns to each of the three shares.

	s_1	s_2	s_3
Henry	32%	31%	37%
Tom	34%	31%	35%
Fred	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%

TABLE 3-1 Players' valuation of shares for Examples 3.1 and 3.2

- **Fair shares.** To Henry, s_3 is the only fair share (the other two shares fall below the fair-share threshold of $33\frac{1}{3}$%); to Tom, s_1 and s_3 are both fair shares; to Fred, who values all three shares equally, they are all fair shares.
- **Fair divisions.** To assign each player a fair share we start with Henry because with him we have only one option—we have to assign him s_3. With Tom we seem to have, at least in principle, two options—s_1 and s_3 are both fair shares—but s_3 is gone, so our only choice is to assign him s_1. This leaves the final share s_2 for Fred. Since all three players received a fair share, we have achieved a fair division. (Notice that this does not mean that all players are equally happy—Tom would have been happier with s_3 rather than s_1, but such is life. All that a fair division promises the players is that they will get a fair share, not the best share.)

Fair-Division Methods

A **fair-division method** is a set of rules that, when properly used by the players, *guarantees* that at the end of the game each player will have received a fair share of the assets. The key requirement is the guarantee—no matter what the circumstances, the method should produce a fair division of the assets. The next example illustrates a seemingly reasonable method that fails this test.

EXAMPLE 3.2 DRAWING LOTS

Once again we go back to Henry, Tom, and Fred and their intention to split their partnership fairly. The method they adopt for the split seems pretty reasonable: One of the brothers divides the assets into three shares (s_1, s_2, and s_3), and then they draw lots to determine the order in which they get to choose. The values of the shares are as shown in Table 3-1. Say the order is Henry first, Tom second, and Fred last. In this case Henry will undoubtedly choose s_3, Tom will then choose s_1, and Fred will end up with s_2. This works out fine, as this gives a fair division of S. So far, so good.

Suppose, however, that the when they draw lots the order is Tom first, Henry second, and Fred last. In this case Tom will start by choosing s_3. Now Henry has to choose between s_1 and s_2, neither of which is a fair share. In this case the method fails to produce a fair division. Since drawing lots does not guarantee that the final result will always be a fair division, it is not a fair-division method.

There are many different fair-division methods known, but in this chapter we will discuss only a few of the classic ones. Depending on the nature of the set S, a fair-division game can be classified as one of three types: *continuous, discrete*, or *mixed*, and the fair-division methods used depend on which of these types we are facing.

- **Continuous fair division.** Here the set S is divisible in infinitely many ways, and shares can be increased or decreased by arbitrarily small amounts. Typical examples of continuous fair-division games involve the division of land, a cake, a pizza, and so forth.
- **Discrete fair division.** Here the set S is made up of objects that are indivisible, like paintings, houses, cars, boats, jewelry, and so on. (What about pieces of candy? One might argue that with a sharp enough knife a piece of candy could be chopped up into smaller and smaller pieces, but nobody really does that—it's messy. As a semantic convenience let's agree that candy is indivisible, and, therefore, dividing candy is a discrete fair-division game.)
- **Mixed fair division.** This is the case where some of the assets are continuous and some are discrete. Dividing an estate consisting of jewelry, a house, and a parcel of land is a mixed fair-division game.

Fair-division methods are classified according to the nature of the problem involved. Thus, there are *discrete fair-division* methods (used when the set S is made up of indivisible, discrete objects), and there are *continuous fair-division* methods (used when the set S is an infinitely divisible, continuous set). Mixed fair-division games can usually be solved by dividing the continuous and discrete parts separately, so we will not discuss them in this chapter. We will start our excursion into fair-division methods with a classic method for continuous fair division.

3.2 The Divider-Chooser Method

When two players are dividing a continuous asset, the standard method used is the **divider-chooser method**. This method is also known as "you cut; I choose," and most of us have used it at some time or another when dividing a piece of cake, a sandwich, or a bowl of ice cream. As the name suggests, one player, called the *divider*, divides S into two shares, and the second player, called the *chooser*, picks the share he or she wants, leaving the other share to the divider.

Under the *rationality* and *privacy* assumptions introduced in Section 3.1, this method guarantees that both divider and chooser will get a share worth 50% or more of the total value of *S*. Not knowing the chooser's likes and dislikes (privacy assumption), the divider can only guarantee himself a 50% share by dividing *S* into two halves of equal value (rationality assumption); the chooser is guaranteed a 50% or better share by choosing the piece she likes best.

EXAMPLE 3.3 DIVIDING A CHEESECAKE

On their first date, Damian and Cleo go to the county fair. They buy jointly a raffle ticket, and, as luck would have it, they win a half chocolate–half strawberry cheesecake.

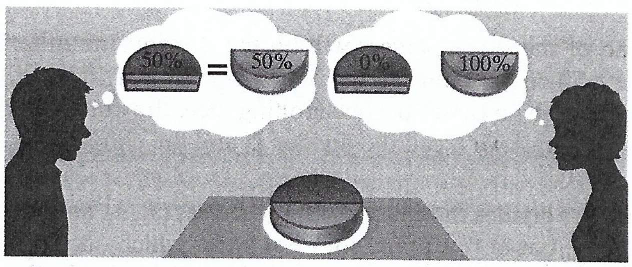
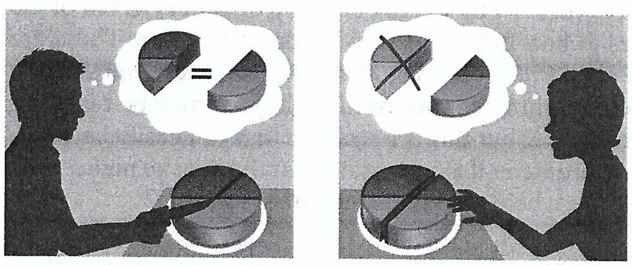


FIGURE 3-1



(a) (b)

FIGURE 3-2 (a) Damian cuts. (b) Cleo picks.

Damian likes chocolate and strawberry equally well, so in his eyes the chocolate and strawberry halves are equal in value (Fig. 3-1). On the other hand, Cleo hates chocolate—she is allergic to it and gets sick if she eats any—so in her eyes the value of the cake is 0% for the chocolate half, 100% for the strawberry part (Fig. 3-1). Once again, to ensure a fair division, we will assume that neither of them knows anything about the other's likes and dislikes.

Let's now see how Damian and Cleo might divide this cake using the divider-chooser method. Damian volunteers to go first and be the divider. His cut is shown in Fig. 3-2(a). Notice that this is a perfectly rational division of the cake based on Damian's value system—each piece is half of the cake and to him worth one-half of the total value of the cake. It is now Cleo's turn to choose, and her choice is obvious—she will pick the piece having the larger strawberry part [Fig. 3-2(b)].

The final outcome of this division is that Damian gets a piece that in his own eyes is worth exactly half of the cake, but Cleo ends up with a much sweeter deal—a piece that in her own eyes is worth about two-thirds of the cake. This is, nonetheless, a fair division of the cake—both players get pieces worth 50% or more. Mission accomplished!

Example 3.3 illustrates why, given a choice, it is always *better to be the chooser than the divider*—the divider is guaranteed a share worth exactly 50% of the total value of *S*, but with just a little luck the chooser can end up with a share worth more than 50%. Since a fair-division method should treat all players equally, both players should have an equal chance of being the chooser. This is best done by means of a coin toss, with the winner of the coin toss getting the privilege of making the choice.

The divider-chooser method goes all the way back to the Old Testament: When Lot and Abraham argued over grazing rights, Abraham proposed, "Let us divide the land into left and right. If you go left, I will go right; and if you go right, I will go left" (Genesis 13:1–9).

But how do we implement the same idea when the division is among three, or four, or *N* players? In the next two sections we will discuss two different ways to extend the divider-chooser method to the case of three or more players.

3.3 The Lone-Divider Method

The first important breakthrough in the mathematics of fair division came in 1943, when the Polish mathematician Hugo Steinhaus came up with a clever way to extend some of the ideas in the divider-chooser method to the case of *three* players, one of whom plays the role of the *divider* and the other two who play the role of *choosers*. Steinhaus's approach was subsequently generalized to any number of players N (one divider and $N-1$ choosers) by Princeton mathematician Harold Kuhn. In either case we will refer to this method as the **lone-divider method**.

We start this section with a description of Steinhaus's *lone-divider method* for the case of $N = 3$ players. We will describe the process in terms of dividing a cake, a commonly used and convenient metaphor for a continuous asset.

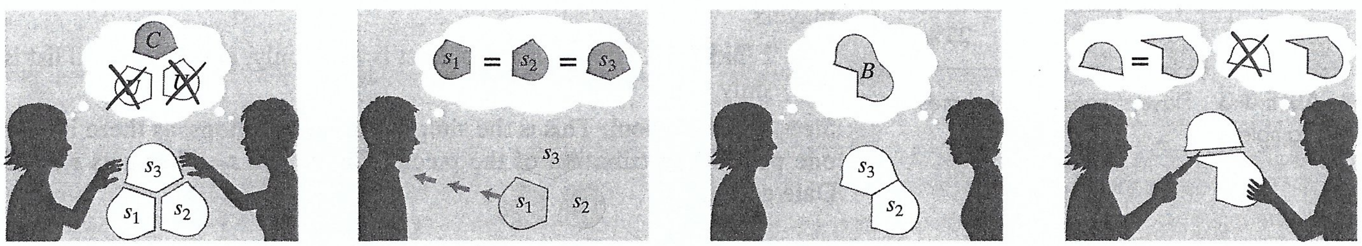
The Lone-Divider Method for Three Players

- **Step 0 (Preliminaries).** One of the three players will be the divider; the other two players will be choosers. Since it is better to be a chooser than a divider, the decision of who is what is made by a random draw (rolling dice, drawing cards from a deck, etc.). For convenience we'll call the divider D and the choosers C_1 and C_2.
- **Step 1 (Division).** The divider D divides the cake into three shares (s_1, s_2, and s_3). D will get one of these shares, but at this point does not know which one. Not knowing which share will be his (privacy assumption) forces D to divide the cake into three shares of equal value (rationality assumption).
- **Step 2 (Bidding).** C_1 declares (usually by writing on a slip of paper) which of the three pieces are fair shares to her. Independently, C_2 does the same. These are the *bids*. A chooser's bid *must list every single piece that he or she considers to be a fair share* (i.e., worth one-third or more of the cake)—it may be tempting to bid only for the very best piece, but this is a strategy that can easily backfire. To preserve the privacy requirement, it is important that the bids be made independently, without the choosers being privy to each other's bids.
- **Step 3 (Distribution).** Who gets which piece? The answer, of course, depends on which pieces are listed in the bids. For convenience, we will separate the pieces into two types: C-pieces (these are pieces *chosen* by either one or both of the choosers) and U-pieces (these are *unwanted* pieces that did not appear in either of the bids). Expressed in terms of value, a U-piece is a piece that *both* choosers value at less than $33\frac{1}{3}\%$ of the cake, and a C-piece is a piece that *at least* one of the choosers (maybe both) values at $33\frac{1}{3}\%$ or more. Depending on the number of C-pieces, there are two separate cases to consider.

  Case 1. When there are two or more C-pieces, there is always a way to give each chooser a different piece from among the pieces listed in her bid. (The details will be covered in Examples 3.4 and 3.5.) Once each chooser gets her piece, the divider gets the last remaining piece. At this point every player has received a fair share, and a fair division has been accomplished. (Sometimes we might end up in a situation in which C_1 likes C_2's piece better than her own and vice versa. In that case it is perfectly reasonable to let them swap pieces—this would make each of them happier than they already were, and who could be against that?)

  Case 2. When there is only one C-piece, we have a bit of a problem because it means that both choosers are bidding for the very same piece [Fig. 3-3(a)]. The solution here requires a little more creativity. First, we take care of the divider D—to whom all pieces are equal in value—by giving him one of the pieces that neither chooser wants [Fig. 3-3(b).] (If the two choosers can agree on the least desirable piece, then so much the better; if they can't agree, then the choice of which piece to give the divider can be made randomly.)

After D gets his piece, the two pieces left (the C-piece and the remaining U-piece) are recombined into one piece that we call the B-piece [Fig. 3-3(c)]. Now we have one piece and two players and we can revert to the *divider-chooser method* to finish the fair division: One player cuts the B-piece into two pieces; the other player chooses the piece she likes better [Fig. 3-3(d)].



(a) (b) (c) (d)

FIGURE 3-3 Case 2 in the lone-divider method (three players). (a) Both choosers covet the same piece. (b) The divider walks away with one of the U-pieces. (c) The two remaining pieces are recombined into the B-piece. (d) The B-piece is divided by the remaining two players using the divider-chooser method.

This process results in a fair division of the cake because it guarantees fair shares for all players. We know that D ends up with a fair share by the very fact that D did the original division, but what about C_1 and C_2? The key observation is that in the eyes of both C_1 and C_2 the B-piece is worth *more than two-thirds of the value of the original cake* (think of the B-piece as 100% of the original cake minus a U-piece worth less than $33\frac{1}{3}\%$), so when we divide *it* fairly into *two* shares, each party is guaranteed *more than one-third* of the original cake. We will come back to this point in Example 3.6.

We will now illustrate the details of Steinhaus's lone-divider method for three players with several examples. In all these examples we will assume that the divider has already divided the cake into three shares s_1, s_2, and s_3. In each example the values that each of the three players assigns to the shares, expressed as percentages of the total value of the cake, are shown in a table. The reader should remember, however, that this information is never available in full to the players—an individual player only knows the percentages on his or her row.

EXAMPLE 3.4 LONE DIVIDER WITH 3 PLAYERS (CASE 1, VERSION 1)

Dale, Cindy, and Cher are dividing a cake using Steinhaus's lone-divider method. They draw cards from a well-shuffled deck of cards, and Dale draws the low card (bad luck!) and has to be the divider.

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	35%	10%	55%
Cher	40%	25%	35%

TABLE 3-2 Players' valuation of shares for Example 3.4

Step 1 (Division). Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-2 shows the values of the three pieces in the eyes of each of the players.

Step 2 (Bidding). From Table 3-2 we can assume that Cindy's bid list is $\{s_1, s_3\}$ and Cher's bid list is also $\{s_1, s_3\}$.

Step 3 (Distribution). The C-pieces are s_1 and s_3. There are two possible distributions. One distribution would be: Cindy gets s_1, Cher gets s_3, and Dale gets s_2. An even better distribution (the *optimal* distribution) would be: Cindy gets s_3, Cher gets s_1, and Dale gets s_2. In the case of the first distribution, both Cindy and Cher would benefit by swapping pieces, and there is no rational reason why they would not do so. Thus, using the rationality assumption, we can conclude that in either case the final result will be the same: Cindy gets s_3, Cher gets s_1, and Dale gets s_2.

EXAMPLE 3.5 LONE DIVIDER WITH 3 PLAYERS (CASE 1, VERSION 2)

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	30%	40%	30%
Cher	60%	15%	25%

■ **TABLE 3-3** Players' valuation of shares for Example 3.5

We'll use the same setup as in Example 3.4—Dale is the divider, Cindy and Cher are the choosers.

Step 1 (Division). Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-3 shows the values of the three pieces in the eyes of each of the players.

Step 2 (Bidding). Here Cindy's bid list is $\{s_2\}$ only, and Cher's bid list is $\{s_1\}$ only.

Step 3 (Distribution). This is the simplest of all situations, as there is only one possible distribution of the pieces: Cindy gets s_2, Cher gets s_1, and Dale gets s_3.

EXAMPLE 3.6 LONE DIVIDER WITH 3 PLAYERS (CASE 2)

The gang of Examples 3.4 and 3.5 is back at it again.

	s_1	s_2	s_3
Dale	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Cindy	20%	30%	50%
Cher	10%	20%	70%

■ **TABLE 3-4** Players' valuation of shares for Example 3.6

Step 1 (Division). Dale divides the cake into three pieces s_1, s_2, and s_3. Table 3-4 shows the values of the three pieces in the eyes of each of the players.

Step 2 (Bidding). Here Cindy's and Cher's bid lists consist of just $\{s_3\}$.

Step 3 (Distribution). The only C-piece is s_3. Cindy and Cher talk it over, and without giving away any other information agree that of the two U-pieces, s_1 is the least desirable, so they all agree that Dale gets s_1. (Dale doesn't care which of the three pieces he gets, so he has no rational objection.) The remaining pieces (s_2 and s_3) are then recombined to form the B-piece, to be divided between Cindy and Cher using the divider-chooser method (one of them divides the B-piece into two shares, the other one chooses the share she likes better). Regardless of how this plays out, both of them will get a very healthy share of the cake: Cindy will end up with a piece worth at least 40% of the original cake (the B-piece is worth 80% of the original cake to Cindy), and Cher will end up with a piece worth at least 45% of the original cake (the B-piece is worth 90% of the original cake to Cher).

The Lone-Divider Method for More Than Three Players

In 1967 Harold Kuhn, a mathematician at Princeton University, extended Steinhaus's lone-divider method to more than three players. The first two steps of Kuhn's method are a straightforward generalization of Steinhaus's lone-divider method for three players, but the distribution step requires some fairly sophisticated mathematical ideas and is rather difficult to describe in full generality. We will only give an outline here and will illustrate the details with a couple of examples.

- **Step 0 (Preliminaries).** One of the players is chosen to be the divider D, and the remaining $N-1$ players are all going to be choosers. As always, it's better to be a chooser than a divider, so the decision should be made by a random draw.
- **Step 1 (Division).** The divider D divides the set S into N shares $s_1, s_2, s_3, \ldots, s_N$. D is guaranteed of getting one of these shares, but doesn't know which one.
- **Step 2 (Bidding).** Each of the $N-1$ choosers independently submits a bid list consisting of *every share that he or she considers to be a fair share* (in this case, this means any share worth $\frac{1}{N}$th or more of the total).

- **Step 3 (Distribution).** The bid lists are opened. Much as we did with three players, we will have to consider two separate cases, depending on how these bid lists turn out.

  Case 1. If there is a way to assign a different share to each of the $N-1$ choosers, then that should be done. (Needless to say, the share assigned to a chooser should be from his or her bid list.) The divider, to whom all shares are presumed to be of equal value, gets the last unassigned share. At the end, players may choose to swap pieces if they want.

  Case 2. There is a *standoff*—in other words, there are two choosers both bidding for just one share, or three choosers bidding for just two shares, or K choosers bidding for less than K shares. This is a much more complicated case, and what follows is a rough sketch of what to do. To resolve a standoff, we first set aside the shares involved in the standoff from the remaining shares. Likewise, the players involved in the standoff are temporarily separated from the rest. Each of the remaining players (including the divider) can be assigned a fair share from among the remaining shares and sent packing. All the shares left are recombined into a new piece to be divided among the players involved in the standoff, and the process starts all over again.

The following two examples will illustrate some of the ideas behind the lone-divider method in the case of four players. The first example is one without a standoff; the second example involves a standoff.

EXAMPLE 3.7 LONE DIVIDER WITH 4 PLAYERS (CASE 1)

We have one divider, Demi, and three choosers, Chan, Chloe, and Chris.

Step 1 (Division). Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-5 shows how each of the players values each of the four shares. Remember that the information on each row of Table 3-5 is private and known only to that player.

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	30%	20%	35%	15%
Chloe	20%	20%	40%	20%
Chris	25%	20%	20%	35%

■ **TABLE 3-5** Players' valuation of shares for Example 3.7

Step 2 (Bidding). Chan's bid list is $\{s_1, s_3\}$; Chloe's bid list is $\{s_3\}$ only; Chris's bid list is $\{s_1, s_4\}$. (Keep in mind that with 4 players the threshold for a fair share is 25%.)

Step 3 (Distribution). The bid lists are opened. It is clear that for starters Chloe must get s_3—there is no other option. This forces the rest of the distribution: s_1 must then go to Chan, and s_4 goes to Chris. Finally, we give the last remaining piece, s_2, to Demi.

This distribution results in a fair division of the cake, although it is not entirely "envy-free"—Chan wishes he had Chloe's piece (35% is better than 30%) but Chloe is not about to trade pieces with him, so he is stuck with s_1. (From a strictly rational point of view, Chan has no reason to gripe—he did not get the best piece, but got a piece worth 30% of the total, better than the 25% he is entitled to.) ■

EXAMPLE 3.8 LONE DIVIDER WITH 4 PLAYERS (CASE 2)

Once again, we will let Demi be the divider and Chan, Chloe, and Chris be the three choosers (same players, different game).

Step 1 (Division). Demi divides the cake into four shares s_1, s_2, s_3, and s_4. Table 3-6 shows how each of the players values each of the four shares.

Step 2 (Bidding). Chan's bid list is $\{s_4\}$; Chloe's bid list is $\{s_2, s_3\}$; Chris's bid list is $\{s_4\}$.

	s_1	s_2	s_3	s_4
Demi	25%	25%	25%	25%
Chan	20%	20%	20%	40%
Chloe	15%	35%	30%	20%
Chris	22%	23%	20%	35%

TABLE 3-6 Players' valuation of shares for Example 3.8

Step 3 (Distribution). The bid lists are opened, and the players can see that there is a standoff brewing on the horizon—Chan and Chris are both bidding for s_4. The first step is to set s_4 aside and assign Chloe and Demi a fair share from s_1, s_2, and s_3. Chloe could be given either s_2 or s_3. (She would rather have s_2, of course, but it's not for her to decide.) A coin toss is used to determine which one. Let's say Chloe ends up with s_3 (bad luck!). Demi could be now given either s_1 or s_2. Another coin toss, and Demi ends up with s_1. The final move is . . . you guessed it!—recombine s_2 and s_4 into a single piece to be divided between Chan and Chris using the divider-chooser method. Since $(s_2 + s_4)$ is worth 60% to Chan and 58% to Chris (you can check it out in Table 3-6), regardless of how this final division plays out they are both guaranteed a final share worth more than 25% of the cake.

Mission accomplished! We have produced a fair division of the cake.

3.4 The Lone-Chooser Method

A completely different approach for extending the divider-chooser method was proposed in 1964 by A. M. Fink, a mathematician at Iowa State University. In this method one player plays the role of chooser, all the other players start out playing the role of dividers. For this reason, the method is known as the **lone-chooser method**. Once again, we will start with a description of the method for the case of three players.

The Lone-Chooser Method for Three Players

- **Step 0 (Preliminaries).** We have one chooser and two dividers. Let's call the chooser C and the dividers D_1 and D_2. As usual, we decide who is what by a random draw.
- **Step 1 (Division).** D_1 and D_2 divide S [Fig. 3-4(a)] between themselves into *two* fair shares. To do this, they use the divider-chooser method. Let's say that D_1 ends up with s_1 and D_2 ends up with s_2 [Fig. 3-4(b)].
- **Step 2 (Subdivision).** Each divider divides his share into three subshares. Thus, D_1 divides s_1 into three subshares, which we will call s_{1a}, s_{1b}, and s_{1c}. Likewise, D_2 divides s_2 into three subshares, which we will call s_{2a}, s_{2b}, and s_{2c} [Fig. 3-4(c)].
- **Step 3 (Selection).** The chooser C now selects one of D_1's three subshares and one of D_2's three subshares [Fig. 3-4(d)]. These two subshares make up C's final share. D_1 then keeps the remaining two subshares from s_1, and D_2 keeps the remaining two subshares from s_2.

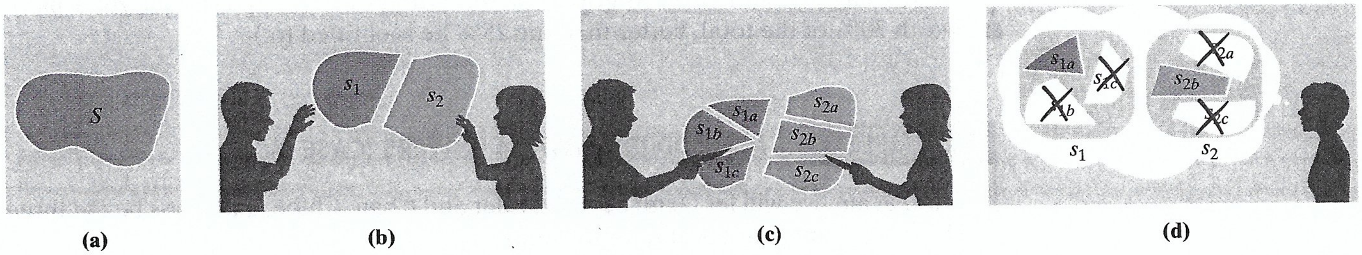


FIGURE 3-4 (a) The original S, (b) first division, (c) second division, and (d) selection.

Why is this a fair division of S? D_1 ends up with two-thirds of s_1. To D_1, s_1 is worth at least one-half of the total value of S, so two-thirds of s_1 is at least one-third—a fair share. The same argument applies to D_2. What about the chooser's share? We don't know what s_1 and s_2 are each worth to C, but it really doesn't matter—a one-third or better share of s_1 plus a one-third or better share of s_2 equals a one-third or better share of $(s_1 + s_2)$ and, thus, a fair share of the cake.

The following example illustrates in detail how the lone-chooser method works with three players.

EXAMPLE 3.9 LONE CHOOSER WITH 3 PLAYERS

David, Dinah, and Cher are dividing the orange-pineapple cake shown in Fig. 3-5 using the lone-chooser method. The cake is valued by each of them at $27, so each of them expects to end up with a share worth at least $9.

Their individual value systems (not known to one another, but available to us as outside observers) are as follows:

- David likes pineapple and orange the same [Fig. 3-5(a)].
- Dinah likes orange but hates pineapple [Fig. 3-5(b)].
- Cher likes pineapple twice as much as she likes orange [Fig. 3-5(c)].

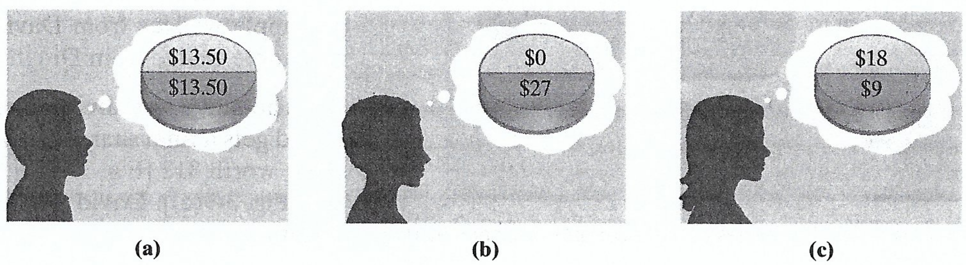


FIGURE 3-5 The values of the pineapple and orange parts in the eyes of each player.

After a random selection, Cher gets to be the chooser and, thus, gets to sit out Steps 1 and 2.

Step 1 (Division). David and Dinah start by dividing the cake between themselves using the divider-chooser method. After a coin flip, David cuts the cake into two pieces, as shown in Fig. 3-6(a). Since Dinah doesn't like pineapple, she will take the share with the most orange [Fig. 3-6(c)].

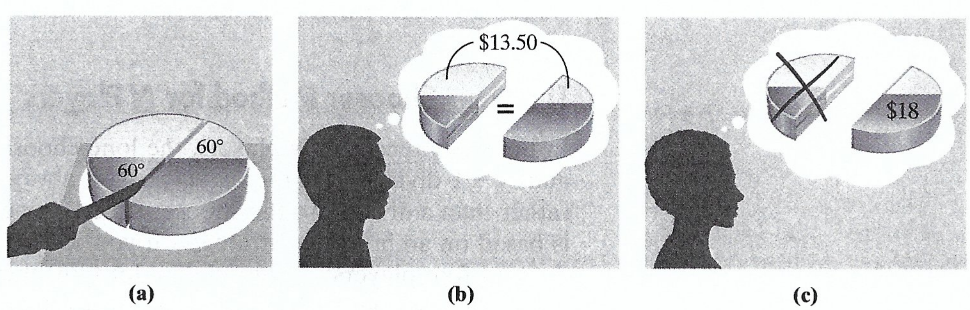
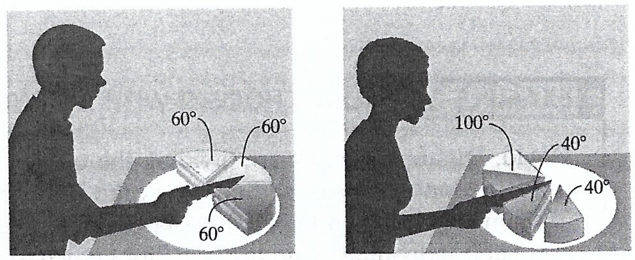


FIGURE 3-6 The first cut and the values of the shares in the eyes of each player.

Step 2 (Subdivision). David divides his share into three subshares that in his opinion are of equal value. Notice that the subshares [Fig. 3-7(a)] are all the same size. Dinah also divides her share into three smaller subshares that in her opinion are of equal value [Fig. 3-7(b)]. (Remember that Dinah hates pineapple. Thus, she has made her cuts in such a way as to have one-third of the orange in each of the subshares.)



(a) (b)

FIGURE 3-7 (a) David cuts his share. (b) Dinah cuts her share.

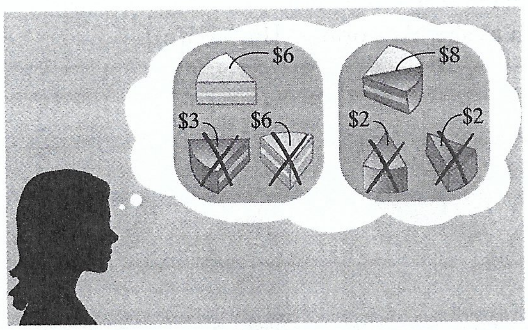
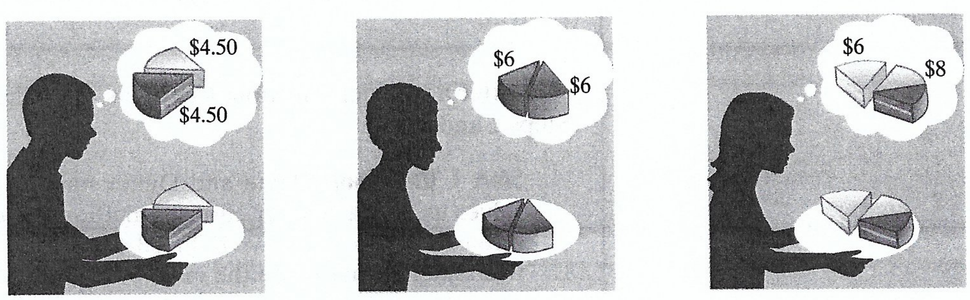


FIGURE 3-8 The values of the subshares in Cher's eyes.

Step 3 (Selection). It's now Cher's turn to choose one subshare from David's three and one subshare from Dinah's three. Figure 3-8 shows the values of the subshares in Cher's eyes. It's clear what her choices will be: She will choose one of the two pineapple wedges from David's subshares and the big orange-pineapple wedge from Dinah's subshares.

The final fair division of the cake is shown in Fig. 3-9: David gets a final share worth $9 [Fig. 3-9(a)], Dinah gets a final share worth $12 [Fig. 3-9(b)], and Cher gets a final share worth $14 [Fig. 3-9(c)]. David is satisfied, Dinah is happy, and Cher is ecstatic.



(a) (b) (c)

FIGURE 3-9 Each player's final fair share.

The Lone-Chooser Method for *N* Players

In the general case of N players, the lone-chooser method involves one chooser C and $N-1$ dividers $D_1, D_2, \ldots, D_{N-1}$. As always, it is preferable to be a chooser rather than a divider, so the chooser is determined by a random draw. The method is based on an inductive strategy—if you can do it for three players, then you can do it for four players; if you can do it for four, then you can do it for five; and so on. Thus, when we get to N players, we can assume that we can use the lone-chooser method with $N-1$ players.

- **Step 1 (Division).** $D_1, D_2, \ldots, D_{N-1}$ divide fairly the set S among themselves, as if C didn't exist. This is a fair division among $N-1$ players, so each one gets a share he or she considers worth at least $\frac{1}{(N-1)}$th of S.
- **Step 2 (Subdivision).** Each divider subdivides his or her share into N subshares.

- **Step 3 (Selection).** The chooser C finally gets to play. C selects one subshare from each divider—one subshare from D_1, one from D_2, and so on. At the end, C ends up with $N-1$ subshares, which make up C's final share, and each divider gets to keep the remaining $N-1$ subshares in his or her subdivision.

When properly played, the lone-chooser method guarantees that everyone, dividers and chooser alike, ends up with a fair share (see Exercise 71).

In the next two sections we will discuss *discrete* fair-division methods—methods for dividing assets consisting of indivisible objects such as houses, cars, art, jewels, or candy. As a general rule of thumb, discrete fair division is harder to achieve than continuous fair division because there is a lot less flexibility in the division process, and discrete fair divisions that are truly fair are only possible under a limited set of conditions. Thus, it is important to keep in mind that while both of the methods we will discuss in the next two sections have limitations, they still are the best methods we have available. Moreover, when they work, both methods work remarkably well and produce surprisingly good fair divisions.

3.5 The Method of Sealed Bids

The **method of sealed bids** was originally proposed by the Polish mathematicians Hugo Steinhaus and Bronislaw Knaster around 1948. The best way to illustrate how this method works is by means of an example.

EXAMPLE 3.10 SETTLING GRANDMA'S ESTATE

In her last will and testament, Grandma plays a little joke on her four grandchildren (Art, Betty, Carla, and Dave) by leaving just three items—a cabin in the mountains, a vintage 1955 Rolls Royce, and a Picasso painting—with the stipulation that the items must remain with the grandchildren (not sold to outsiders) and must be divided fairly among them. How can we possibly resolve this conundrum? The method of sealed bids will give an ingenious and elegant solution.

- **Step 1 (Bidding).** Each of the players makes a bid (in dollars) for each of the items in the estate, giving his or her honest assessment of the actual value of each item. To satisfy the privacy assumption, it is important that the bids are done independently, and no player should be privy to another player's bids before making his or her own. The easiest way to accomplish this is for each player to submit his or her bid in a sealed envelope. When all the bids are in, they are opened. The top three rows in Table 3-7 show each player's bid on each item in the estate.

	Art	Betty	Carla	Dave
Cabin	420,000	450,000	411,000	398,000
Vintage Rolls	80,000	70,000	87,000	92,000
Painting	680,000	640,000	634,000	590,000
Total value	1,180,000	1,160,000	1,132,000	1,080,000
Fair-share value	295,000	290,000	283,000	270,000

TABLE 3-7 The original bids and fair-share values for each player in Example 3.10

- **Step 2 (Allocation).** Each item will go to the highest bidder for that item. (If there is a tie, the tie can be broken with a coin flip.) In this example the cabin goes to Betty, the vintage Rolls Royce goes to Dave, and the Picasso painting goes to Art. Notice that Carla gets nothing. Not to worry—it all works out at the end! (In this method it is possible for one player to get none of the items and another player to get many or all of the items. Much like in a silent auction, it's a matter of who bids the highest.)

- **Step 3 (First Settlement).** It's now time to settle things up. Depending on what items (if any) a player gets in Step 2, he or she will owe money to or be owed money by the estate. To determine how much a player owes or is owed, we first calculate each player's *fair-share* value of the estate (bottom row of Table 3.7). A player's *fair-share* value is found by adding that player's bids and dividing the total by the number of players (in this case 4).

  The fair-share values are the baseline for the settlements—if the total value of the items that the player gets in Step 2 is more than his or her fair-share value, then the player *pays* the estate the difference. If the total value of the items that the player gets is *less* than his or her fair-share value, then the player *gets* the difference in cash. Here are the details of how the settlement works out for each of our four players.

  Table 3-8 illustrates how the first settlements are calculated. Take Art, for example. Art receives the Picasso painting, which he valued at $680,000. But his fair-share value is only $295,000, so Art must make up the difference of $385,000 by paying it to the estate. Likewise, Betty gets the cabin for $450,000 but has to pay $160,000 to the estate. Carla is not getting any items, so she receives the full value of her share ($283,000) in cash *from* the estate. Dave gets the vintage Rolls, which he values at $92,000. Since he is entitled to a fair share valued at $270,000, the difference is given to him in cash from the estate.

	Art	Betty	Carla	Dave	
Item(s) received	Picasso	Cabin	none	Rolls	
Value received	$680,000	$450,000	0	$92,000	
Fair-share value	$295,000	$290,000	$283,000	$270,000	Surplus
To (from) estate	$385,000	$160,000	($283,000)	($178,000)	$84,000

TABLE 3-8 First settlement of the estate and surplus for Example 3.10

  At this point each of the four heirs has received a fair share, and we might consider our job done, but this is not the end of the story—there is more to come (good news mostly!). If we add Art and Betty's payments to the estate and subtract the payments made by the estate to Carla and Dave, we discover that there is a *surplus* of $84,000! ($385,000 and $160,000 came in from Art and Betty respectively; $283,000 and $178,000 went out to Carla and Dave.)

- **Step 4 (Division of the Surplus).** The surplus is common money that belongs to the estate, and thus is to be divided equally among the players. In our example each player's share of the $84,000 surplus is $21,000.
- **Step 5 (Final Settlement).** Everything done up to this point could be done on paper, but now, finally, real money needs to change hands! Art gets the Picasso painting and pays the estate $364,000 ($385,000 – $21,000). Betty gets the cabin and has to pay the estate $139,000 ($160,000 – $21,000). Carla gets $284,000 in cash ($263,000 + $21,000). Dave gets the vintage Rolls Royce plus $199,000 ($78,000 + $21,000).

When two players are dividing a commonly owned discrete asset, one of them will end up with the asset and the other one must get an equivalent share of something, usually money. The method of sealed bids handles this situation nicely and provides a solution that makes both players happy (assuming, of course, that they are rational players). Consider the following example.

EXAMPLE 3.11 SPLITTING UP THE HOUSE

Al and Betty are getting a divorce. The only joint property of any value is their house. Rather than hiring attorneys and going to court to figure out how to split up the house, they agree to give the method of sealed bids a try.

Al's bid on the house is $340,000; Betty's bid is $364,000. The fair-share value of the "estate" is $170,000 to Al and $182,000 to Betty. Since Betty is the higher bidder, she gets to keep the house and must pay Al cash for his share. The computation of how much cash Betty pays Al can be done in two steps. In the first settlement, Betty owes the estate $182,000. Of this money, $170,000 pays for Al's fair share, leaving a surplus of $12,000 to be split equally between them. The bottom line is that Betty ends up paying $176,000 to Al for his share of the house. Table 3-9 summarizes the split.

	Al	Betty
Bid for house	$340,000	$364,000
Fair-share value	$170,000	$182,000
To (from) estate	($170,000)	$182,000
Share of surplus	$6,000	$6,000
Final settlement	gets $176,000	gets house, pays $176,000

TABLE 3-9 Method of sealed bids for Example 3.11

The method of sealed bids can provide an excellent solution not only to settlements of property in a divorce but also to the equally difficult and often contentious issue of splitting up a partnership. The catch is that in these kinds of splits we can rarely count on the rationality assumption to hold. A divorce or partnership split devoid of emotion, spite, and hard feelings is a rare thing indeed! ■

Auctions, Reverse Auctions, and Negative Bidding

Most people are familiar with the principle behind a standard **auction** (we can thank eBay for that): A seller has an item of *positive* value (say a vintage car) that he is willing to give up in exchange for money, and there are bidders who are willing to part with their money to get the item. The *highest* bidder gets the item. In some situations the bidders are allowed to make only one secret bid (usually in writing) and the highest bidder gets the item. These types of auctions are called **sealed-bid auctions**. A good way to think of the method of sealed bids is as a variation of a sealed-bid auction.

The key idea behind the method of sealed bids is that for each item there is a sealed-bid auction in which the players are simultaneously sellers and bidders. The highest bidder for the item ends up being the buyer (of the other players' shares), the other players are all sellers (of their shares). The twist is that a player doesn't know whether he will end up a buyer or a seller until the bids are opened. Not knowing forces the player to make honest bids (if he bids too high he runs the risk of buying an overpriced item; if he bids too low he runs the risk of selling an item too cheaply).

The above interpretation of the method of sealed bids helps us understand how the method can also be used to divide negative-valued items such as chores and other unpleasant responsibilities (for lack of a better word let's just call such items "baddies"). The only difference is that now we will model the process after a *reverse auction*. A **reverse auction** is the flip side of a regular auction: There is a buyer who wants to buy a service (for example repairing a leaky roof) and bidders who are willing to provide for this service in exchange for cash. Here the *lowest* bidder gets the job (i.e., gets to sell his services to the buyer). The buyer pays because he doesn't want to climb on the roof and risk breaking his neck, and the low bidder does the job because he wants the cash.

When we combine the basic rules for the method of sealed bids with the idea of a reverse auction, we have a method for dividing any discrete set of baddies. We will illustrate how this works in our next example.

EXAMPLE 3.12 APARTMENT CLEANUP 101

	Anne	Belinda	Clara
1. Clean bathrooms	$120	$80	$75
2. Clean kitchen	$85	$80	$70
3. Patch walls and paint	$180	$170	$210
4. Shampoo carpets	$80	$110	$100
5. Wash windows	$90	$70	$100
Total value	$555	$510	$555
Fair-share value	$185	$170	$185

TABLE 3-10 Bids and fair-share values for each player in Example 3.12

Anne, Belinda, and Clara are college roommates. The school year just ended, and they are getting ready to move out of their apartment. They talk to their landlord about getting their $1200 cleaning deposit back, and he tells them that there are five major chores that they need to do to get their cleaning deposit (or at least most of it) back. Table 3-10 lists the five chores as well as the bids that each roommate made for each chore. Remember that now the bids represent bids for services provided. (When Anne bids $90 for washing the windows she is saying something like "I am willing to wash the windows if I can get $90 in credit toward the final settlement on the cleaning deposit. If someone else wants to do it for less that's fine—let them do it.")

The fair division of the chores is summarized in Table 3-11. Since this is a reverse auction, each chore is assigned to the lowest bidder. Anne ends up with chore 4, Belinda ends up with chores 3 and 5, and Clara ends up with chores 1 and 2. In the first settlement Anne has to "pay" $105 (no money is changing hands yet—this is just a debit on the final settlement of the cleaning deposit); Belinda gets a credit of $70; and Clara has a debit of $40. The details of the calculations are shown in Table 3-11. There is a surplus of $75, so each roommate gets a surplus credit of $25.

	Anne	Belinda	Clara	
Chores assigned (low bidder)	4	3 and 5	1 and 2	
Value of service provided	$80	$240	$145	
Fair-share value of duties	$185	$170	$185	
Debit (credit) toward final settlement	$105	($70)	$40	Surplus $75

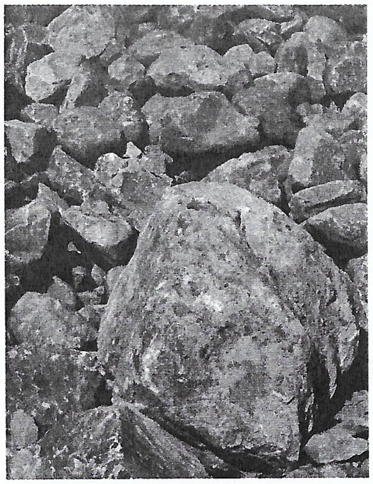
TABLE 3-11 First settlement and surplus for Example 3.12

The final settlement: Because they did such a good job, Anne, Belinda, and Clara get their full $1200 cleaning deposit back. The money is divided as follows:

- Anne, who shampooed the carpets, gets $320.
  [$400 (share of cleaning deposit) + $25 (surplus credit) − $105 (settlement debit)]
- Belinda, who painted walls and cleaned windows, gets $495.
  [$400 (share of cleaning deposit) + $25 (surplus credit) + $70 (settlement credit)]
- Clara, who cleaned the bathrooms and the kitchen, gets $385.
  [$400 (share of cleaning deposit) + $25 (surplus credit) − $40 (settlement debit)]

Whether we are dividing assets ("goodies") or chores ("baddies"), the method of sealed bids offers an elegant and effective solution to the problem of fair division. There are, however, two limitations to the method of sealed bids:

1. Players must have enough money to properly play the game. A player that is cash strapped is at a disadvantage—he may not be able to make the right bids if he cannot back them up. (So, if you are the poor grandson dividing grandma's estate with a bunch of fat-cat cousins, you may want to argue for something other than the method of sealed bids.)
2. Players must accept the fact that all assets are marketable commodities (i.e., the principle that "everything has a price"). When there is an item that more than one player considers "priceless" (some kind of family heirloom, for example), then there are serious issues that the method of sealed bids cannot resolve.



Fine-Grained, Discrete Fair Division

Under the right set of circumstances, a discrete fair-division problem can be solved using continuous fair-division methods. This happens when the set of assets consists of many items with plenty of relatively low-valued items to "smooth" out the division. We call these situations "fine-grained" fair-division games. To clarify the idea, imagine that you have a large pile of rocks and you want to split the rocks into three piles of equal weight. If all you have is big boulders then it is not very likely you will be able to do it, but if you also have a big supply of small rocks (the fine grain, so to speak) then your chances of making three piles of equal weight are quite good.

Using continuous fair-division methods to solve discrete fair-division problems is very helpful—continuous fair-division is easier and has fewer restrictions than discrete fair division. Our next example illustrates how we can use a continuous fair-division method to solve a fine-grained, discrete fair-division problem. It also brings us back full circle to the story of the Elghanayan brothers.

EXAMPLE 3.13 DIVIDING $3 BILLION WORTH OF REAL ESTATE

In 2009, Henry, Tom and Fred Elghanayan agreed to break up Rockrose Development Corporation, the $3-billion New York City real estate empire they jointly owned (about 8000 apartments, 9 office towers, and 9 major development projects). The breakup was reasonably amicable (there were a few bruised egos but no hard feelings), and each of the three brothers ended up with a share that he considered fair. How did they do it?

First, notice that because of the large number of apartments (the small rocks), this is an example of a fine-grained, discrete, fair-division problem. Thus, it was possible to divide the assets into three shares using a variation of the lone-divider method. Fred ended up as the divider (how he ended up with that dubious honor will be explained soon), and his task was to divide the assets into three "piles" of equal value. After Fred was finished with the division (it took him about 60 days to make up the three piles) a coin flip was used to determine which of the other two brothers (the choosers) would get to choose first. Henry won the coin flip, and this determined the order of choice: Henry first, Tom second, Fred last.

As in any lone-divider strategy, being the divider is the least desirable role, and to determine who would end up in that role the brothers used a variation of the method of sealed bids: a reverse auction in which the lowest bidder for the assets (Fred) got to be the divider. This insured that both choosers valued the assets at a higher rate than the divider and guaranteed that Fred, knowing that he would choose last, would divide the assets into three approximately equal shares.

The most important postscript to this story is that all three brothers were happy with the outcome of the split, a very uncommon ending to such a big stakes division.

3.6 The Method of Markers

The **method of markers** is a discrete fair-division method proposed in 1975 by William F. Lucas, a mathematician at the Claremont Graduate School. The method has the great virtue that it does not require the players to put up any of their own money. On the other hand, unlike the method of sealed bids, this method can only be used effectively in the case of a fine-grained, discrete, fair-division game.

In this method we start with the items lined up in a random but fixed sequence called an *array*. Each of the players then gets to make an independent *bid* on the items in the array. A player's bid consists of dividing the array into segments of consecutive items (as many segments as there are players) so that each of the segments represents a fair share of the entire set of items.

For convenience, we might think of the array as a string. Each player then "cuts" the string into N segments, each of which he or she considers an acceptable share. (Notice that to cut a string into N sections, we need $N - 1$ cuts.) In practice, one way to make the "cuts" is to lay markers in the places where the cuts are made. Thus, each player can make his or her bids by placing $N - 1$ markers so that they divide the array into N segments. To ensure privacy, no player should see the markers of another player before laying down his or her own.

The final step is to give to each player one of the segments in his or her bid. The easiest way to explain how this can be done is with an example.

EXAMPLE 3.14 DIVIDING THE POST-HALLOWEEN STASH

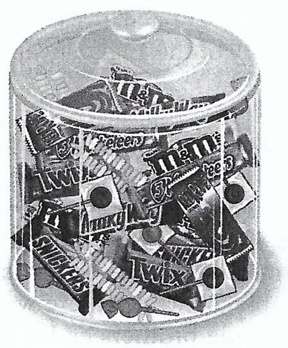


FIGURE 3-10 The Halloween leftovers.

Alice, Bianca, Carla, and Dana want to divide their jointly owned post-Halloween stash of candy (Fig. 3-10). Their mother offers to divide the candy for them, but the girls reply that they just learned about a cool fair-division game they want to try, and they can do it themselves, thank you.

As a preliminary step, the 20 pieces are arranged in an array (Fig. 3-11). For convenience, we will label the pieces of candy 1 through 20. The order in which the pieces are lined up should be random (the easiest way to do this is to dump the pieces into a paper bag, shake the bag, and take the pieces out of the bag one at a time).

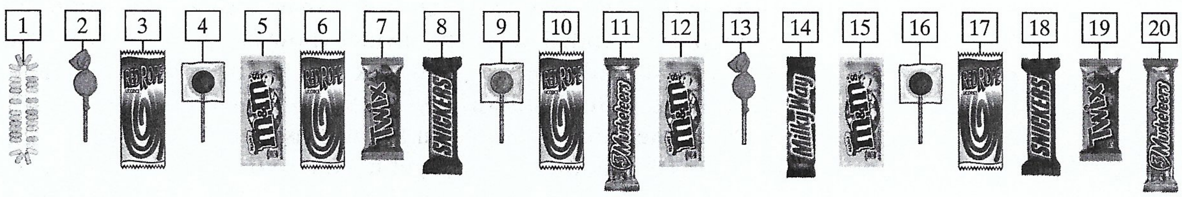


FIGURE 3-11 The items lined up in an array.

- **Step 1 (Bidding).** Each player writes down independently on a piece of paper exactly where she wants to place her three markers. (Three markers divide the array into four sections.) The bids are opened, and the results are shown in Fig. 3-12. The A-labels indicate the position of Alice's markers (A_1 denotes her first marker, A_2 her second marker, and A_3 her third and last marker). Alice's bid means that she is willing to accept one of the following as a fair share of the candy: (1) pieces 1 through 5 (first segment), (2) pieces 6 through 11 (second segment), (3) pieces 12 through 16 (third segment), or (4) pieces 17 through 20 (last segment). Bianca's bid is shown by the B-markers and indicates how she would break up the array into four segments that are fair shares; same for Carla's bid (shown by the C-markers) and Dana's bid (shown by the D-markers).

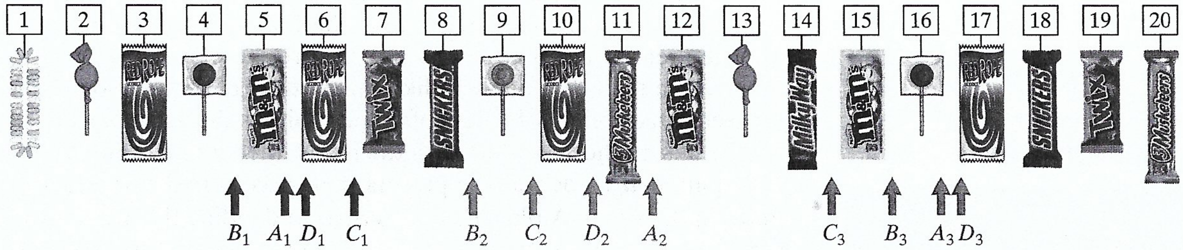


FIGURE 3-12 The bids.

- **Step 2 (Allocations).** This is the tricky part, where we are going to give to each player one of the segments in her bid. Here is how to do it: Scan the array from left to right until the first *first marker* comes up. Here the first *first marker* is Bianca's B_1. This means that Bianca will be the first player to get her fair share, consisting of the first segment in her bid (pieces 1 through 4, Fig. 3-13).

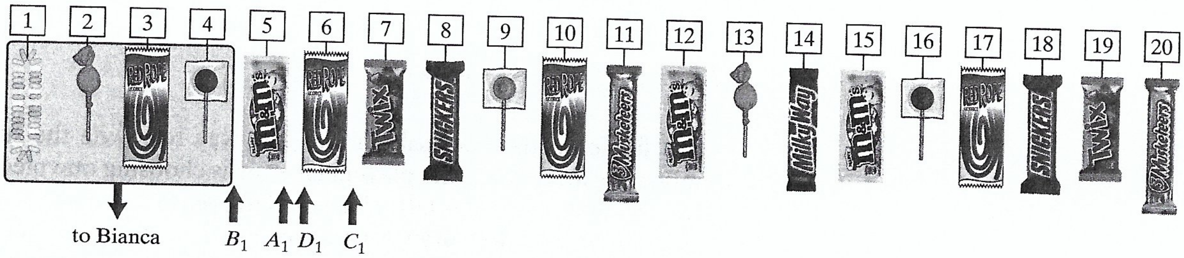


FIGURE 3-13 B_1 is the first 1-marker. Bianca goes first, gets her first segment.

Bianca is done now, and her markers can be removed since they are no longer needed. Continue scanning from left to right looking for the first *second marker*. Here the first second marker is Carla's C_2, so Carla will be the second player taken care of. Carla gets the second segment in her bid (pieces 7 through 9, Fig. 3-14). Carla's remaining markers can now be removed.

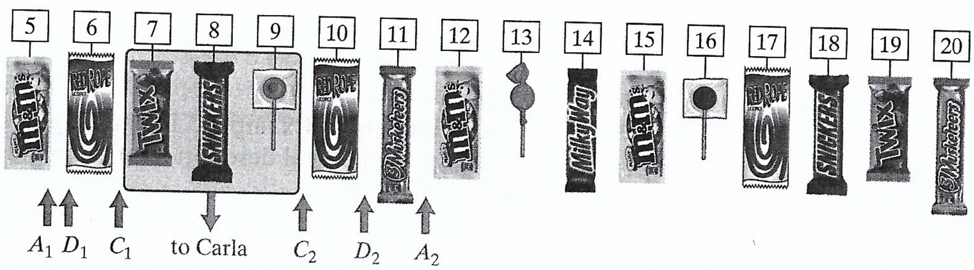


FIGURE 3-14 C_2 is the first 2-marker (among A's, C's, and D's). Carla goes second, gets her second segment.

Continue scanning from left to right looking for the first *third marker*. Here there is a tie between Alice's A_3 and Dana's D_3. As usual, a coin toss is used to break the tie and Alice will be the third player to go—she will get the third segment in her bid (pieces 12 through 16, Fig. 3-15). Dana is the last player

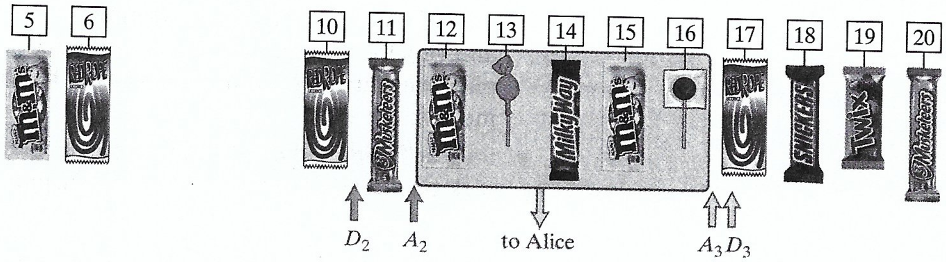


FIGURE 3-15 A_3 and D_3 are tied as the first 3-marker. After a coin toss, Alice gets her third segment.

and gets the last segment in her bid (pieces 17 through 20, Fig. 3-16). At this point each player has gotten a fair share of the 20 pieces of candy. The amazing part is that there is *leftover candy*!

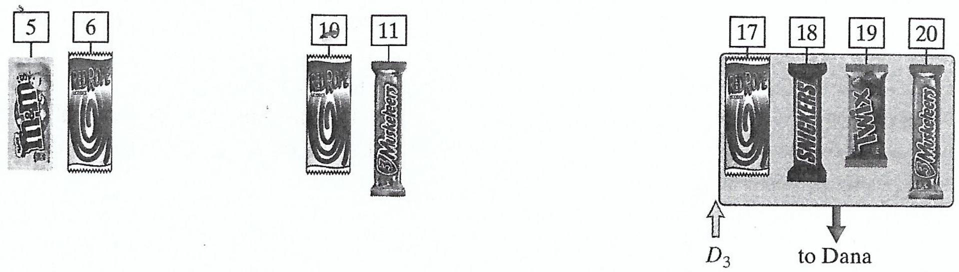


FIGURE 3-16 Dana is last, gets her last segment.

- **Step 3 (Dividing the Surplus).** The easiest way to divide the surplus is to randomly draw lots and let the players take turns choosing one piece at a time until there are no more pieces left. Here the leftover pieces are 5, 6, 10, and 11 (Fig. 3-17). The players now draw lots; Carla gets to choose first and takes piece 11. Dana chooses next and takes piece 5. Bianca and Alice receive pieces 6 and 10, respectively.

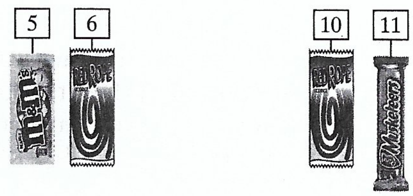


FIGURE 3-17 The surplus pieces (to be given randomly to the players one at a time) are a bonus.

The ideas behind Example 3.14 can be generalized to any number of players. We now give the general description of the method of markers with N players and M indivisible items.

- **Preliminaries.** The items are arranged randomly into an array. For convenience, label the items 1 through M, going from left to right.
- **Step 1 (Bidding).** Each player independently divides the array into N segments (segments $1, 2, \ldots, N$) by placing $N-1$ markers along the array. These segments are assumed to represent the fair shares of the array in the opinion of that player.
- **Step 2 (Allocations).** Scan the array from left to right until the first *first marker* is located. The player owning that marker (let's call him P_1) goes first and gets the first segment in his bid. (In case of a tie, break the tie randomly.) P_1's markers are removed, and we continue scanning from left to right, looking for the first *second marker*. The player owning that marker (let's call her P_2) goes second and gets the second segment in her bid. Continue this process, assigning to each player in turn one of the segments in her bid. The last player gets the last segment in her bid.
- **Step 3 (Dividing the Surplus).** The players take turns in some random order and pick one item at a time until all the surplus items are given out.

Despite its simple elegance, the method of markers can be used only under some fairly restrictive conditions. In particular, the method assumes that every player is able to divide the array of items into segments in such a way that each of the segments has approximately equal value. This is usually possible in a fine-grained, fair-division game, but almost impossible to accomplish when there is a combination of expensive and inexpensive items (good luck trying to divide fairly 19 candy bars plus an iPod using the method of markers!).

Conclusion

The Judgment of Solomon, by Julius Schnorr von Carolsfeld, 1794–1874

Problems of fair division are as old as humankind. One of the best-known and best-loved biblical stories is built around one such problem. Two women, both claiming to be the mother of the same baby, make their case to King Solomon. As a solution King Solomon proposes to cut the baby in two and give each woman a share. (Basically, King Solomon was proposing a continuous solution to a discrete fair-division problem!) This solution is totally unacceptable to the true mother, who would rather see the baby go to the other woman than have it "divided" into two shares. The final settlement, of course, is that the baby is returned to its rightful mother.

The problem of dividing an object or a set of objects among the members of a group is a practical problem that comes up regularly in our daily lives. When we are dividing a pizza, a cake, or a bunch of candy, we don't always pay a great deal of attention to the issue of fairness, but when we are dividing real estate, jewelry, or some other valuable asset, dividing things fairly becomes a critical issue.

This chapter gave an overview of an interesting and surprising application of mathematics to one of the fundamental questions of social science—how to get humans to share their common assets in a rational and fair way.

KEY CONCEPTS

3.1 Fair-Division Games

- **assets:** the items or things being divided; usually of positive value, but can also be things with negative value such as liabilities, obligations, or chores, **70**
- **players:** the parties that jointly own the assets and wish to divide them among themselves, **70**
- **value system:** the opinion that each player has regarding the value of the assets or any part thereof, **70**
- **fair share (proportional fair share):** to a player, a share that in the player's opinion is worth at least $(\frac{1}{N})$ of the total value of the assets (where N denotes the number of players), **71**
- **fair division:** a division of the assets that gives each player a fair share, **71**
- **fair-division method:** a procedure that guarantees as its outcome a fair division of the assets, **71**
- **continuous fair-division:** a division involving assets that can be divided in infinite ways and by making arbitrarily small changes, **72**
- **discrete fair-division:** a division involving assets consisting of indivisible objects or objects that can only be divided up to a point, **72**

3.2 The Divider-Chooser Method

- **divider-chooser method:** two players; one cuts the assets into two shares, and the other one chooses one of the shares, **72**

3.3 The Lone-Divider Method

- **lone-divider method:** N players $(N \geq 2)$; the lone divider cuts the assets into N shares; the others (choosers) declare which shares they consider to be fair, **74**

3.4 The Lone-Chooser Method

- **lone-chooser method:** N players $(N \geq 2)$; all but one player (the dividers) divide the assets fairly among themselves, and each then divides his or her share into N sub-shares; the remaining player (chooser) picks one sub-share from each divider, **78**

3.5 The Method of Sealed Bids

- **method of sealed bids:** each player bids a dollar value for each item with the item going to the highest bidder; the other players get cash from the winning bid for their equity on the item, **81**
- **reverse auction:** an auction in which the item being auctioned is a job or a chore; the lowest bidder gets the amount of his or her bid as payment for doing that job or chore, **83**

3.6 The Method of Markers

- **method of markers:** each player bids on how to split an array of the items into N sections; the player with the "smallest" bid for the first section gets it; among the remaining players, the player with the "smallest" bid for the second section gets it, and so on until every player gets one of his or her sections, **85**

EXERCISES

WALKING

3.1 Fair-Division Games

1. Henry, Tom, and Fred are dividing among themselves a set of common assets equally owned by the three of them. The assets are divided into three shares (s_1, s_2, and s_3). Table 3-12 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Henry?

(b) Which of the shares are fair shares to Tom?

(c) Which of the shares are fair shares to Fred?

(d) Find all possible fair divisions of the assets using s_1, s_2, and s_3 as shares.

(e) Of the fair divisions found in (d), which one is the best?

	s_1	s_2	s_3
Henry	25%	40%	35%
Tom	28%	35%	37%
Fred	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%

■ **TABLE 3-12**

2. Alice, Bob, and Carlos are dividing among themselves a set of common assets equally owned by the three of them. The assets are divided into three shares (s_1, s_2, and s_3). Table 3-13 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Alice?

(b) Which of the shares are fair shares to Bob?

(c) Which of the shares are fair shares to Carlos?

(d) Find all possible fair divisions of the assets using s_1, s_2, and s_3 as shares.

(e) Of the fair divisions found in (d), which one is the best?

	s_1	s_2	s_3
Alice	38%	28%	34%
Bob	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%
Carlos	34%	40%	26%

■ **TABLE 3-13**

3. Angie, Bev, Ceci, and Dina are dividing among themselves a set of common assets equally owned by the four of them. The assets are divided into four shares (s_1, s_2, s_3, and s_4). Table 3-14 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Angie?

(b) Which of the shares are fair shares to Bev?

(c) Which of the shares are fair shares to Ceci?

(d) Which of the shares are fair shares to Dina?

(e) Find all possible fair divisions of the assets using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Angie	22%	26%	28%	24%
Bev	25%	26%	22%	27%
Ceci	20%	30%	27%	23%
Dina	25%	25%	25%	25%

■ **TABLE 3-14**

4. Mark, Tim, Maia, and Kelly are dividing among themselves a set of common assets equally owned by the four of them. The assets are divided into four shares (s_1, s_2, s_3, and s_4). Table 3-15 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Mark?

(b) Which of the shares are fair shares to Tim?

(c) Which of the shares are fair shares to Maia?

(d) Which of the shares are fair shares to Kelly?

(e) Find all possible fair divisions of the assets using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Mark	20%	32%	28%	20%
Tim	25%	25%	25%	25%
Maia	15%	15%	30%	40%
Kelly	24%	26%	24%	26%

■ **TABLE 3-15**

5. Allen, Brady, Cody, and Diane are sharing a cake. The cake had previously been divided into four slices (s_1, s_2, s_3, and s_4). Table 3-16 shows the values of the slices in the eyes of each player.

(a) Which of the slices are fair shares to Allen?

(b) Which of the slices are fair shares to Brady?

(c) Which of the slices are fair shares to Cody?

(d) Which of the slices are fair shares to Diane?

(e) Find all possible fair divisions of the cake using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Allen	$4.00	$5.00	$6.00	$5.00
Brady	$3.00	$3.50	$4.00	$5.50
Cody	$6.00	$4.50	$3.50	$4.00
Diane	$7.00	$4.00	$4.00	$5.00

■ **TABLE 3-16**

6. Carlos, Sonya, Tanner, and Wen are sharing a cake. The cake had previously been divided into four slices (s_1, s_2, s_3, and s_4). Table 3-17 shows the values of the slices in the eyes of each player.

(a) Which of the slices are fair shares to Carlos?

(b) Which of the slices are fair shares to Sonya?

(c) Which of the slices are fair shares to Tanner?

(d) Which of the slices are fair shares to Wen?

(e) Find all possible fair divisions of the cake using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Carlos	$3.00	$5.00	$5.00	$3.00
Sonya	$4.50	$3.50	$4.50	$5.50
Tanner	$4.25	$4.50	$3.50	$3.75
Wen	$5.50	$4.00	$4.50	$6.00

■ **TABLE 3-17**

7. Alex, Betty, and Cindy are sharing a cake. The cake had previously been divided into three slices (s_1, s_2, and s_3). Table 3-18 shows the values of s_1 and s_2 to each player expressed as a percent of the total value of the cake.

(a) Which of the slices are fair shares to Alex?

(b) Which of the slices are fair shares to Betty?

(c) Which of the slices are fair shares to Cindy?

(d) Find a fair division of the cake using s_1, s_2, and s_3 as shares. If no such fair division is possible, explain why.

	s_1	s_2
Alex	30%	40%
Betty	31%	35%
Cindy	30%	35%

■ TABLE 3-18

8. Alex, Betty, and Cindy are sharing a cake. The cake had previously been divided into three slices (s_1, s_2, and s_3). Table 3-19 shows the values of s_1 and s_2 to each player expressed as a percent of the total value of the cake.

(a) Which of the slices are fair shares to Alex?

(b) Which of the slices are fair shares to Betty?

(c) Which of the slices are fair shares to Cindy?

(d) Find a fair division of the cake using s_1, s_2, and s_3 as shares. If no such fair division is possible, explain why.

	s_1	s_2
Alex	30%	34%
Betty	28%	36%
Cindy	30%	$33\frac{1}{3}$%

■ TABLE 3-19

9. Four partners (Adams, Benson, Cagle, and Duncan) jointly own a piece of land with a market value of $400,000. Suppose that the land is subdivided into four parcels s_1, s_2, s_3, and s_4. The partners are planning to split up, with each partner getting one of the four parcels.

(a) To Adams, s_1 is worth $40,000 more than s_2, s_2 and s_3 are equal in value, and s_4 is worth $20,000 more than s_1. Determine which of the four parcels are fair shares to Adams.

(b) To Benson, s_1 is worth $40,000 more than s_2, s_4 is $8,000 more than s_3, and together s_4 and s_3 have a combined value equal to 40% of the value of the land. Determine which of the four parcels are fair shares to Benson.

(c) To Cagle, s_1 is worth $40,000 more than s_2 and $20,000 more than s_4, and s_3 is worth twice as much as s_4. Determine which of the four parcels are fair shares to Cagle.

(d) To Duncan, s_1 is worth $4,000 more than s_2; s_2 and s_3 have equal value; and s_1, s_2, and s_3 have a combined value equal to 70% of the value of the land. Determine which of the four parcels are fair shares to Duncan.

(e) Find a fair division of the land using the parcels s_1, s_2, s_3, and s_4 as fair shares.

10. Four players (Abe, Betty, Cory, and Dana) are sharing a cake. Suppose that the cake is divided into four slices s_1, s_2, s_3, and s_4.

(a) To Abe, s_1 is worth $3.60, s_4 is worth $3.50, s_2 and s_3 have equal value, and the entire cake is worth $15.00. Determine which of the four slices are fair shares to Abe.

(b) To Betty, s_2 is worth twice as much as s_1, s_3 is worth three times as much as s_1, and s_4 is worth four times as much as s_1. Determine which of the four slices are fair shares to Betty.

(c) To Cory, s_1, s_2, and s_4 have equal value, and s_3 is worth as much as s_1, s_2, and s_4 combined. Determine which of the four slices are fair shares to Cory.

(d) To Dana, s_1 is worth $1.00 more than s_2, s_3 is worth $1.00 more than s_1, s_4 is worth $3.00, and the entire cake is worth $18.00. Determine which of the four slices are fair shares to Dana.

(e) Find a fair division of the cake using s_1, s_2, s_3, and s_4 as fair shares.

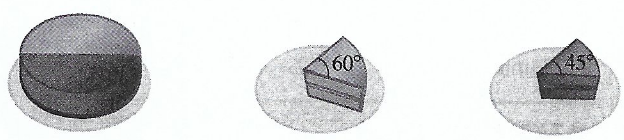
11. Angelina and Brad jointly buy the chocolate-strawberry mousse cake shown in Fig. 3-18(a) for $36. Suppose that Angelina values chocolate cake *twice* as much as she values strawberry cake. Find the dollar value to Angelina of each of the following pieces of cake:

(a) the strawberry half of the cake

(b) the chocolate half of the cake

(c) the slice of strawberry cake shown in Fig. 3-18(b)

(d) the slice of chocolate cake shown in Fig. 3-18(c)



(a) (b) (c)

FIGURE 3-18

12. Brad and Angelina jointly buy the chocolate-strawberry mousse cake shown in Fig. 3-18(a) for $36. Suppose that Brad values strawberry cake *three* times as much as he values chocolate cake. Find the dollar value to Brad of each of the following pieces of cake:

(a) the strawberry half of the cake

(b) the chocolate half of the cake

(c) the slice of strawberry cake shown in Fig. 3-18(b)

(d) the slice of chocolate cake shown in Fig. 3-18(c)

13. Karla and five other friends jointly buy the chocolate-strawberry-vanilla cake shown in Fig. 3-19(a) for \$30 and plan to divide the cake fairly among themselves. After much discussion, the cake is divided into the six equal-sized slices $s_1, s_2, \ldots, s_6$ shown in Fig. 3-19(b). Suppose that Karla values strawberry cake *twice* as much as vanilla cake and chocolate cake *three* times as much as vanilla cake.

(a) Find the dollar value to Karla of each of the slices s_1 through s_6.

(b) Which of the slices s_1 through s_6 are fair shares to Karla?

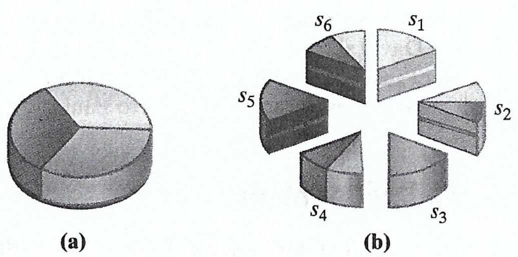


FIGURE 3-19

14. Marla and five other friends jointly buy the chocolate-strawberry-vanilla cake shown in Fig. 3-19(a) for \$30 and plan to divide the cake fairly among themselves. After much discussion, the cake is divided into the six equal-sized slices $s_1, s_2, \ldots, s_6$ shown in Fig. 3-19(b). Suppose that Marla values vanilla cake *twice* as much as chocolate cake and chocolate cake *three* times as much as strawberry cake.

(a) Find the dollar value to Marla of each of the slices s_1 through s_6.

(b) Which of the slices s_1 through s_6 are fair shares to Marla?

3.2 The Divider-Chooser Method

Exercises 15 and 16 refer to the following situation: Jared and Karla jointly bought the half meatball–half vegetarian foot-long sub shown in Fig. 3-20 for \$8.00. They plan to divide the sandwich fairly using the divider-chooser method. Jared likes meatball subs three times as much as vegetarian subs; Karla is a strict vegetarian and does not eat meat at all. Assume that Jared just met Karla and has no idea that she is a vegetarian. Assume also that when the sandwich is cut, the cut is made perpendicular to the length of the sandwich. (You can describe different shares of the sandwich using the ruler and interval notation. For example, [0, 6] *describes the vegetarian half,* [6, 8] *describes one-third of the meatball half, etc.).*

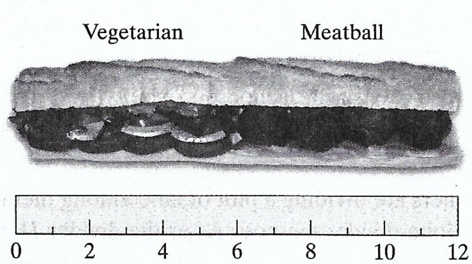


FIGURE 3-20

15. Suppose that they flip a coin and Jared ends up being the divider.

(a) Describe how Jared should cut the sandwich into two shares s_1 and s_2.

(b) After Jared cuts, Karla gets to choose. Specify which of the two shares Karla should choose and give the value of the share to Karla.

16. Suppose they flip a coin and Karla ends up being the divider.

(a) Describe how Karla should cut the sandwich into two shares s_1 and s_2.

(b) After Karla cuts, Jared gets to choose. Specify which of the two shares Jared should choose and give the value of the share to Jared.

Exercises 17 and 18 refer to the following situation: Martha and Nick jointly bought the giant 28-in. sub sandwich shown in Fig. 3-21 for \$9. They plan to divide the sandwich fairly using the divider-chooser method. Martha likes ham subs twice as much as she likes turkey subs, and she likes turkey and roast beef subs the same. Nick likes roast beef subs twice as much as he likes ham subs, and he likes ham and turkey subs the same. Assume that Nick and Martha just met and know nothing of each other's likes and dislikes. Assume also that when the sandwich is cut, the cut is made perpendicular to the length of the sandwich. (You can describe different shares of the sandwich using the ruler and interval notation. For example, [0, 8] *describes the ham part,* [8, 12] *describes one-third of the turkey part, etc.).*

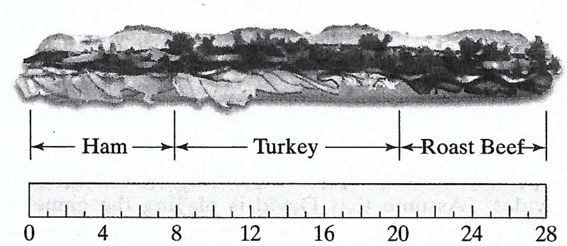


FIGURE 3-21

17. Suppose that they flip a coin and Martha ends up being the divider.

(a) Describe how Martha would cut the sandwich into two shares s_1 and s_2.

(b) After Martha cuts, Nick gets to choose. Specify which of the two shares Nick should choose, and give the value of the share to Nick.

18. Suppose that they flip a coin and Nick ends up being the divider.

(a) Describe how Nick would cut the sandwich into two shares s_1 and s_2.

(b) After Nick cuts, Martha gets to choose. Specify which of the two shares Martha should choose and give the value of the share to Martha.

Exercises 19 and 20 refer to the following situation: David and Paula are planning to divide the pizza shown in Fig. 3-22(a) using the divider-chooser method. David likes pepperoni and sausage pizza equally well, and he likes sausage pizza twice as much as mushroom pizza. Paula likes sausage and mushroom pizza equally well, but she hates pepperoni pizza.

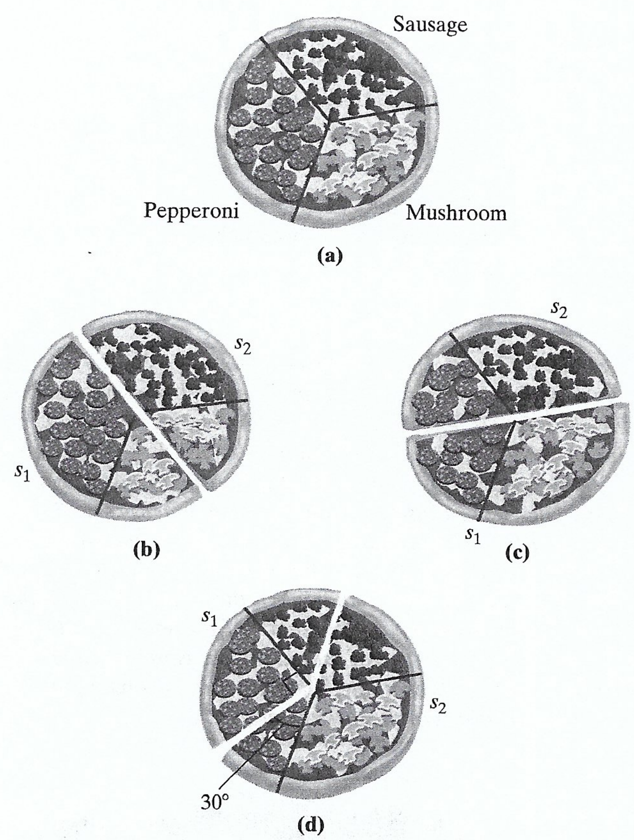


FIGURE 3-22

19. Suppose that they flip a coin and David ends up being the divider. (Assume that David is playing the game by the rules and knows nothing about Paula's likes and dislikes.)

(a) Is the cut shown in Fig. 3-22(b) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

(b) Is the cut shown in Fig. 3-22(c) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

(c) Is the cut shown in Fig. 3-22(d) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

20. Suppose that they flip a coin and Paula ends up being the divider. (Assume that Paula is playing the game by the rules and knows nothing about David's likes and dislikes.)

(a) Is the cut shown in Fig. 3-22(b) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

(b) Is the cut shown in Fig. 3-22(c) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

(c) Is the cut shown in Fig. 3-22(d) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

3.3 The Lone-Divider Method

21. Three partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into three shares s_1, s_2, and s_3, the choosers C_1 and C_2 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_2, s_3\}$. Describe *two* different fair divisions of the land.

(b) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_1, s_3\}$. Describe *three* different fair divisions of the land.

22. Three partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into three shares s_1, s_2, and s_3, the choosers C_1 and C_2 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$. Describe *two* different fair divisions of the land.

(b) Suppose that the choosers' bids are C_1: $\{s_1, s_2\}$; C_2: $\{s_2, s_3\}$. Describe *three* different fair divisions of the land.

23. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_2, s_3\}$. Find a fair division of the land. Explain why this is the only possible fair division.

(b) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_2\}$. Describe *two* different fair divisions of the land.

(c) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_4\}$. Describe *three* different fair divisions of the land.

24. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_2, s_3\}$. Find a fair division of the land. Explain why this is the only possible fair division.

(b) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_4\}$. Describe *three* different fair divisions of the land.

(c) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_2, s_3\}$; C_3: $\{s_2, s_3, s_4\}$. Describe *three* different fair divisions of the land.

25. Mark, Tim, Maia, and Kelly are dividing a cake among themselves using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, and s_4). Table 3-20 shows the values of the slices to each player expressed as a percent of the total value of the cake.

(a) Who was the divider?

(b) Find a fair division of the cake.

	s_1	s_2	s_3	s_4
Mark	20%	32%	28%	20%
Tim	25%	25%	25%	25%
Maia	15%	15%	30%	40%
Kelly	24%	24%	24%	28%

■ **TABLE 3-20**

26. Allen, Brady, Cody, and Diane are sharing a cake valued at \$20 using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, and s_4). Table 3-21 shows the values of the slices in the eyes of each player.

(a) Who was the divider?

(b) Find a fair division of the cake.

	s_1	s_2	s_3	s_4
Allen	\$4.00	\$5.00	\$4.00	\$7.00
Brady	\$6.00	\$6.50	\$4.00	\$3.50
Cody	\$5.00	\$5.00	\$5.00	\$5.00
Diane	\$7.00	\$4.50	\$4.00	\$4.50

■ **TABLE 3-21**

27. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit the following bids: C_1: $\{s_2\}$; C_2: $\{s_1, s_2\}$; C_3: $\{s_1, s_2\}$. For each of the following possible divisions, determine if it is a fair division or not. If not, explain why not.

(a) D gets s_3; s_1, s_2, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(b) D gets s_1; s_2, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(c) D gets s_4; s_1, s_2, and s_3 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(d) D gets s_3; C_1 gets s_2; and s_1, s_4 are recombined into a single piece that is then divided fairly between C_2 and C_3 using the divider-chooser method.

28. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit the following bids: C_1: $\{s_3, s_4\}$; C_2: $\{s_4\}$; C_3: $\{s_3\}$. For each of the following possible divisions, determine if it is a fair division or not. If not, explain why not.

(a) D gets s_1; s_2, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(b) D gets s_3; s_1, s_2, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(c) D gets s_2; s_1, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(d) C_2 gets s_4; C_3 gets s_3; s_1, s_2 are recombined into a single piece that is then divided fairly between C_1 and D using the divider-chooser method.

29. Five players are dividing a cake among themselves using the lone-divider method. After the divider D cuts the cake into five slices (s_1, s_2, s_3, s_4, s_5), the choosers C_1, C_2, C_3, and C_4 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_4\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_3, s_5\}$; C_4: $\{s_2, s_3, s_4\}$. Describe *two* different fair divisions of the cake. Explain why that's it—why there are no others.

(b) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_3, s_5\}$; C_4: $\{s_2, s_3, s_4\}$. Find a fair division of the cake. Explain why that's it—there are no others.

30. Five players are dividing a cake among themselves using the lone-divider method. After the divider D cuts the cake into five slices (s_1, s_2, s_3, s_4, s_5), the choosers C_1, C_2, C_3, and C_4 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_1, s_2\}$; C_4: $\{s_1, s_3, s_4\}$. Describe *three* different fair divisions of the land. Explain why that's it—why there are no others.

(b) Suppose that the choosers' bids are C_1: $\{s_1, s_4\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_4, s_5\}$; C_4: $\{s_2\}$. Find a fair division of the land. Explain why that's it—why there are no others.

31. Four partners (Egan, Fine, Gong, and Hart) jointly own a piece of land with a market value of \$480,000. The partnership is breaking up, and the partners decide to divide the land among themselves using the lone-divider method. Using a map, the divider divides the property into four parcels s_1, s_2, s_3, and s_4. Table 3-22 shows the value of some of the parcels in the eyes of each partner.

	s_1	s_2	s_3	s_4
Egan	\$80,000	\$85,000		\$195,000
Fine		\$100,000	\$135,000	\$120,000
Gong			\$120,000	
Hart	\$95,000	\$100,000		\$110,000

TABLE 3-22

(a) Who was the divider? Explain.

(b) Determine each chooser's bid.

(c) Find a fair division of the property.

32. Four players (Abe, Betty, Cory, and Dana) are dividing a pizza worth \$18.00 among themselves using the lone-divider method. The divider divides the pizza into four shares s_1, s_2, s_3, and s_4. Table 3-23 shows the value of some of the slices in the eyes of each player.

	s_1	s_2	s_3	s_4
Abe	\$5.00	\$5.00	\$3.50	
Betty		\$4.50		
Cory	\$4.80	\$4.20	\$4.00	
Dana	\$4.00	\$3.75	\$4.25	

TABLE 3-23

(a) Who was the divider? Explain.

(b) Determine each chooser's bid.

(c) Find a fair division of the pizza.

Exercises 33 and 34 refer to the following situation: Jared, Karla, and Lori are planning to divide the half vegetarian–half meatball foot-long sub sandwich shown in Fig. 3-23 among themselves using the lone-divider method. Jared likes the meatball and vegetarian parts equally well; Karla is a strict vegetarian and does not eat meat at all; Lori likes the meatball part twice as much as the vegetarian part. (Assume that when the sandwich is cut, the cuts are always made perpendicular to the length of the sandwich. You can describe different shares of the sandwich using the ruler and interval notation—for example, [0, 6] *describes the vegetarian half,* [6, 8] *describes one-third of the meatball half, etc.)*

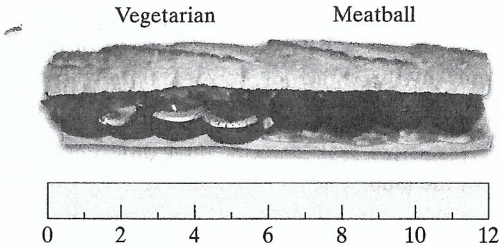


FIGURE 3-23

33. Suppose that Jared is the divider.

(a) Describe how Jared should cut the sandwich into three shares. Label the three shares s_1 for the leftmost piece, s_2 for the middle piece, and s_3 for the rightmost piece. Use the ruler and interval notation to describe the three shares. (Assume that Jared knows nothing about Karla and Lori's likes and dislikes.)

(b) Which of the three shares are fair shares to Karla?

(c) Which of the three shares are fair shares to Lori?

(d) Find three different fair divisions of the sandwich.

34. Suppose that Lori ends up being the divider.

(a) Describe how Lori should cut the sandwich into three shares. Label the three shares s_1 for the leftmost piece, s_2 for the middle piece, and s_3 for the rightmost piece. Use the ruler and interval notation to describe the three shares. (Assume that Lori knows nothing about Karla and Jared's likes and dislikes.)

(b) Which of the three shares are fair shares to Jared?

(c) Which of the three shares are fair shares to Karla?

(d) Suppose that Lori gets s_3. Describe how to proceed to find a fair division of the sandwich.

3.4 The Lone-Chooser Method

Exercises 35 through 38 refer to the following situation: Angela, Boris, and Carlos are dividing the vanilla-strawberry cake shown in Fig. 3-24(a) using the lone-chooser method. Figure 3-24(b) shows how each player values each half of the cake. In your answers assume that all cuts are normal "cake cuts" from the center to the edge of the cake. You can describe each piece of cake by giving the angles of the vanilla and strawberry parts, as in "15° strawberry–40° vanilla" or "60° vanilla only."

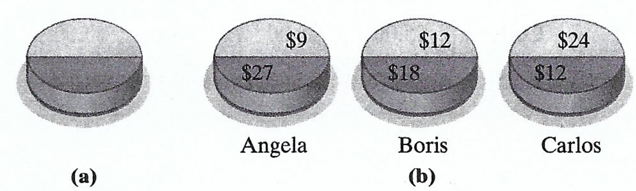


FIGURE 3-24

35. Suppose that Angela and Boris are the dividers and Carlos is the chooser. In the first division, Boris cuts the cake vertically through the center as shown in Fig. 3-25, with Angela choosing s_1 (the left half) and Boris s_2 (the right half). In the second division, Angela subdivides s_1 into three pieces and Boris subdivides s_2 into three pieces.

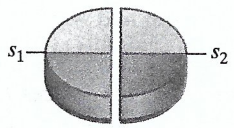


FIGURE 3-25

(a) Describe how Angela would subdivide s_1 into three pieces.

(b) Describe how Boris would subdivide s_2 into three pieces.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

36. Suppose that Carlos and Angela are the dividers and Boris is the chooser. In the first division, Carlos cuts the cake vertically through the center as shown in Fig. 3-25, with Angela choosing s_1 (the left half) and Carlos s_2 (the right half). In the second division, Angela subdivides s_1 into three pieces and Carlos subdivides s_2 into three pieces.

(a) Describe how Carlos would subdivide s_2 into three pieces.

(b) Describe how Angela would subdivide s_1 into three pieces.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

37. Suppose that Angela and Boris are the dividers and Carlos is the chooser. In the first division, Angela cuts the cake into two shares: s_1 (a 120° strawberry-only piece) and s_2 (a 60° strawberry–180° vanilla piece) as shown in Fig. 3-26. Boris picks the share he likes best, and Angela gets the other share. In the second division, Angela subdivides her share of the cake into three pieces and Boris subdivides his share of the cake into three pieces.

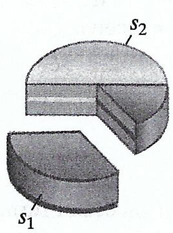


FIGURE 3-26

(a) Describe which share (s_1 or s_2) Boris picks and how he might subdivide it.

(b) Describe how Angela would subdivide her share of the cake.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

38. Suppose that Carlos and Angela are the dividers and Boris is the chooser. In the first division, Carlos cuts the cake into two shares: s_1 (a 135° vanilla-only piece) and s_2 (a 45° vanilla–180° strawberry piece) as shown in Fig. 3-27. Angela picks the share she likes better and Carlos gets the other share. In the second division, Angela subdivides her share of the cake into three pieces and Carlos subdivides his share of the cake into three pieces.

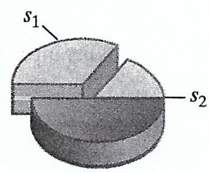


FIGURE 3-27

(a) Describe which share (s_1 or s_2) Angela picks and how she might subdivide it.

(b) Describe how Carlos might subdivide his share of the cake.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

Exercises 39 and 40 refer to the following: Arthur, Brian, and Carl are dividing the cake shown in Fig. 3-28 using the lone-chooser method. Arthur loves chocolate cake and orange cake equally but hates strawberry cake and vanilla cake. Brian loves chocolate cake and strawberry cake equally but hates orange cake and vanilla cake. Carl loves chocolate cake and vanilla cake equally but hates orange cake and strawberry cake. In your answers, assume all cuts are normal "cake cuts" from the center to the edge of the cake. You can describe each piece of cake by giving the angles of its parts, as in "15° strawberry–40° chocolate" or "60° orange only."

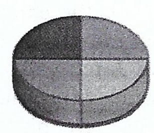


FIGURE 3-28

39. Suppose that Arthur and Brian are the dividers and Carl is the chooser. In the first division, Arthur cuts the cake vertically through the center as shown in Fig. 3-29 and Brian picks the share he likes better. In the second division, Brian subdivides the share he chose into three pieces and Arthur subdivides the other share into three pieces.

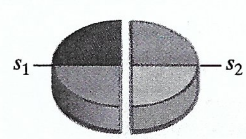


FIGURE 3-29

(a) Describe which share (s_1 or s_2) Brian picks and how he might subdivide it.

(b) Describe how Arthur might subdivide the other share.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value of each share (as a percentage of the total value of the cake) in the eyes of the player receiving it.

40. Suppose that Carl and Arthur are the dividers and Brian is the chooser. In the first division, Carl makes the cut shown in Fig. 3-30 and Arthur picks the share he likes better. In the second division, Arthur subdivides the share he chose into three pieces and Carl subdivides the other share into three pieces.

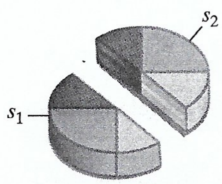


FIGURE 3-30

(a) Describe which share (s_1 or s_2) Arthur picks and how he might subdivide it.

(b) Describe how Carl might subdivide the other share.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value of each share (as a percentage of the total value of the cake) in the eyes of the player receiving it.

41. Jared, Karla, and Lori are dividing the foot-long half meatball–half vegetarian sub shown in Fig. 3-31 using the lone-chooser method. Jared likes the vegetarian and meatball parts equally well, Karla is a strict vegetarian and does not eat meat at all, and Lori likes the meatball part twice as much as she likes the vegetarian part. Suppose that Karla and Jared are the dividers and Lori is the chooser. In the first division, Karla divides the sub into two shares (a left share s_1 and a right share s_2) and Jared picks the share he likes better. In the second division, Jared subdivides the share he picks into three pieces (a "left" piece J_1, a "middle" piece J_2, and a "right" piece J_3) and Karla subdivides the other share into three pieces (a "left" piece K_1, a "middle" piece K_2, and a "right" piece K_3). Assume that all cuts are perpendicular to the length of the sub. (You can describe the pieces of sub using the ruler and interval notation, as in [3, 7] for the piece that starts at inch 3 and ends at inch 7.)

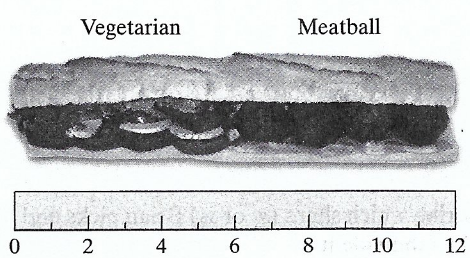


FIGURE 3-31

(a) Describe Karla's first division into s_1 and s_2.

(b) Describe which share (s_1 or s_2) Jared picks and how he would then subdivide it into the three pieces J_1, J_2, and J_3.

(c) Describe how Karla would subdivide her share into three pieces K_1, K_2, and K_3.

(d) Based on the subdivisions in (a), (b), and (c), describe the final fair division of the sub and give the value of each player's share (as a percentage of the total value of the sub) in the eyes of the player receiving it.

42. Jared, Karla, and Lori are dividing the foot-long half meatball–half vegetarian sub shown in Fig. 3-31 using the lone-chooser method. Jared likes the vegetarian and meatball parts equally well, Karla is a strict vegetarian and does not eat meat at all, and Lori likes the meatball part twice as much as she likes the vegetarian part. Suppose that Karla and Lori are the dividers and Jared is the chooser. In the first division, Lori divides the sub into two shares (a left share s_1 and a right share s_2) and Karla picks the share she likes better. In the second division, Karla subdivides the share she picks into three pieces (a "left" piece K_1, a "middle" piece K_2, and a "right" piece K_3) and Lori subdivides the other share into three pieces (a "left" piece L_1, a "middle" piece L_2, and a "right" piece L_3). Assume that all cuts are perpendicular to the length of the sub. (You can describe the pieces of sub using the ruler and interval notation, as in [3, 7] for the piece that starts at inch 3 and ends at inch 7.)

(a) Describe Lori's first division into s_1 and s_2.

(b) Describe which share (s_1 or s_2) Karla picks and how she would then subdivide it into the three pieces K_1, K_2, and K_3.

(c) Describe how Lori would subdivide her share into three pieces L_1, L_2, and L_3.

(d) Based on the subdivisions in (a), (b), and (c), describe the final fair division of the sub and give the value of each player's share (as a percentage of the total value of the sub) in the eyes of the player receiving it.

3.5 The Method of Sealed Bids

43. Ana, Belle, and Chloe are dividing four pieces of furniture using the method of sealed bids. Table 3-24 shows the players' bids on each of the items.

	Ana	Belle	Chloe
Dresser	\$150	\$300	\$275
Desk	\$180	\$150	\$165
Vanity	\$170	\$200	\$260
Tapestry	\$400	\$250	\$500

TABLE 3-24

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

44. Andre, Bea, and Chad are dividing an estate consisting of a house, a small farm, and a painting using the method of sealed bids. Table 3-25 shows the players' bids on each of the items.

	Andre	Bea	Chad
House	$150,000	$146,000	$175,000
Farm	$430,000	$425,000	$428,000
Painting	$50,000	$59,000	$57,000

TABLE 3-25

(a) Describe the first settlement of this fair division and compute the surplus.

(b) Describe the final settlement of this fair-division problem.

45. Five heirs (A, B, C, D, and E) are dividing an estate consisting of six items using the method of sealed bids. The heirs' bids on each of the items are given in Table 3-26.

	A	B	C	D	E
Item 1	$352	$295	$395	$368	$324
Item 2	$98	$102	$98	$95	$105
Item 3	$460	$449	$510	$501	$476
Item 4	$852	$825	$832	$817	$843
Item 5	$513	$501	$505	$505	$491
Item 6	$725	$738	$750	$744	$761

TABLE 3-26

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

46. Oscar, Bert, and Ernie are using the method of sealed bids to divide among themselves four items they commonly own. Table 3-27 shows the bids that each player makes for each item.

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

	Oscar	Bert	Ernie
Item 1	$8600	$5500	$3700
Item 2	$3500	$4200	$5000
Item 3	$2300	$4400	$3400
Item 4	$4800	$2700	$2300

TABLE 3-27

47. Anne, Bette, and Chia jointly own a flower shop. They can't get along anymore and decide to break up the partnership using the method of sealed bids, with the understanding that one of them will get the flower shop and the other two will get cash. Anne bids $210,000, Bette bids $240,000, and Chia bids $225,000. How much money do Anne and Chia each get from Bette for their third share of the flower shop?

48. Al, Ben, and Cal jointly own a fruit stand. They can't get along anymore and decide to break up the partnership using the method of sealed bids, with the understanding that one of them will get the fruit stand and the other two will get cash. Al bids $156,000, Ben bids $150,000, and Cal bids $171,000. How much money do Al and Ben each get from Cal for their one-third share of the fruit stand?

49. Ali, Briana, and Caren are roommates planning to move out of their apartment. They identify four major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-28 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

	Ali	Briana	Caren
Chore 1	$65	$70	$55
Chore 2	$100	$85	$95
Chore 3	$60	$50	$45
Chore 4	$75	$80	$90

TABLE 3-28

50. Anne, Bess, and Cindy are roommates planning to move out of their apartment. They identify five major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-29 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

	Anne	Bess	Cindy
Chore 1	$20	$30	$40
Chore 2	$50	$10	$22
Chore 3	$30	$20	$15
Chore 4	$30	$20	$10
Chore 5	$20	$40	$15

TABLE 3-29

3.6 The Method of Markers

51. Three players (A, B, and C) are dividing the array of 13 items shown in Fig. 3-32 using the method of markers. The players' bids are as indicated in the figure.

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

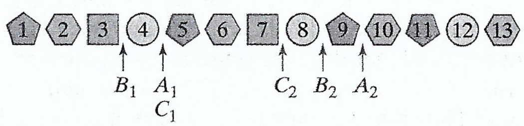


FIGURE 3-32

52. Three players (A, B, and C) are dividing the array of 13 items shown in Fig. 3-33 using the method of markers. The players' bids are as indicated in the figure.

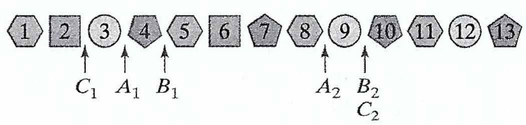


FIGURE 3-33

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

53. Three players (A, B, and C) are dividing the array of 12 items shown in Fig. 3-34 using the method of markers. The players' bids are as indicated in the figure.

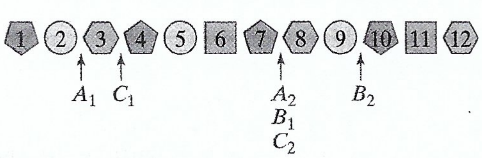


FIGURE 3-34

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

54. Three players (A, B, and C) are dividing the array of 12 items shown in Fig. 3-35 using the method of markers. The players' bids are indicated in the figure.

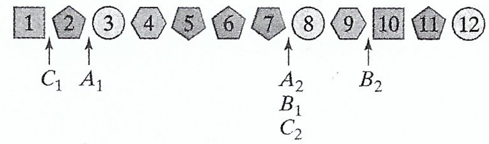


FIGURE 3-35

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

55. Five players (A, B, C, D, and E) are dividing the array of 20 items shown in Fig. 3-36 using the method of markers. The players' bids are as indicated in the figure.

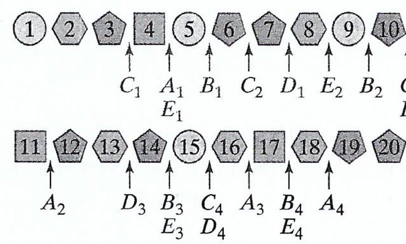


FIGURE 3-36

(a) Describe the allocation of items to each player.

(b) Which items are left over?

56. Four players (A, B, C, and D) are dividing the array of 15 items shown in Fig. 3-37 using the method of markers. The players' bids are as indicated in the figure.

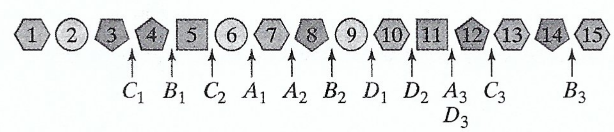


FIGURE 3-37

(a) Describe the allocation of items to each player.

(b) Which items are left over?

57. Quintin, Ramon, Stephone, and Tim are dividing a collection of 18 classic superhero comic books using the method of markers. The comic books are randomly lined up in the array shown below. (The W's are Wonder Woman comic books, the S's are Spider-Man comic books, the G's are Green Lantern comic books, and the B's are Batman comic books.)

W S S G S W W B G G G S G S G S B B

The value of the comic books in the eyes of each player is shown in Table 3-30.

	Quintin	Ramon	Stephone	Tim
Each W is worth	$12	$9	$8	$5
Each S is worth	$7	$5	$7	$4
Each G is worth	$4	$5	$6	$4
Each B is worth	$6	$11	$14	$7

TABLE 3-30

(a) Describe the placement of each player's markers. (Use Q_1, Q_2, and Q_3 for Quintin's markers, R_1, R_2, and R_3 for Ramon's markers, and so on.)

(b) Describe the allocation of comic books to each player and describe what comic books are left over.

58. Queenie, Roxy, and Sophie are dividing a set of 15 CDs—3 Beach Boys CDs, 6 Grateful Dead CDs, and 6 opera CDs using the method of markers. Queenie loves the Beach Boys but hates the Grateful Dead and opera. Roxy loves the Grateful Dead and the Beach Boys equally well but hates opera. Sophie loves the Grateful Dead and opera equally well but hates the Beach Boys. The CDs are lined up in an array as follows:

O O O GD GD GD BB BB BB GD GD GD O O O

(O represents the opera CDs, GD the Grateful Dead CDs, and BB the Beach Boys CDs.)

(a) Describe the placement of each player's markers. (Use Q_1 and Q_2 for Queenie's markers, R_1 and R_2 for Roxy's markers, etc.)

(b) Describe the allocation of CDs to each player and describe what CDs are left over.

(c) Suppose that the players agree that each one gets to pick an extra CD from the leftover CDs. Suppose that Queenie picks first, Sophie picks second, and Roxy picks third. Describe which leftover CDs each one would pick.

59. Ana, Belle, and Chloe are dividing 3 Snickers bars, 3 Milky Way bars, and 3 candy necklaces among themselves using the method of markers. The players' value systems are as follows: (1) Ana likes all candy bars the same; (2) Belle loves Milky Way bars but hates Snickers bars and candy necklaces; (3) Chloe likes candy necklaces twice as much as she likes Snickers or Milky Way bars. Suppose that the candy is lined up exactly as shown in Fig. 3-38.

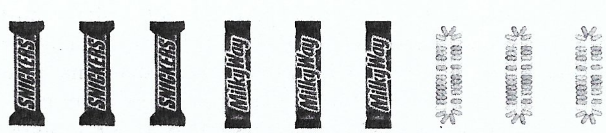


FIGURE 3-38

(a) Describe the placement of each player's markers. (Use A_1 and A_2 for Ana's markers, B_1 and B_2 for Belle's markers, and C_1 and C_2 for Chloe's markers.) (*Hint*: For each player, compute the value of each piece as a fraction of the value of the booty first. This will help you figure out where the players would place their markers.)

(b) Describe the allocation of candy to each player and which pieces of candy are left over.

(c) Suppose that the players decide to divide the leftover pieces by a random lottery in which each player gets to choose one piece. Suppose that Belle gets to choose first, Chloe second, and Ana last. Describe the division of the leftover pieces.

60. Arne, Bruno, Chloe, and Daphne are dividing 3 Snickers bars, 3 Milky Way bars, 3 candy necklaces, and 6 3Musketeers among themselves using the method of markers. Arne hates Snickers Bars but likes Milky Way bars, candy necklaces, and 3Musketeers equally well. Bruno hates Milky Way bars but likes Snickers bars, candy necklaces, and 3Musketeers equally well. Chloe hates candy necklaces and 3Musketeers and likes Snickers three times as much as Milky Way bars. Daphne hates Snickers and Milky Way bars and values a candy necklaces as equal to two-thirds the value of a 3Musketeers (i.e., 2 3Musketeers bars equal 3 candy necklaces). Suppose the candy is lined up exactly as shown in Fig. 3-39.

FIGURE 3-39

(a) Describe the placement of each player's markers. (*Hint*: For each player, compute the value of each piece as a fraction of the value of the booty first. This will help you figure out where the players would place their markers.)

(b) Describe the allocation of candy to each player and which pieces of candy are left over.

(c) After the allocation, each player is allowed to pick one piece of candy. Will there be any arguments?

JOGGING

61. Consider the following method for dividing a continuous asset S among three players (two dividers and one chooser):

Step 1. Divider 1 (D_1) cuts S into two pieces s_1 and s_2 that he considers to be worth, $\frac{1}{3}$ and $\frac{2}{3}$ of the value of S, respectively.

Step 2. Divider 2 (D_2) cuts the second piece s_2 into two halves s_{21} and s_{22} that she considers to be of equal value.

Step 3. The chooser C chooses one of the three pieces (s_1, s_{21}, or s_{22}), D_1 chooses next, and D_2 gets the last piece.

(a) Explain why under this method C is guaranteed a fair share.

(b) Explain why under this method D_1 is guaranteed a fair share.

(c) Illustrate with an example why under this method D_2 is not guaranteed a fair share.

62. Consider the following method for dividing a continuous asset S among three players:

Step 1. Divider 1 (D_1) cuts S into two pieces s_1 and s_2 that he considers to be worth, $\frac{1}{3}$ and $\frac{2}{3}$ of the value of S, respectively.

Step 2. Divider 2 (D_2) gets a shot at s_1. If he thinks that s_1 is worth $\frac{1}{3}$ of S or less, he can pass (case 1); if he thinks that s_1 is worth more than $\frac{1}{3}$, he can trim the piece to a smaller piece s_{11} that he considers to be worth exactly $\frac{1}{3}$ of S (case 2).

Step 3. The chooser C gets a shot at either s_1 (in case 1) or at s_{11} (in case 2). If she thinks the piece is a fair share, she gets to keep it (case 3). Otherwise, the piece goes to the divider that considers it to be worth $\frac{1}{3}$ (D_1 in case 1, D_2 in case 2).

Step 4. The two remaining players (D_2 and C in case 1, D_1 and C in case 2, D_1 and D_2 in case 3) get to divide the "remainder" (whatever is left of S) between themselves using the divider-chooser method.

Explain why the above is a fair-division method that guarantees that if played properly, each player will get a fair share.

63. Two partners (A and B) jointly own a business but wish to dissolve the partnership using the method of sealed bids. One of the partners will keep the business; the other will get cash for his half of the business. Suppose that A bids \$$x$ and B bids \$$y$. Assume that B is the high bidder. Describe the final settlement of this fair division in terms of x and y.

64. Three partners (A, B, and C) jointly own a business but wish to dissolve the partnership using the method of sealed bids. One of the partners will keep the business; the other two will each get cash for their one-third share of the business. Suppose that A bids \$$x$, B bids \$$y$, and C bids \$$z$. Assume that C is the high bidder. Describe the final settlement of this fair division in terms of x, y, and z. (*Hint*: Try Exercises 47 and 48 first.)

65. Three players (A, B, and C) are sharing the chocolate-strawberry-vanilla cake shown in Fig. 3-40(a). Figure 3-40(b) shows the relative value that each player gives to each of the three parts of the cake. There is a way to divide this cake into three pieces (using just three cuts) so that each player ends up with a piece that he or she will value at exactly 50% of the value of the cake. Find such a fair division.

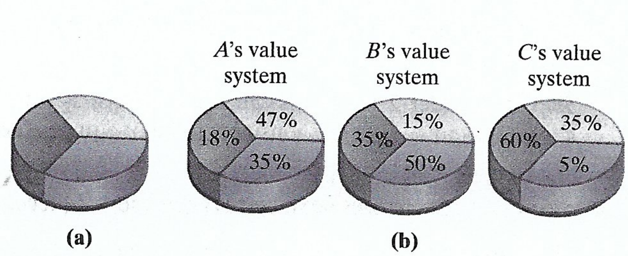


FIGURE 3-40

66. Angelina and Brad are planning to divide the chocolate-strawberry cake shown in Fig. 3-41(a) using the divider-chooser method, with Angelina being the divider. Suppose that Angelina values chocolate cake *three* times as much as she values strawberry cake. Figure 3-41(b) shows a generic cut made by Angelina dividing the cake into two shares s_1 and s_2 of equal value to her. Think of s_1 as an "$x°$ chocolate–$y°$ strawberry" share. For each given measure of the angle x, find the corresponding measure of the angle y.

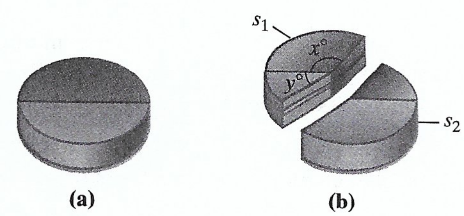


FIGURE 3-41

(a) $x = 60°$.

(b) $x = 72°$.

(c) $x = 108°$.

(d) $x = 120°$.

67. Standoffs in the lone-divider method. In the lone-divider method, a *standoff* occurs when a set of k choosers are bidding for less than k items. The types of *standoffs* possible depend on the number of players. With $N = 3$ players, there is only *one* type of standoff—the two choosers are bidding for the same item. With $N = 4$ players, there are *three* possible types of standoffs: two choosers are bidding on the same item, or three choosers are bidding on the same item, or three choosers are bidding on just two items. With $N = 5$ players, there are *six* possible types of standoffs, and the number of possible types of standoffs increases rapidly as the number of players increases.

(a) List the six possible types of *standoffs* with $N = 5$ players.

(b) What is the number of possible types of *standoffs* with $N = 6$ players?

(c) What is the number of possible types of *standoffs* with N players? Give your answer in terms of N. (*Hint*: You will need to use the formula given in Chapter 1 for the number of pairwise comparisons among a set of objects.)

68. Efficient and envy-free fair divisions. A fair division is called **efficient** if there is no other fair division that gives *every* player a share that is as good or better (i.e., any other fair division that gives some players a better share must give some other players a worse share). A fair division is called **envy-free** if every player ends up with a share that he or she feels is *as good as or better than* that of any other player.

Suppose that three partners (A, B, and C) jointly own a piece of land that has been subdivided into six parcels ($s_1, s_2, \ldots, s_6$). The partnership is splitting up, and the partners are going to divide fairly the six parcels among themselves. Table 3-31 shows the value of each parcel in the eyes of each partner.

	A	B	C
s_1	\$20,000	\$16,000	\$19,000
s_2	\$19,000	\$18,000	\$18,000
s_3	\$18,000	\$19,000	\$15,000
s_4	\$16,000	\$20,000	\$12,000
s_5	\$15,000	\$15,000	\$20,000
s_6	\$12,000	\$12,000	\$16,000

TABLE 3-31

(a) Find a fair division of the six parcels among the three partners that is *efficient*.

(b) Find a fair division of the six parcels among the three partners that is *envy-free*.

(c) Find a fair division of the six parcels among the three partners that is *efficient but not envy-free*.

(d) Find a fair division of the six parcels among the three partners that is *envy-free but not efficient*.

RUNNING

69. Suppose that N players bid on M items using the method of sealed bids. Let T denote the table with M rows (one for each item) and N columns (one for each player) containing all the players' bids (i.e., the entry in column j, row k represents player j's bid for item k). Let $c_1, c_2, \dots, c_N$ denote, respectively, the sum of the entries in column 1, column 2, ..., column N of T, and let $r_1, r_2, \dots, r_M$ denote, respectively, the sum of the entries in row 1, row 2, ..., row M of T. Let $w_1, w_2, \dots, w_M$ denote the winning bids for items $1, 2, \dots, M$, respectively (i.e., w_1 is the largest entry in row 1 of T, w_2 is the largest entry in row 2, etc.). Let S denote the surplus money left after the first settlement.

(a) Show that

$$S = (w_1 + w_2 + \cdots + w_M) - \frac{(c_1 + c_2 + \cdots + c_N)}{N}.$$

(b) Using (a), show that

$$S = \left(w_1 - \frac{r_1}{N}\right) + \left(w_2 - \frac{r_2}{N}\right) + \cdots + \left(w_M - \frac{r_M}{N}\right).$$

(c) Using (b), show that $S \geq 0$.

(d) Describe the conditions under which $S = 0$.

70. Asymmetric method of sealed bids. Suppose that an estate consisting of M indivisible items is to be divided among N heirs $(P_1, P_2, \dots, P_N)$ using the method of sealed bids. Suppose that Grandma's will stipulates that P_1 is entitled to x_1% of the estate, P_2 is entitled to x_2% of the estate, ..., P_N is entitled to x_N% of the estate. The percentages add up to 100%, but they are not all equal (Grandma loved some grandchildren more than others). Describe a variation of the method of sealed bids that ensures that each player receives a "fair share" (i.e., P_1 receives a share that she considers to be worth at least x_1% of the estate, P_2 receives a share that he considers to be worth at least x_2% of the estate, etc.).

71. Lone-chooser is a fair-division method. Suppose that N players divide a cake using the *lone-chooser method.* The chooser is C and the dividers are $D_1, D_2, \dots, D_{N-1}$. Explain why, when properly played, the method guarantees to each player a fair share. (You will need one argument for the dividers and a different argument for the chooser.)

PROJECTS AND PAPERS

1 Envy-Free Fair Division

An *envy-free fair division* is a fair division in which each player ends up with a share that he or she feels is as good as or better than that of any other player. Thus, in an envy-free fair division a player would never envy or covet another player's share. Over the last 20 years several envy-free fair-division methods have been developed.

Write a paper discussing the topic of envy-free fair division.

Notes: Some ideas of topics for your paper: (1) Discuss how envy-free fair division differs from the (proportional) type of fair division discussed in this chapter. (2) Describe a continuous envy-free fair-division method for $N = 3$ players. (3) Give an outline of the Brams-Taylor method for continuous envy-free fair division for any number of players.

2 Fair Divisions with Unequal Shares

All the fair-division problems we discussed in this chapter were based on the assumption of *symmetry* (i.e., all players have equal rights in the division). Sometimes, players are not all equal and are entitled to larger or smaller shares than other players. This type of fair-division problem is called an *asymmetric* fair division (*asymmetric* means that the players are not all equal in their rights). For example, Grandma's will may stipulate that her estate is to be divided as follows: Art is entitled to 25%, Betty is entitled to 35%, Carla is entitled to 30%, and Dave is entitled to 10%. (After all, it is her will, and if she wants to be difficult, she can!)

Write a paper discussing how some of the fair-division methods discussed in this chapter can be adapted for the case of asymmetric fair division. Discuss at least one discrete and one continuous asymmetric fair-division method.

2.4 Subsets and Permutations

39. (a) $2^{10} = 1024$ **(b)** 1023 **(c)** 10 **(d)** 1013

41. (a) 1023 **(b)** 1013

43. (a) 63 **(b)** 31 **(c)** 31 **(d)** 15 **(e)** 16

45. (a) 6,227,020,800
(b) 6,402,373,705,728,000
(c) 15,511,210,043,330,985,984,000,000
(d) 491,857 years

47. (a) 1,037,836,800 **(b)** 286 **(c)** 715 **(d)** 1287

49. (a) 362,880 **(b)** 11 **(c)** 110 **(d)** 504 **(e)** 10,100

51. (a) $7! = 5040$ **(b)** $6! = 720$ **(c)** $6! = 720$ **(d)** 4320

53. (a) 4320 **(b)** $\frac{4320}{5040} = \frac{6}{7}$
(c) $\sigma_1 = \frac{6}{7}; \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = \sigma_7 = \frac{1}{42}$

JOGGING

55. If x is even, then $q = \frac{15x}{2} + 1 \le 8x$; if x is odd, then $q = \frac{15x+1}{2} \le 8x$.

57. $\beta_1 = \frac{3}{5}, \beta_2 = \frac{1}{5}, \beta_3 = \frac{1}{5}$

59. (a) $q = 4, h = 2, a = 1$. (Any answer with multiples of these numbers is also correct.)
(b) *SS* power index of the head coach is $\frac{1}{2}$; *SS* power index of each assistant coach is $\frac{1}{6}$

61. (a) Suppose that a winning coalition that contains P is not a winning coalition without P. Then P would be a critical player in that coalition, contradicting the fact that P is a dummy.
(b) P is a dummy $\Leftrightarrow$ P is never critical $\Leftrightarrow$ the numerator of its Banzhaf power index is 0 $\Leftrightarrow$ its Banzhaf power index is 0.
(c) Suppose that P is not a dummy. Then P is critical in some winning coalition. Let S denote the other players in that winning coalition. The sequential coalition with the players in S first (in any order) followed by P and then followed by the remaining players has P as its pivotal player. Thus, P's Shapley-Shubik power index is not zero. Conversely, if P's Shapley-Shubik power index is not zero, then P is pivotal in some sequential coalition. A coalition consisting of P together with the players preceding P in that sequential coalition is a winning coalition, and P is a critical player in it. Thus, P is not a dummy.

63. (a) $7 \le q \le 13$
(b) For $q = 7$ or $q = 8$, P_1 is a dictator because $\{P_1\}$ is a winning coalition.
(c) For $q = 9$, only P_1 has veto power because P_2 and P_3 together have only five votes.
(d) For $10 \le q \le 12$, both P_1 and P_2 have veto power because no motion can pass without both of their votes. For $q = 13$, all three players have veto power.
(e) For $q = 7$ or $q = 8$, both P_2 and P_3 are dummies because P_1 is a dictator. For $10 \le q \le 12$, P_3 is a dummy because all winning coalitions contain $\{P_1, P_2\}$, which is itself a winning coalition.

65. (a) Both have $\beta_1 = \frac{2}{5}, \beta_2 = \frac{1}{5}, \beta_3 = \frac{1}{5}$, and $\beta_4 = \frac{1}{5}$.
(b) In the weighted voting system $[q: w_1, w_2, \ldots, w_N]$, if P_k is critical in a coalition, then the sum of the weights of all the players in that coalition (including P_k) is at least q but the sum of the weights of all the players in the coalition except P_k is less than q. Consequently, if the weights of all the players in that coalition are multiplied by $c > 0$ ($c = 0$ would make no sense), then the sum of the weights of all the players in the coalition (including P_k) is at least cq but the sum of the weights of all the players in the coalition except P_k is less than cq. Therefore, P_k is critical in the same coalitions in the weighted voting system $[cq: cw_1, cw_2, \ldots, cw_N]$.

67. The senior partner has Shapley-Shubik power index $\frac{N}{N+1}$; each of the junior partners has Shapley-Shubik power index $\frac{1}{N(N+1)}$.

69. You should buy from P_3.

71. (a) You should buy from P_2.
(b) You should buy two votes from P_2.
(c) Buying a single vote from P_2 raises your power from $\frac{1}{25} = 4\%$ to $\frac{3}{25} = 12\%$. Buying a second vote from P_2 raises your power to $\frac{2}{13} \approx 15.4\%$.

73. (a) The losing coalitions are $\{P_1\}$, $\{P_2\}$, and $\{P_3\}$. The complements of these coalitions are $\{P_2, P_3\}$, $\{P_1, P_3\}$, and $\{P_1, P_2\}$, respectively, all of which are winning coalitions.
(b) The losing coalitions are $\{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}, \{P_2, P_3\}, \{P_2, P_4\}$, and $\{P_3, P_4\}$. The complements of these coalitions are $\{P_2, P_3, P_4\}, \{P_1, P_3, P_4\}, \{P_1, P_2, P_4\}, \{P_1, P_2, P_3\}, \{P_1, P_4\}, \{P_1, P_3\}$, and $\{P_1, P_2\}$, respectively, all of which are winning coalitions.
(c) If P is a dictator, then the losing coalitions are all the coalitions without P; the winning coalitions are all the coalitions that include P.
(d) 2^{N-1}

75. (a) approximately 1.115 (relative voting weight $= \frac{55}{538}$; BPI $= 0.114$)
(b) $\frac{115}{93}, \frac{115}{93}, \frac{115}{84}, 0, 0, 0$

Chapter 3

WALKING

3.1 Fair-Division Games

1. (a) s_2 and s_3 **(b)** s_2 and s_3 **(c)** s_1, s_2, and s_3
(d) 1. Henry gets s_2; Tom gets s_3; Fred gets s_1.
2. Henry gets s_3; Tom gets s_2; Fred gets s_1.
(e) Henry gets s_2; Tom gets s_3; Fred gets s_1.

3. (a) s_2, s_3 **(b)** s_1, s_2, s_4 **(c)** s_2, s_3 **(d)** s_1, s_2, s_3, s_4
(e) 1. Angie gets s_2; Bev gets s_1; Ceci gets s_3; Dina gets s_4.
2. Angie gets s_2; Bev gets s_4; Ceci gets s_3; Dina gets s_1.
3. Angie gets s_3; Bev gets s_1; Ceci gets s_2; Dina gets s_4.
4. Angie gets s_3; Bev gets s_4; Ceci gets s_2; Dina gets s_1.

5. **(a)** s_2, s_3, s_4 **(b)** s_3, s_4 **(c)** s_1, s_2 **(d)** s_1, s_4
(e) 1. Allen gets s_2; Brady gets s_3; Cody gets s_1; Diane gets s_4.
2. Allen gets s_3; Brady gets s_4; Cody gets s_2; Diane gets s_1.
3. Allen gets s_4; Brady gets s_3; Cody gets s_2; Diane gets s_1.

7. **(a)** s_2 **(b)** s_2, s_3 **(c)** s_2, s_3
(d) It is not possible. No player considers s_1 to be a fair share.

9. **(a)** s_1, s_4 **(b)** s_1, s_2 **(c)** s_1, s_3 **(d)** s_4
(e) Adams: s_1; Benson: s_2; Cagle: s_3; Duncan: s_4

11. **(a)** \$12 **(b)** \$24 **(c)** \$4 **(d)** \$6

13. **(a)** s_1: \$2.50; s_2: \$3.75; s_3: \$5.00; s_4: \$6.25; s_5: \$7.50; s_6: \$5.00
(b) s_3, s_4, s_5, s_6

3.2 The Divider-Chooser Method

15. **(a)** s_1: $[0, 8]$; s_2: $[8, 12]$ **(b)** s_1; \$8.00

17. **(a)** s_1: $[0, 10]$; s_2: $[10, 28]$ **(b)** s_2; \$6.50

19. **(a)** yes; s_2 (75%) **(b)** no (David values s_2 at 60%, s_1 at 40%.)
(c) yes; s_2 (75%)

3.3 The Lone-Divider Method

21. **(a)** C_1: s_2, C_2: s_3, D: s_1 or C_1: s_3, C_2: s_2, D: s_1
(b) C_1: s_2, C_2: s_1, D: s_3 or C_1: s_2, C_2: s_3, D: s_1 or C_1: s_3, C_2: s_1, D: s_2

23. **(a)** C_1: s_2, C_2: s_1, C_3: s_3, D: s_4 (First, C_1 must receive s_2. Then, C_3 must receive s_3. So, C_2 receives s_1 and D receives s_4.)
(b) C_1: s_2, C_2: s_3, C_3: s_1, D: s_4 or C_1: s_3, C_2: s_1, C_3: s_2, D: s_4
(c) C_1: s_2, C_2: s_1, C_3: s_4, D: s_3 or C_1: s_2, C_2: s_3, C_3: s_1, D: s_4 or C_1: s_2, C_2: s_3, C_3: s_4, D: s_1

25. **(a)** Tim **(b)** Mark gets s_2; Tim gets s_1; Maia gets s_3; Kelly gets s_4

27. **(a)** fair **(b)** not fair ($\{s_2, s_3, s_4\}$ may be worth less than 75% to C_2 and C_3). **(c)** fair **(d)** not fair ($\{s_1, s_4\}$ may be worth less than 50% to C_2 and C_3).

29. **(a)** C_1: s_2, C_2: s_4, C_3: s_5, C_4: s_3, D: s_1; C_1: s_4, C_2: s_2; C_3: s_5, C_4: s_3, D: s_1 (If C_1 is to receive s_2, then C_2 must receive s_4 (and conversely). So, C_4 must receive s_3. It follows that C_3 must receive s_5. This leaves D with s_1.)
(b) C_1: s_2, C_2: s_4, C_3: s_5, C_4: s_3, D: s_1 (If C_1 is to receive s_2, then C_2 must receive s_4. So, C_4 must receive s_3. It follows that C_3 must receive s_5. This leaves D with s_1.)

31. **(a)** Gong (he is the only player who could value each piece equally).
(b) Egan: $\{s_3, s_4\}$; Fine: $\{s_1, s_3, s_4\}$; Hart: $\{s_3\}$
(c) Egan: s_4; Fine: s_1; Gong: s_2; Hart: s_3

33. **(a)** s_1: $[0, 4]$; s_2: $[4, 8]$; s_3: $[8, 12]$ **(b)** s_1, s_2 **(c)** s_2, s_3
(d) Three possible fair divisions are
1. Jared: s_2; Karla; s_1; Lori: s_3
2. Jared: s_1; Karla; s_2; Lori: s_3
3. Jared: s_3; Karla: s_1; Lori: s_2

3.4 The Lone-Chooser Method

35. **(a)** 40° strawberry; 40° strawberry; 90° vanilla–10° strawberry
(b) 75° vanilla; 15° vanilla–40° strawberry; 50° strawberry
(c) Angela: 80° strawberry; Boris: 15° vanilla–90° strawberry; Carlos: 165° vanilla–10° strawberry
(d) Angela: \$12.00; Boris: \$10.00; Carlos: \$22.67

37. **(a)** s_2; 90° vanilla; 90° vanilla; 60° strawberry
(b) 40° strawberry; 40° strawberry; 40° strawberry
(c) Angela: 80° strawberry; Boris: 90° vanilla–60° strawberry; Carlos: 90° vanilla–40° strawberry
(d) Angela: \$12.00; Boris: \$12.00; Carlos: \$14.67

39. **(a)** s_1 (60° chocolate, 30° chocolate–30° strawberry, 60° strawberry)
(b) 30° orange; 30° orange; 30° orange–90° vanilla
(c) Arthur: 60° orange; Brian: 90° strawberry–30° chocolate; Carl: 90° vanilla–30° orange–60° chocolate
(d) Arthur: $33\frac{1}{3}\%$; Brian: $66\frac{2}{3}\%$; Carl: $83\frac{1}{3}\%$

41. **(a)** s_1: $[0; 3]$; s_2: $[3, 12]$
(b) Jared picks s_2 (J_1: $[3, 6]$; J_2: $[6, 9]$; J_3: $[9, 12]$)
(c) K_1: $[0, 1]$; K_2: $[1, 2]$; K_3: $[2, 3]$
(d) Jared: $[3, 9]$; Karla: $[0, 2]$; Lori: $[2, 3]$ and $[9, 12]$
Jared: 50%; Karla: $33\frac{1}{3}\%$; Lori: $38\frac{8}{9}\%$

3.5 The Method of Sealed Bids

43. **(a)** Ana: \$300; Belle: \$300; Chloe: \$400
(b) In the first settlement, Ana gets the desk and receives \$120 in cash, Belle gets the dresser, and Chloe gets the vanity and the tapestry and pays \$360.
(c) \$240
(d) In the final settlement, Ana gets the desk and receives \$200 in cash, Belle gets the dresser and receives \$80, and Chloe gets the vanity and the tapestry and pays \$280.

45. **(a)** A: \$600; B: \$582; C: \$618; D: \$606; E : \$600
(b) In the first settlement, A gets items 4 and 5 and pays \$765, B gets \$582, C gets items 1 and 3 and pays \$287, D gets \$606, and E gets items 2 and 6 and pays \$266.
(c) \$130
(d) In the final settlement, A gets items 4 and 5 and pays \$739, B gets \$608, C gets items 1 and 3 and pays \$261, D gets \$632, and E gets items 2 and 6 and pays \$240.

47. Anne gets \$75,000 and Chia gets \$80,000.

49. Ali gets to do Chore 4; Briana gets to do Chore 2; Caren gets to do Chores 1 and 3. Ali has to pay \$15 to Caren; Briana is even (neither pays nor gets money).

3.6 The Method of Markers

51. **(a)** 10, 11, 12, 13 **(b)** 1, 2, 3 **(c)** 5, 6, 7 **(d)** 4, 8, 9

53. **(a)** 1, 2 **(b)** 10, 11, 12 **(c)** 4, 5, 6, 7 **(d)** 3, 8, 9

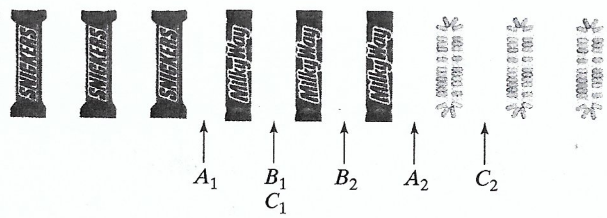
55. **(a)** A: 19, 20; B: 15, 16, 17; C: 1, 2, 3; D: 11, 12, 13; E: 5, 6, 7, 8
(b) 4, 9, 10, 14, 18

57. **(a)** W S S G S W W B G G G S G S G S B B

↑ ↑ (after positions 3, 4): Q_1 R_1; S_1; T_1
↑ ↑ ↑ (after positions 7, 8, 9): Q_2 R_2 S_2; T_2
↑ ↑ ↑ (after positions 14, 15, 16): Q_3 R_3 S_3; T_3

(b) Quintin gets a W, two S's, and a G; Ramon gets two W's and a B; Tim gets three G's and two S's; and Stephone gets an S and two B's. An S and two G's are left over.

59. **(a)**



(b) Ana gets three Snickers, Belle gets one Milky Way, and Chloe gets two candy necklaces. Two Milky Way bars and one candy necklace are left over.

(c) Belle would select one of the Milky Way bars, Chloe would then select the candy necklace, and then Ana would be left with the last Milky Way bar.

JOGGING

61. **(a)** C gets to choose one of the three pieces. One of the three must be worth at least $\frac{1}{3}$ of S.

(b) If C chooses either s_{21} or s_{22}, D_1 gets to choose s_1 (a fair share to him). If C chooses s_1, D_1 gets to choose between s_{21} and s_{22}. Together they are worth $\frac{2}{3}$ of S, so one of the two must be worth at least $\frac{1}{3}$ of S.

(c) Suppose that in D_2's opinion, the first cut by D_1 splits S into a 40%–60% split. In this case the value of the three pieces to D_2 is $s_1 = 40\%$, $s_{21} = 30\%$, and $s_{22} = 30\%$. If C or D_1 choose s_1, D_2 will not get a fair share.

63. B must pay $\dfrac{x+y}{4}$ dollars to A.

65. B receives the strawberry part. C will receive $\frac{5}{6}$ of the chocolate part. A receives the vanilla part and $\frac{1}{6}$ of the chocolate part.

67. **(a)** two, three, or four choosers bidding on the same item; three or four choosers bidding on the same two items; four choosers bidding on the same three items

(b) 10

(c) $\dfrac{(N-1)(N-2)}{2}$

Chapter 4

WALKING

4.1 Apportionment Problems and Apportionment Methods

1. **(a)** 50,000
(b) Apure: 66.2; Barinas: 53.4; Carabobo: 26.6; Dolores: 13.8

3. **(a)** 1000
(b) the average ridership per bus each day
(c) A: 45.30; B: 31.07; C: 20.49; D: 14.16; E: 10.26; F: 8.72

5. **(a)** 137 **(b)** 200,000
(c) A: 8,240,000; B: 6,380,000; C: 4,960,000; D: 4,520,000; E: 3,300,000

7. 35.41

9. **(a)** 0.5%
(b) A: 22.74; B: 16.14; C: 77.24; D: 29.96; E: 20.84; F: 33.08

4.2 Hamilton's Method

11. A: 66; B: 53; C: 27; D: 14

13. A: 45; B: 31; C: 21; D: 14; E: 10; F: 9

15. A: 41; B: 32; C: 25; D: 23; E: 16

17. A: 23; B: 16; C: 77; D: 30; E: 21; F: 33

19. **(a)** Dunes: 3; Smithville: 6; Johnstown: 15
(b) Dunes: 2; Smithville: 7; Johnstown: 16
(c) Dunes's apportionment went from 3 social workers to 2 as the total number of social workers increased from 24 to 25.

4.3 Jefferson's Method

21. **(a)** $SD = 200{,}000$; standard quotas are 22.5 (Arcadia), 24.5 (Belarmine), 19.5 (Crowley), 33.5 (Dandia).
(b) 98 **(c)** 99 **(d)** 102 **(e)** 100 **(f)** 100
(g) We know that $d = 195{,}800$ and $d = 196{,}000$ work from parts (e) and (f). Any other divisor in between (for example $d = 195{,}900$) will also work.

23. A: 67; B: 54; C: 26; D: 13

25. A: 46; B: 31; C: 21; D: 14; E: 10; F: 8

27. A: 41; B: 32; C: 25; D: 23; E: 16

29. A: 22; B: 16; C: 78; D: 30; E: 21; F: 33