1 The Mathematics of Elections

The Paradoxes of Democracy

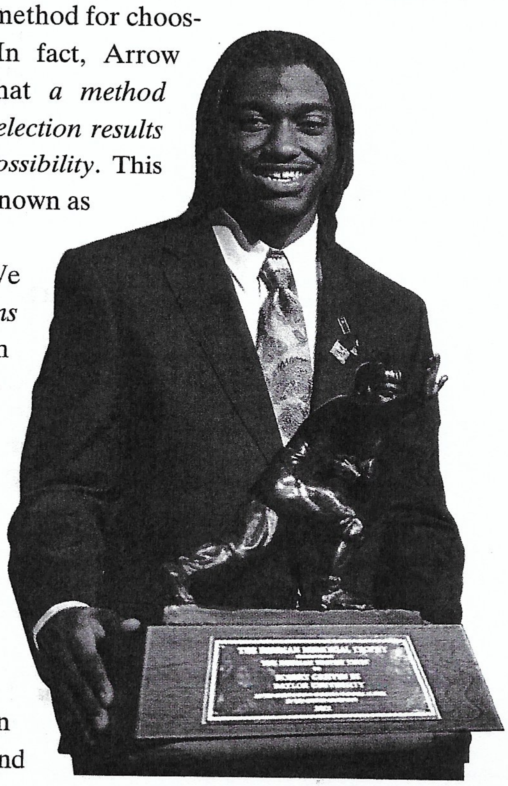
Whether we like it or not, we are all affected by the outcomes of elections. Our president, senators, governors, and mayors make decisions that impact our lives in significant ways, and they all get to be in that position because an election made it possible. But elections touch our lives not just in politics. Academy Awards, *American Idol*, Heisman trophies, football rankings—they are all decided by means of an election. Even something as simple as deciding where to go for dinner might require a mini-election.

We have elections because we don't all think alike. Since we cannot all have things our way, we vote. But *voting* is only the first half of the story, the one we are most familiar with. As playwright Tom Stoppard suggests, it's the second half of the story—the *counting*—that is at the heart of the democratic process. How do we sift through the many choices of individual voters to find the collective choice of the group? More important, how well does the process work? Is the process always fair? Answering these questions and explaining a few of the many intricacies and subtleties of *voting theory* are the purpose of this chapter.

But wait just a second! Voting theory? Why do we need a fancy theory to figure out how to count the votes? It all sounds pretty simple: We have an election; we count the ballots. Based on that count, we decide the outcome of the election in a consistent and fair manner. Surely, there must be a reasonable way to accomplish this! Surprisingly, there isn't.

> “It's not the voting that's democracy; it's the counting.”
>
> – Tom Stoppard

In the late 1940s the American economist Kenneth Arrow discovered a remarkable fact: For elections involving three or more candidates, there is no consistently fair democratic method for choosing a winner. In fact, Arrow demonstrated that *a method for determining election results that is always fair is a mathematical impossibility*. This fact, the most famous in voting theory, is known as *Arrow's Impossibility Theorem*.



This chapter is organized as follows. We will start with a general discussion of *elections* and *ballots* in Section 1.1. This discussion provides the backdrop for the remaining sections, which are the heart of the chapter. In Sections 1.2 through 1.5 we will explore four of the most commonly used *voting methods*—how they work and how they are used in real-life applications. In Section 1.6 we will introduce some basic principles of fairness for voting methods and apply these *fairness criteria* to the voting methods discussed in Sections 1.2 through 1.5. The section concludes with a discussion of the meaning and significance of Arrow's Impossibility Theorem.

1.1 The Basic Elements of an Election

Big or small, important or trivial, *all* elections share a set of common elements.

- **The candidates.** The purpose of an election is to choose from a set of *candidates* or *alternatives* (at least two—otherwise it is not a real election). Typically, the word *candidate* is used for people and the word *alternative* is used for other things (movies, football teams, pizza toppings, etc.), but it is acceptable to use the two terms interchangeably. In the case of a generic choice (when we don't know if we are referring to a person or a thing), we will use the term *candidate*. While in theory there is no upper limit on the number of candidates, for most elections (in particular the ones we will discuss in this chapter) the number of candidates is small.
- **The voters.** These are the people who get a say in the outcome of the election. In most democratic elections the presumption is that all voters have an equal say, and we will assume this to be the case in this chapter. (This is not always true, as we will see in great detail in Chapter 2.) The number of voters in an election can range from very small (as few as 3 or 4) to very large (hundreds of millions). In this section we will see examples of both.
- **The ballots.** A ballot is the device by means of which a voter gets to express his or her opinion of the candidates. The most common type is a paper ballot, but a voice vote, a text message, or a phone call can also serve as a "ballot" (see Example 1.5 *American Idol*). There are many different forms of ballots that can be used in an election, and Fig. 1-1 shows a few common examples. The simplest form is the **single-choice ballot**, shown in Fig. 1-1(a). Here very little is being asked of the voter ("pick the candidate you like best, and keep the rest of your opinions to yourself!"). At the other end of the spectrum is the **preference ballot**, where the voter is asked to rank *all* the candidates in order of preference. Figure 1-1(b) shows a typical preference ballot in an election with five candidates. In this ballot, the voter has entered the candidates' names in order of preference. An alternative version of the same preference ballot is shown in Fig. 1-1(c). Here the names of the candidates are already printed on the ballot and the voter simply has to mark first, second, third, etc. In elections where there are a large number of candidates, a **truncated preference ballot** is often used. In a truncated preference ballot the voter is asked to rank some, but not all, of the candidates. Figure 1-1(d) shows a truncated preference ballot for an election with dozens of candidates.

Choose one candidate
O Beyoncé
O Lady Gaga
O Rihanna
● Taylor Swift
O Katy Perry

(a)

List all candidates in order of preference	
1st	James Franco
2nd	Javier Bardem
3rd	Colin Firth
4th	Jeff Bridges
5th	Jesse Eisenberg

(b)

Rank all candidates in order of preference	
Javier Bardem	2nd
Jeff Bridges	4th
Jesse Eisenberg	5th
Colin Firth	3rd
James Franco	1st

(c)

List the top 3 candidates in order of preference	
1st	LaMichael James
2nd	Cam Newton
3rd	Andrew Luck

(d)

FIGURE 1-1 (a) Single-choice ballot, (b) preference ballot, (c) a different version of the same preference ballot, and (d) truncated preference ballot.

- **The outcome.** The purpose of an election is to use the information provided by the ballots to produce some type of outcome. But what types of outcomes are possible? The most common is **winner-only**. As the name indicates, in a winner-only election all we want is to find a winner. We don't distinguish among the nonwinners. There are, however, situations where we want a broader outcome than just a winner—say we want to determine a first-place, second-place, and third-place candidate from a set of many candidates (but we don't care about fourth place, fifth place, etc.). We call this type of outcome a **partial ranking**. Finally, there are some situations where we want to rank *all* the candidates in order: first, second, third, . . . , last. We call this type of outcome a **full ranking**, or just a **ranking** for short.

- **The voting method.** The final piece of the puzzle is the method that we use to tabulate the ballots and produce the outcome. This is the most interesting (and complicated) part of the story, but we will not dwell on the topic here, as we will discuss voting methods throughout the rest of the chapter.

It is now time to illustrate and clarify the above concepts with some examples.

We start with a simple example of a fictitious election. This is an important example, and we will revisit it many times throughout the chapter. You may want to think of Example 1.1 as a mathematical parable, its importance being not in the story itself but in what lies hidden behind it. (As you will soon see, there is a lot more to Example 1.1 than first meets the eye.)

EXAMPLE 1.1 THE MATH CLUB ELECTION (WINNER-ONLY)

The Math Appreciation Society (MAS) is a student club dedicated to an unsung but worthy cause: that of fostering the enjoyment and appreciation of mathematics among college students. The MAS chapter at Tasmania State University is holding its annual election for club president, and there are four *candidates* running: Alisha, Boris, Carmen, and Dave (A, B, C, and D for short).

Every member of the club is eligible to vote, and the vote takes the form of a *preference ballot*. Each voter is asked to rank each of the four candidates in order of preference. There are 37 *voters* who submit their ballots, and the 37 *preference ballots* submitted are shown in Fig. 1-2.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *B*	1st *A*	1st *C*	1st *B*	1st *C*	1st *A*	1st *B*	1st *C*	1st *A*	1st *C*	1st *D*	1st *A*	1st *A*	1st *C*	1st *A*	1st *C*	1st *D*
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *C*
3rd *C*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*
4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*	4th *A*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *D*	1st *D*	1st *C*	1st *C*	1st *D*	1st *A*	1st *D*	1st *C*	1st *A*	1st *D*	1st *B*	1st *A*	1st *C*	1st *A*	1st *A*	1st *D*	1st *A*
2nd *B*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *D*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*
3rd *D*	3rd *C*	3rd *B*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *C*	3rd *B*	3rd *D*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*
4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*

FIGURE 1-2 The 37 preference ballots for the Math Club election.

Last but not least, what about the *outcome* of the election? Since the purpose of the election is to choose a club president, it is pointless to discuss or consider which candidate comes in second place, third place, etc. This is a *winner-only* election.

EXAMPLE 1.2 THE MATH CLUB ELECTION (FULL RANKING)

Suppose now that we have pretty much the same situation as in Example 1.1 (same candidates, same voters, same preference ballots), but that the election is to choose not only the president of the club but also a vice-president, a treasurer, and a secretary. According to the club bylaws, the president is the candidate who comes in first, the vice-president is the candidate who comes in second, the treasurer is the candidate who comes in third, and the secretary is the candidate who comes in fourth.

Given that there are four candidates, each candidate will get to be an officer, but there is a big difference between being elected president and being elected treasurer (the president gets status and perks; the treasurer gets to collect the dues and balance the budget). In this version how you place matters, and the outcome should be a full *ranking* of the candidates.

EXAMPLE 1.3 THE ACADEMY AWARDS

The Academy Awards (also known as the Oscars) are given out each year by the Academy of Motion Picture Arts and Sciences for Best Picture, Best Actress, Best Actor, Best Director, and many other, lesser categories (Sound Mixing, Makeup, etc.). The election process is not the same for all awards, so for the sake of simplicity we will just discuss the selection of Best Picture.

The *voters* in this election are all the eligible members of the Academy (approximately 6000 voters—5755 for the 2011 awards). After a complicated preliminary round (a process that we won't discuss here) ten films are selected as the nominees—these are our *candidates*. (For most other awards there are only five nominees.) Each voter is asked to submit a preference ballot ranking the ten candidates. There is only a winner (the other candidates are not ranked), with the winner determined by a voting method called plurality-with-elimination that we will discuss in detail in Section 1.4. (The winner of the 2011 Best Picture Award was *The Artist*.)

The part with which people are most familiar comes after the ballots are submitted and tabulated—the annual Academy Awards ceremony, held each year in late February. How many viewers realize that behind one of the most extravagant and glamorous events in pop culture lies an election?

EXAMPLE 1.4 THE HEISMAN TROPHY

The Heisman Memorial Trophy Award is given annually to the "most outstanding player in collegiate football." The Heisman, as it is usually known, is not only a very prestigious award but also a very controversial award. With so many players playing so many different positions, how do you determine who is the most "outstanding"?

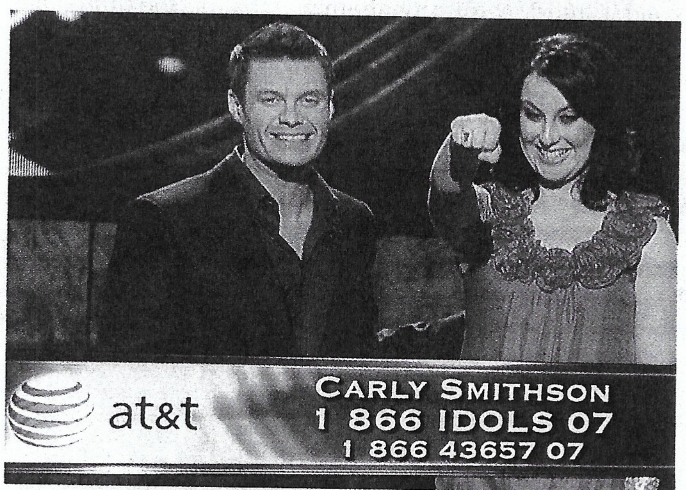
In theory, any football player in any division of college football is a potential *candidate* for the award. In practice, the real candidates are players from Division I programs and are almost always in the glamour positions—quarterback or running back. (Since its inception in 1935, only once has the award gone to a defensive player—Charles Woodson of Michigan.)

There are approximately 930 *voters* (the exact number of voters varies each year). The *voters* are members of the media plus all past Heisman award winners still living, plus one vote from the public (as determined by a survey conducted by ESPN). Each voter submits a *truncated preference ballot* consisting of a first, second, and third choice (see Fig. 1-1[d]). A first-place vote is worth 3 points, a second-place vote 2 points, and a third-place vote 1 point. The candidate with the most total points from all the ballots is awarded the Heisman trophy in a televised ceremony held each December at the Downtown Athletic Club in New York.

While only one player gets the award, the finalists are ranked by the number of total points received, in effect making the *outcome* of the Heisman trophy a *partial ranking* of the top four (sometimes five) candidates. (For the 2011 season, the Heisman Trophy went to Robert Griffin III, Baylor; second place to Andrew Luck, Stanford; third place to Trent Richardson, Alabama; fourth place to Montee Ball, Wisconsin; and fifth place to Tyrann Mathieu, LSU.)

EXAMPLE 1.5 AMERICAN IDOL

The single most watched program in the history of American television is *American Idol*, a singing competition for individuals (as opposed to *The X-Factor*, a similar singing competition that allows for groups as well as individuals). Each year, the winner of *American Idol* gets a big recording contract, and many past winners have gone on to become famous recording artists (Kelly Clarkson, Carrie Underwood, Taylor Hicks). While there is a lot at stake and a big reward for winning, *American Idol* is not a winner-only competition, and there is indeed a ranking of all the finalists. In fact, some nonwinners (Clay Aiken, Jennifer Hudson) have gone on to become great recording artists in their own right.



The 12 (sometimes 13) candidates who reach the final rounds of the competition compete in a weekly televised show. During and immediately after each weekly show the voters cast their votes. The candidate with the fewest number of votes gets eliminated from the competition, and the following week the process starts all over again with one fewer candidate (on rare occasions two candidates are eliminated in the same week—see Table 1-11). And who are the *voters* responsible for deciding the fate of these candidates? Anyone and everyone—you, me, Aunt Betsie—we are all potential voters. All one has to do to vote for a particular candidate is to text or call a toll-free number specific to that candidate ("to vote for Carly, call 1-866-IDOLS07," etc.). *American Idol* voting is an example of democracy run amok—you can vote for a candidate even if you never heard her sing, and you can vote as many times as you want.

By the final week of the competition there are only two finalists left, and after one last frenzied round of voting, the winner is determined. (For the 2011 *American Idol* competition there were nearly 750 million votes cast. Table 1-11 shows a summary of the results.)

Examples 1.1 through 1.5 represent just a small sample of how elections can be structured, both in terms of the ballots (think of these as the *inputs* to the election) and the types of outcomes we look for (the *outputs* of the election). We will revisit some of these examples and many others as we wind our way through the chapter.

Preference Ballots and Preference Schedules

Let's focus now on elections where the balloting is done by means of preference ballots, as in Examples 1.1 and 1.2. The great advantage of preference ballots (compared with, for example, single-choice ballots) is that they provide a great deal of useful information about an individual voter's preferences—in both direct and indirect ways.

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

FIGURE 1-3

To illustrate what we mean, consider the preference ballot shown in Fig. 1-3. This ballot directly tells us that the voter likes candidate *C* best, *B* second best, *D* third best, and *A* last. But, in fact, this ballot tells us a lot more—it tells us unequivocally which candidate the voter would choose if it came down to a choice between just two candidates. For example, if it came down to a choice between, say, *A* and *B*, which one would this voter choose? Of course she would choose *B*—she has *B* above *A* in her ranking. Thus, a preference ballot allows us to make relative comparisons between any two candidates—*the candidate higher on the ballot is always preferred over the candidate in the lower position*. Please take note of this simple but important idea, as we will use it repeatedly later in the chapter.

The second important idea we will use later is the fact that the relative preferences in a preference ballot do not change if one of the candidates withdraws or is eliminated. Once again, we can illustrate this using Fig. 1-3. What would happen if for some unforeseen reason candidate *B* drops out of the race right before the ballots are tabulated? Do we have to have a new election? Absolutely not—the old ballot simply becomes the ballot shown on the left in Fig. 1-4. The candidates above *B* stay put and each of the candidates below *B* moves up a spot.

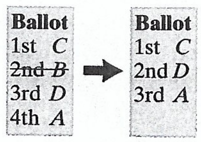


FIGURE 1-4

In an election with many voters, some voters will vote exactly the same way—for the same candidates in the same order of preference. If we take a careful look at the 37 ballots submitted for the Math Club election shown in Fig. 1-2, we see that 14 ballots look exactly the same (*A* first, *B* second, *C* third, *D* fourth), another 10 ballots look the same, and so on. So, if you were going to tabulate the 37 ballots, it might make sense to put all the *A-B-C-D* ballots in one pile, all the *C-B-D-A* ballots in another pile, and so on. If you were to do this you would get the five piles shown in Fig. 1-5 (the order in which you list the piles from left to right is irrelevant). Better yet, you can make the whole idea a little more formal by putting all the ballot information in a table such as Table 1-1, called the **preference schedule** for the election.

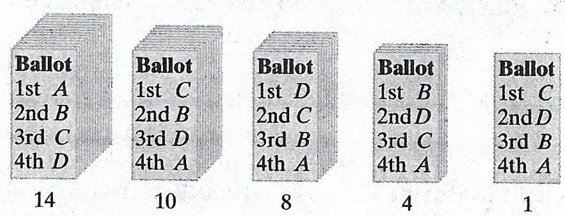


FIGURE 1-5 The 37 Math Club election ballots organized into piles.

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

■ **TABLE 1-1** Preference schedule for the Math Club election

We will be working with preference schedules throughout the chapter, so it is important to emphasize that a preference schedule is nothing more than a convenient bookkeeping tool—it summarizes all the elements that constitute the input to an election: the candidates, the voters, and the balloting. Just to make sure this is clear, we conclude this section with a quick example of how to read a preference schedule.

EXAMPLE 1.6 THE CITY OF KINGSBURG MAYORAL ELECTION

Number of voters	93	44	10	30	42	81
1st	*A*	*B*	*C*	*C*	*D*	*E*
2nd	*B*	*D*	*A*	*E*	*C*	*D*
3rd	*C*	*E*	*E*	*B*	*E*	*C*
4th	*D*	*C*	*B*	*A*	*A*	*B*
5th	*E*	*A*	*D*	*D*	*B*	*A*

■ **TABLE 1-2** Preference schedule for the Kingsburg mayoral election

Table 1-2 shows the preference schedule summarizing the results of the most recent election for mayor of the city of Kingsburg (there actually is a city by that name, but the election is fictitious). Just by looking at the preference schedule we can answer all of the relevant input questions:

- *Candidates*: there were five candidates (*A*, *B*, *C*, *D*, and *E*, which are just abbreviations for their real names).

- *Voters*: there were 300 voters that submitted ballots (add the numbers at the head of each column: $93+44+10+30+42+81=300$).
- *Balloting*: the 300 preference ballots were organized into six piles as shown in Table 1-2.

The question that still remains unanswered: Who is the winner of the election? In the next four sections we will discuss different ways in which such output questions can be answered.

Ties

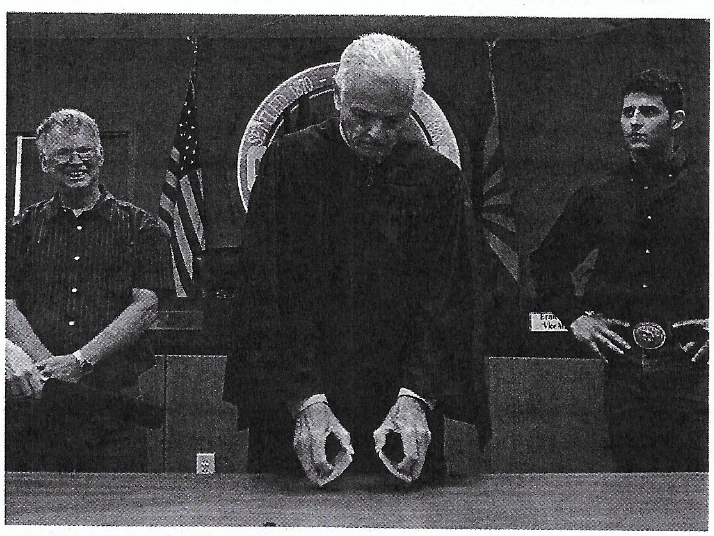
In any election, be it a *winner-only* election or a *ranking* of the candidates, ties can occur. What happens then?

In some elections the rule is that ties are allowed to stand and need not be broken. Here are a few interesting examples:

- In the 1968 Academy Awards, Katharine Hepburn and Barbra Streisand tied for Best Actress. Both received Oscars.
- In the 1992 Grammy Awards, Lisa Fischer and Patti La Belle tied for Best Female R&B Vocal Performance and shared the award.
- In the 1979 National League MVP balloting, Keith Hernandez and Willie Stargell tied for first and shared the award.
- In the 2011 *American Idol* competition, Thia Megia and Naima Adepapo tied for 10th place. They were declared as sharing the 10th–11th position (see Table 1-11).

In other situations, especially in elections for political office (president, senator, mayor, city council, etc.), ties cannot be allowed (can you imagine having co-mayors?), and then a tie-breaking rule must be specified. The Constitution, for example, stipulates how a tie in the Electoral College is broken, and most elections have a set rule for breaking ties. The most common method for breaking a tie for political office is through a runoff election, but runoff elections are expensive and take time, so many other tie-breaking procedures are used. Here are a few interesting examples:

- In the 2009 election for a seat in the Cave Creek, Arizona, city council, Thomas McGuire and Adam Trenk tied with 660 votes each. The winner was decided by drawing from a deck of cards. Mr. McGuire drew first—a six of hearts. Mr. Trenk (the young man with the silver belt buckle) drew next and drew a king of hearts. This is how Mr. Trenk became a city councilman.



- In the 2010 election for mayor of Jefferson City, Tennessee, Rocky Melton and Mark Potts tied with 623 votes each. The decision then went to a vote of the city council. Mr. Potts became the mayor.
- In the 2011 election for trustees of the Island Lake, Illinois, village board, Allen Murvine and Charles Cernak tied for one of the three seats with 576 votes each. The winner was decided by a coin toss. Mr. Cernak called tails and won the seat.

Ties and tie-breaking procedures add another layer of complexity to an already rich subject. To simplify our presentation, in this chapter we will stay away from ties as much as possible. In the rare example where a tie occurs, we will assume that the tie does not have to be broken.

1.2 The Plurality Method

The **plurality method** is arguably the most commonly used and simplest method for determining the outcome of an election. With the plurality method, all that matters is how many first-place votes each candidate gets: In a *winner-only* election the candidate with the most first-place votes is the winner; in a *ranked* election the candidate with the most first-place votes is first, the candidate with the second most is second, and so on.

For an election decided under the plurality method, *preference ballots* are not needed, since the voter's second, third, etc. choices are not used. But, since we already have the preference schedule for the Math Club election (Examples 1.1 and 1.2) let's use it to determine the outcome under the plurality method.

EXAMPLE 1.7 THE MATH CLUB ELECTION UNDER THE PLURALITY METHOD

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

■ **TABLE 1-3** Preference schedule for the Math Club election

We discussed the Math Club election in Section 1.1. Table 1-3 shows once again the preference schedule for the election. Counting only first-place votes, we can see that *A* gets 14, *B* gets 4, *C* gets 11, and *D* gets 8. So there you have it: In the case of a *winner-only* election (see Example 1.1) the winner is *A* (Headline: "Alisha wins presidency of the Math Club"); in the case of a *ranked election* (see Example 1.2) the results are: *A* first (14 votes); *C* second (11 votes); *D* third (8 votes); *B* fourth (4 votes). (Headline: "New board of MAS elected! President—Alisha; VP—Carmen; Treasurer—Dave; Secretary—Boris.")

The vast majority of elections for political office in the United States are decided using the plurality method. The main appeal of the plurality method is its simplicity, but as we will see in our next example, the plurality method has many drawbacks.

EXAMPLE 1.8 THE 2010 MAINE GOVERNOR'S ELECTION

Like many states, Maine chooses its governor using the plurality method. In the 2010 election there were five candidates: Eliot Cutler (Independent), Paul LePage (Republican), Libby Mitchell (Democrat), Shawn Moody (Independent), and Kevin Scott (Independent). Table 1-4 shows the results of the election. Before reading on, take a close look at the numbers in Table 1-4 and draw your own conclusions.

Candidate	Votes	Percent
Eliot Cutler (Independent)	208,270	36.5%
Paul LePage (Republican)	218,065	38.2%
Libby Mitchell (Democrat)	109,937	19.3%
Shawn Moody (Independent)	28,756	5.0%
Kevin Scott (Independent)	5,664	1.0%

Source: The New York Times, *www.elections.nytimes.com/2010/results/governor*

■ **TABLE 1-4** Results of 2010 Maine gubernatorial election

A big problem with the plurality method is that when there are more than two candidates we can end up with a winner that does not have a *majority* (i.e., more than 50%) of the votes. The 2010 Maine gubernatorial election is a case in point. As Table 1-4 shows, Paul LePage became governor with the support of only 38.2% of the voters (which means, of course, that 61.8% of the voters in Maine wanted someone else). A few days after the election, an editorial piece in the *Portland Press Herald* expressed the public concern about the outcome.

> *The election of Paul LePage with 38 percent of the vote means Maine's next governor won't take office with the support of the majority of voters—a situation that has occurred in six of the last seven gubernatorial elections . . . . Some people . . . say it's time to reform the system so Maine's next governor can better represent the consensus of voters.* (*Is Winning an Election Enough?* Portland Press Herald, Nov. 10, 2010)

The second problem with the Maine governor election is the closeness of the election: Out of roughly 571,000 votes cast, less than 10,000 votes separated the winner and the runner-up. This is not the plurality method's fault, but it does raise the possibility that the results of the election could have been *manipulated* by a small number of voters. Imagine for a minute being inside the mind of a voter we call Mr. Insincere: "Of all these candidates, I like Kevin Scott the best. But if I vote for Scott I'm just wasting my vote—he doesn't have a chance. All the polls say that it really is a tight race between LePage and Cutler. I don't much care for either one, but LePage is the better of two evils. I'd better vote for LePage." The same thinking, of course, can be applied in the other direction—voters afraid to "waste" their vote on Scott (or Moody, or Mitchell) and insincerely voting for Cutler over Le Page. The problem is that we don't know how many *insincere votes* went one way or the other, and the possibility that there were enough insincere votes to tip the results of the election cannot be ruled out.

While all voting methods can be manipulated by insincere voters, the plurality method is the one that can be most easily manipulated, and insincere voting is quite common in real-world elections. For Americans, the most significant cases of insincere voting occur in close presidential or gubernatorial races between the two major party candidates and a third candidate ("the spoiler") who has little or no chance of winning. Insincere voting not only hurts small parties and fringe candidates, it has unintended and often negative consequences for the political system itself. The history of American political elections is littered with examples of independent candidates and small parties that never get a fair voice or a fair level of funding (it takes 5% of the vote to qualify for federal funds for the next election) because of the "let's not waste our vote" philosophy of insincere voters. The ultimate consequence of the plurality method is an entrenched two-party system that often gives the voters little real choice.

The last, but not least, of the problems with the plurality method is that a candidate may be preferred by the voters over all other candidates and yet not win the election. We will illustrate how this can happen with the example of the fabulous Tasmania State University marching band.

EXAMPLE 1.9 THE FABULOUS TSU BAND GOES BOWLING

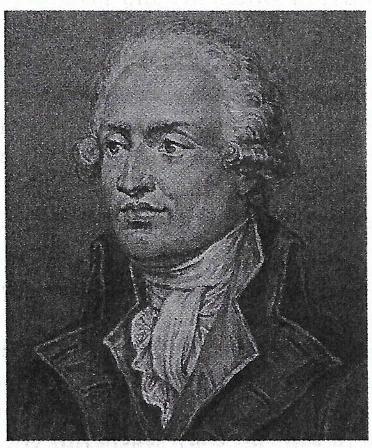
Tasmania State University has a superb marching band. They are so good that this coming bowl season they have invitations to perform at five different bowl games: the Rose Bowl (R), the Hula Bowl (H), the Fiesta Bowl (F), the Orange Bowl (O), and the Sugar Bowl (S). An election is held among the 100 band members to decide in which of the five bowl games they will perform. Each band member

Number of voters	49	48	3
1st	*R*	*H*	*F*
2nd	*H*	*S*	*H*
3rd	*F*	*O*	*S*
4th	*O*	*F*	*O*
5th	*S*	*R*	*R*

TABLE 1-5 Preference schedule for the band election

submits a preference ballot ranking the five choices. The results of the election are shown in Table 1-5.

Under the plurality method the winner of the election is the Rose Bowl (R), with 49 first-place votes. It's hard not to notice that this is a rather bad outcome, as there are 51 voters that have the Rose Bowl as their last choice. By contrast, the Hula Bowl (H) has 48 first-place votes and 52 second-place votes. Simple common sense tells us that the Hula Bowl is a far better choice to represent the wishes of the entire band. In fact, we can make the following persuasive argument in favor of the Hula Bowl: If we compare the Hula Bowl with any other bowl on a *head-to-head* basis, the Hula Bowl is always the preferred choice. Take, for example, a comparison between the Hula Bowl and the Rose Bowl. There are 51 votes for the Hula Bowl (48 from the second column plus the 3 votes in the last column) versus 49 votes for the Rose Bowl. Likewise, a comparison between the Hula Bowl and the Fiesta Bowl would result in 97 votes for the Hula Bowl (first and second columns) and 3 votes for the Fiesta Bowl. And when the Hula Bowl is compared with either the Orange Bowl or the Sugar Bowl, it gets all 100 votes. Thus, no matter with which bowl we compare the Hula Bowl, there is always a majority of the band that prefers the Hula Bowl.



Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743–1794)

A candidate preferred by a majority of the voters over every other candidate when the candidates are compared in head-to-head comparisons is called a **Condorcet candidate** (named after the Marquis de Condorcet, an eighteenth-century French mathematician and philosopher). Not every election has a Condorcet candidate, but if there is one, it is a good sign that this candidate represents the voice of the voters better than any other candidate. In Example 1.9 the Hula Bowl is the Condorcet candidate—it is not unreasonable to expect that it should be the winner of the election. We will return to this topic in Section 1.6.

1.3 The Borda Count Method

The second most commonly used method for determining the winner of an election is the **Borda count method**, named after the Frenchman Jean-Charles de Borda. In this method each place on a ballot is assigned points. In an election with N candidates we give 1 point for *last* place, 2 points for *second from last* place, and so on. At the top of the ballot, a *first-place* vote is worth N points. The points are tallied for each candidate separately, and the candidate with the highest total is the winner. If we are ranking the candidates, the candidate with the second-most points comes in second, the candidate with the third-most points comes in third, and so on. We will start our discussion of the Borda count method by revisiting the Math Club election.

EXAMPLE 1.10 THE MATH CLUB ELECTION (BORDA COUNT)

Table 1-6 shows the preference schedule for the Math Club election with the Borda points for the candidates shown in parentheses to the right of their names. For example, the 14 voters in the first column ranked A first (giving A $14 \times 4 = 56$ points), B second ($14 \times 3 = 42$ points), and so on.

Number of voters	14	10	8	4	1
1st (4 points)	A (56)	C (40)	D (32)	B (16)	C (4)
2nd (3 points)	B (42)	B (30)	C (24)	D (12)	D (3)
3rd (2 points)	C (28)	D (20)	B (16)	C (8)	B (2)
4th (1 point)	D (14)	A (10)	A (8)	A (4)	A (1)

TABLE 1-6 Borda points for the Math Club election

When we tally the points,

A gets $56 + 10 + 8 + 4 + 1 = 79$ points,
B gets $42 + 30 + 16 + 16 + 2 = 106$ points,
C gets $28 + 40 + 24 + 8 + 4 = 104$ points,
D gets $14 + 20 + 32 + 12 + 3 = 81$ points.

The Borda winner of this election is Boris! (Wasn't Alisha the winner of this election under the plurality method?)

If we have to rank the candidates, B is first, C second, D third, and A fourth. To see what a difference the voting method makes, compare this ranking with the ranking obtained under the plurality method (Example 1.7).

EXAMPLE 1.11 THE 2011 HEISMAN AWARD

For general details on the Heisman Award, see Example 1.4. The Heisman is determined using a Borda count, but with *truncated preference ballots*: each voter chooses a first, second, and third choice out of a large list of candidates, with a first-place vote worth 3 points, a second-place vote worth 2 points, and a third-place vote worth 1 point.

Table 1-7 shows a summary of the results of the 2010 Heisman ballot. The table shows the number of first-, second-, and third-place votes for each of the four finalists; the last column shows the total point tally for each. Notice that Table 1-7 is not

Player	1st (3pts.)	2nd (2pts.)	3rd (1 pt.)	Total points
Robert Griffin III	405	168	136	1687
Andrew Luck	247	250	166	1407
Trent Richardson	138	207	150	978
Montee Ball	22	83	116	348
Tyrann Mathieu	34	63	99	327

Source: Heisman Award, *www.heisman.com/winners/r-griffin11.php*

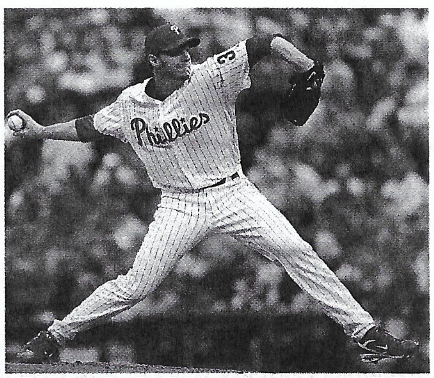
TABLE 1-7 2010 Heisman Trophy: top four finalists a preference schedule. Because the Heisman uses truncated preference ballots and many candidates get votes, it is easier and more convenient to summarize the balloting this way.

The last column of Table 1-7 shows the ranking of the finalists: Robert Griffin III (Baylor) won the Heisman easily, Andrew Luck (Stanford) was second, Trent Richardson (Alabama) was third, Montee Ball (Wisconsin) was fourth, and Tyrann Mathieu (LSU) was fifth. ■

Many variations of the standard Borda count method are possible, the most common being using a different set of values for the positions on the ballot. We will call these **modified Borda count** methods. Example 1.12 illustrates one situation where a modified Borda count is used.

EXAMPLE 1.12 THE 2010 NATIONAL LEAGUE CY YOUNG AWARD

The Cy Young Award is an annual award given by Major League baseball for "the best pitcher" in each league (one award for the American League and one for the National League). For the National League award there are 32 *voters* (they are baseball writers—two from each of the 16 cities having a National League team), and each voter submits a truncated preference ballot with a first, second, third, fourth, and fifth choice. The modification in the Cy Young calculations (in effect for the first time with the 2010 award) is that first place is worth 7 points (rather than 5). The other places in the ballot count just as in a regular Borda count: 4 points for second, 3 points for third, 2 points for fourth, and 1 point for fifth. The idea here is to give extra value to first-place votes—the gap between a first and a second place should be bigger than the gap between a second and a third place.



Pitcher	1st (7 pts.)	2nd (4 pts.)	3rd (3 pts.)	4th (2 pts.)	5th (1 pt.)	Total points
Roy Halladay (PHI)	32	0	0	0	0	224
Adam Wainwright (STL)	0	28	3	0	1	122
Ubaldo Jimenez (COL)	0	4	19	8	1	90
Tim Hudson (ATL)	0	0	3	13	4	39
Josh Johnson (FLA)	0	0	5	5	9	34

Source: Baseball-Reference.com, *www.baseball-reference.com/awards/awards_2010.shtml*

■ **TABLE 1-8** 2010 National League Cy Young Award: top five finalists

Table 1-8 shows the top five finalists for the 2010 National League Cy Young award. An unusual thing happened in this election—Roy Halladay (Phillies) was the unanimous first choice of all 32 voters, thus garnering the maximum possible points ($32 \times 7 = 224$), a very rare event indeed. ■

In real life, the Borda count method (or some variation of it) is widely used in a variety of settings, from individual sport awards to music industry awards to the hiring of school principals, university presidents, and corporate executives. It is generally considered to be a much better method for determining the outcome of an election than the plurality method. In contrast to the plurality method, it takes into account the voter's preferences not just for first place but also for second, third, etc., and then chooses as the winner the candidate with the best average ranking—the best compromise candidate, if you will.

1.4 The Plurality-with-Elimination Method

In the United States most municipal and local elections have a majority requirement—a candidate needs a majority of the votes to get elected. With only two candidates this is rarely a problem (unless they tie, one of the two candidates must have a majority of the votes). When there are three or more candidates running, it can easily happen that no candidate has a majority. Typically, the candidate or candidates with the fewest first-place votes are eliminated, and a runoff election is held. But runoff elections are expensive, and in these times of tight budgets more efficient ways to accomplish the "runoff" are highly desirable.

A very efficient way to implement the runoff process without needing runoff elections is to use preference ballots, since a preference ballot tells us not only which candidate the voter wants to win but also which candidate the voter would choose in a runoff (with one important caveat—we assume the voters are consistent in their preferences and would stick with their original ranking of the candidates). The idea is to use the information in the preference schedule to eliminate the candidates with the fewest first-place votes one at a time until one of them gets a majority. This method has become increasingly popular and is nowadays known under several different names, including *plurality-with-elimination, instant runoff voting* (IRV), *ranked choice voting* (RCV), and the *Hare method.* For the sake of clarity, we will call it the plurality-with-elimination method—it is the most descriptive of all the names.

Here is a formal description of the **plurality-with-elimination method**:

- **Round 1.** Count the first-place votes for each candidate, just as you would in the plurality method. If a candidate has a majority of first-place votes, then that candidate is the winner. Otherwise, eliminate the candidate (or candidates if there is a tie) with the *fewest* first-place votes.
- **Round 2.** Cross out the name(s) of the candidates eliminated from the preference schedule and transfer those votes to the next eligible candidates on those ballots. Recount the votes. If a candidate has a majority then declare that candidate the winner. Otherwise, eliminate the candidate with the fewest votes.
- **Rounds 3, 4, . . .** Repeat the process, each time eliminating the candidate with the fewest votes and transferring those votes to the next eligible candidates. Continue until there is a candidate with a majority. That candidate is the winner of the election.

  In a ranked election the candidates should be ranked in reverse order of elimination: the candidate eliminated in the last round gets second place, the candidate eliminated in the second-to-last round gets third place, and so on.

EXAMPLE 1.13 THE MATH CLUB ELECTION (PLURALITY-WITH-ELIMINATION)

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

■ **TABLE 1-9** Preference schedule for the Math Club election

Let's see how the plurality-with-elimination method works when applied to the Math Club election. For the reader's convenience Table 1-9 shows the preference schedule again.

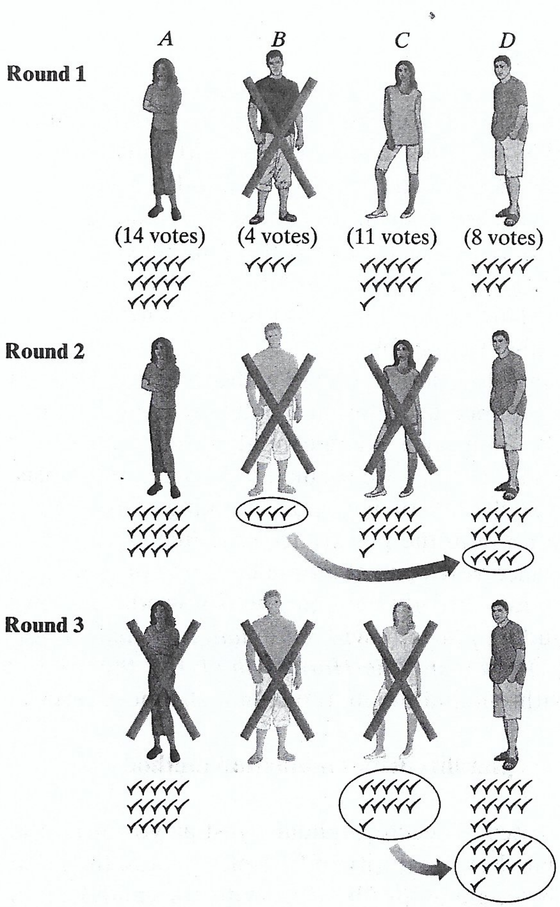


FIGURE 1-6 Boris is eliminated first, then Carmen, and then Alisha. The last one standing is Dave.

Round 1.

Candidate	*A*	*B*	*C*	*D*
Votes	14	4	11	8

B has the fewest first-place votes and is eliminated first (Fig. 1-6).

Round 2. After *B* is eliminated, the four votes that originally went to *B* are transferred to *D* (per column 4 of Table 1-9). The new tally is

Candidate	*A*	*C*	*D*
Votes	14	11	12

In this round *C* has the fewest first-place votes and is eliminated.

Round 3. The 11 votes that went to *C* in round 2 are all transferred to *D* (per columns 2 and 5 of Table 1-9). The new tally is

Candidate	*A*	*D*
Number of first-place votes	14	23

The winner of the election is *D*! For a ranked election we have *D* first, *A* second (eliminated in round 3), *C* third (eliminated in round 2), and *B* last (eliminated in round 1). ■

Our next example illustrates a few subtleties that can come up when applying the plurality-with-elimination method.

EXAMPLE 1.14 THE CITY OF KINGSBURG MAYORAL ELECTION

Table 1-10 shows the preference schedule for the Kingsburg mayoral election first introduced in Example 1.6. To save money Kingsburg has done away with runoff elections and now uses plurality-with-elimination for all local elections. (Notice that since there are 300 voters voting in this election, a candidate needs 151 or more votes to win.)

Number of voters	93	44	10	30	42	81
1st	*A*	*B*	*C*	*C*	*D*	*E*
2nd	*B*	*D*	*A*	*E*	*C*	*D*
3rd	*C*	*E*	*E*	*B*	*E*	*C*
4th	*D*	*C*	*B*	*A*	*A*	*B*
5th	*E*	*A*	*D*	*D*	*B*	*A*

■ **TABLE 1-10** Preference schedule for the Kingsburg mayoral election

Round 1.

Candidate	*A*	*B*	*C*	*D*	*E*
Votes	93	44	40	42	81

Here *C* has the fewest number of first-place votes and is eliminated first. Of the 40 votes originally cast for *C*, 10 are transferred to *A* (per column 3 of Table 1-10) and 30 are transferred to *E* (per column 4 of Table 1-10).

Round 2.

Candidate	*A*	*B*	*D*	*E*
Number of first-place votes	103	44	42	111

Now *D* has the fewest first-place votes and is eliminated. The 42 votes originally cast for *D* would be transferred to *C* (per column 5 of Table 1-10), but *C* has already been eliminated, so the next eligible candidate is *E* (column 5 again). Thus, the 42 votes are transferred to *E*.

Round 3.

Candidate	*A*	*B*	*E*
Number of first-place votes	103	44	153

Since this is a winner-only election we are done! *E* has a majority and is declared the winner. (If this were a ranked election, we would continue on to Round 4, only to determine second place between *A* and *B*.)

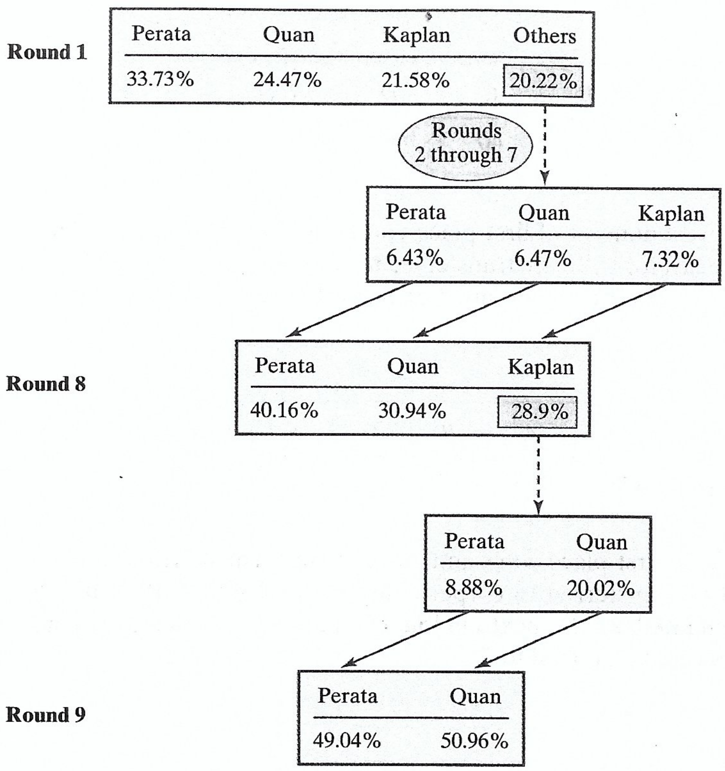
Several variations of the plurality-with-elimination method are used in real-life elections. One of the most popular goes by the name *instant runoff voting* (also called *ranked choice voting* in some places). Instant runoff voting uses a truncated preference ballot (typically asking for just first, second, and third choice). Once the ballots are cast the process works very much like plurality-with-elimination: the candidate with the fewest first-place votes is eliminated and his votes are transferred to the second-place candidates in those ballots; in the next round the candidate with the fewest votes is eliminated and her votes are transferred to the next eligible candidate, and so on. There is one important difference: unlike regular plurality-with-elimination there is a point at which some votes can no longer be transferred (say your vote was for candidates *X*, *Y*, and *Z* —if and when all three of them are eliminated there is no one to transfer your vote to). Such votes are called *exhausted votes* and although perfectly legal, they don't count in the final analysis.

Instant runoff voting is used in several U.S. cities in elections for mayor and city council, including San Francisco, Minneapolis, St. Paul, and Oakland, California, as well as in elections for political office in Australia, Canada, Ireland, and New Zealand. We will illustrate how instant runoff voting works with the 2010 election for mayor of Oakland.

EXAMPLE 1.15 THE 2010 OAKLAND MAYORAL ELECTION

In 2010 the city of Oakland, California introduced instant runoff voting for the first time in all city elections with three or more candidates.

The election for mayor had 10 candidates. The top three candidates were Don Perata, Jean Quan, and Rebecca Kaplan; the remaining seven candidates had relatively few votes and for simplicity they will be lumped together under the name



Source: Alameda County Registrar of Voters

FIGURE 1-7 Results of 2010 Oakland mayoral election

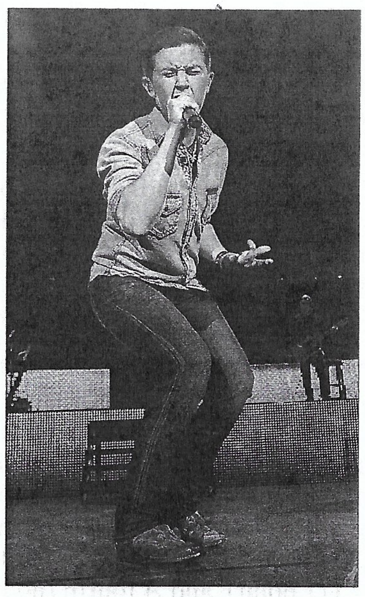
"Others." There were 122,265 ballots cast, with each voter ranking up to three candidates (ballots with just one or two candidates are allowed). Of the original 122,265 ballots, 16,499 were exhausted (more on that later), so the final analysis is based on the 105,766 ballots that really counted. Figure 1-7 shows a summary of how the election went. (To simplify the calculations Fig. 1-7 shows percentages rather than the number of actual votes.) In the first count, Perata had 33.73% of the votes, Quan had 24.47%, Kaplan had 21.58%, and the seven other candidates together had 20.22%. In rounds 1 through 7 all the minor candidates were eliminated one at a time and their votes transferred to the top three candidates or exhausted: 6.43% were transferred to Perata, 6.47% were transferred to Quan, and 7.32% transferred to Kaplan. Round 8 starts with Perata at 40.16%, Quan at 30.94%, and Kaplan at 28.9%. Perata is still the clear leader. Kaplan has the fewest votes (barely) and is eliminated. Kaplan's votes are transferred—8.88% to Perata and 20.02% to Quan. In the final round Quan overtakes Perata 50.96% to 49.04% and is declared mayor of Oakland.

Note that a ballot that did not include either Perata or Quan did not count in the final analysis: at some point before round 9 all of the candidates on that ballot were eliminated, and at that point there was no candidate to transfer that ballot to. This explains the 16,499 exhausted ballots.

As mentioned earlier in this section, the practical advantage of plurality-with-elimination is that it does away with expensive and time-consuming runoff elections. There is one situation, however, where expense is not an issue and delaying the process is part of the game: televised competitions such as *Dancing with the Stars*, *The X-Factor*, and *American Idol*. The longer the competition goes, the higher the ratings—having many runoffs accomplishes this goal. All of these "elections" work under the same variation of the plurality-with-elimination method: have a round of competition, vote, eliminate the candidate (or candidates) with the fewest votes. The following week have another round of competition and repeat the process. This builds up to the last round of competition, when there are two finalists left. Millions of us get caught up in the hoopla. We will illustrate one such election using the 2011 *American Idol* competition.

EXAMPLE 1.16 THE 2011 *AMERICAN IDOL* COMPETITION

We discussed *American Idol* as an election in Example 1.5. Table 1-11 shows the evolution of the 2011 competition. As noted in Example 1.5, the winner is the big deal, but how the candidates place in the competition is also of some relevance, so we consider *American Idol* a ranked election. Working our way up from the bottom of the table, we see that the first candidate eliminated was Ashton Jones. This puts Ashton in 13th place. The second week Karen Rodriguez was eliminated. This puts Karen in 12th place. The third week no one was eliminated (there is a rule that allows the judges to give candidates a free pass to the next round). The fourth week Naima Adepapo and Thia Megia were eliminated in the same round (another mysterious rule). Naima and Thia were declared tied for 10th–11th place.



Winner	→	Scotty McCreery		
Runner-up	→	Lauren Alaina	→	eliminated final week
3rd place	→	Haley Reinhart	→	eliminated week 11
4th place	→	James Durbin	→	eliminated week 10
5th place	→	Jacob Lusk	→	eliminated week 9
6th place	→	Casey Abrams	→	eliminated week 8
7th place	→	Stefano Langone	→	eliminated week 7
8th place	→	Paul McDonald	→	eliminated week 6
9th place	→	Pia Toscano	→	eliminated week 5
10th place 11th place	→	Naima Adepapo Thia Megia (tie)	→	eliminated week 4
12th place	→	Karen Rodriguez	→	eliminated week 2
13th place	→	Ashton Jones	→	eliminated week 1

TABLE 1-11 2011 *American Idol* results

And so it went for a full three months. In the final week, after a lot of controversy, it came down to Lauren Alaina and Scotty McCreery. Lauren was eliminated last, and Scotty became the 2011 *American Idol.*

1.5 The Method of Pairwise Comparisons

One of the most useful features of a preference schedule is that it allows us to find the winner of any **pairwise comparison** between candidates. Specifically, given any two candidates—call them X and Y—we can count how many voters rank X above Y and how many rank Y above X. The one with the most votes wins the pairwise comparison. This is the basis for a method called the **method of pairwise comparisons** (sometimes also called *Copeland's method*). For each possible pairwise comparison between candidates, give 1 point to the winner, 0 points to the loser (if the pairwise comparison ends up in a tie give each candidate $\frac{1}{2}$ point). The candidate with the most points is the winner. (If we are ranking the candidates, the candidate with the second-most points is second, and so on.) The method of pairwise comparisons is very much like a round-robin tournament: (1) every player plays every other player once; (2) the winner of each "match" gets a point and the loser gets no points (if there is a tie each gets $\frac{1}{2}$ point); and (3) the player with the most points wins the tournament.

As usual, we will start with the Math Club election as our first example.

EXAMPLE 1.17 THE MATH CLUB ELECTION (PAIRWISE COMPARISONS)

Number of voters	14	10	8	4	1
1st	A	C	D	B	C
2nd	B	B	C	D	D
3rd	C	D	B	C	B
4th	D	A	A	A	A

TABLE 1-12 Preference schedule for the Math Club election

Table 1-12 shows, once again, the preference schedule for the Math Club election. With four candidates, there are six possible pairwise comparisons to consider: (1) A v B, (2) A v C, (3) A v D, (4) B v C, (5) B v D, and (6) C v D. We'll look at (1) A v B and (6) C v D and leave the details of the other four to the reader.

Pairwise comparison	Votes	Winner
(1) A v B	A (14); B (23)	B
(2) A v C	A (14); C (23)	C
(3) A v D	A (14); D (23)	D
(4) B v C	B (18); C (19)	C
(5) B v D	B (28); D (9)	B
(6) C v D	C (25); D (12)	C
Total points: $C = 3, B = 2, D = 1, A = 0$		

TABLE 1-13 Pairwise comparisons for the Math Club election

- A v B: The first column of Table 1-12 represents 14 votes for A (A is ranked higher than B); the remaining 23 votes are for B (B is ranked higher than A in the last four columns of the table). The winner of this comparison is B.
- C v D: The first, second, and last columns of Table 1-12 represent votes for C (C is ranked higher than D); the third and fourth columns represent votes for D (D is ranked higher than C). Thus, C has 25 votes to D's 12 votes. The winner of this comparison is C.

We continue this way, checking the results of all six possible comparisons (try it now on your own, before you read on!). Once you are done, you should get something along the lines of Table 1-13 with a summary of the results (a sort of scoreboard, if you will.) From Table 1-13 one can immediately figure out the outcome of the election: In a winner-only election the winner is C (with 3 points); in a ranked election C is first (3 points), B second (2 points), D third (1 point), and A fourth (no points).

Method	Winner only	Ranking			
		1st	2nd	3rd	4th
Plurality	A	A	C	D	B
Borda count	B	B	C	D	A
Plurality with elimination	D	D	A	C	B
Pairwise comparisons	C	C	B	D	A

TABLE 1-14 The outcome of the Math Club election under four different voting methods

If you have been paying close attention, you may have noticed that the results of the Math Club election have been different under each of the voting methods we have discussed—both in terms of the winner and in terms of the ranking of the candidates. This can be seen quite clearly in the summary results shown in Table 1-14. It is amazing how much the outcome of an election can depend on the voting method used!

One more important comment about Example 1.17: Notice that C was the *undefeated* champion, as C won each of the pairwise comparisons against the other candidates. (We already saw that there is a name for a candidate that beats all the other candidates in pairwise comparisons—we call such a candidate a *Condorcet* candidate.) The method of pairwise comparisons always chooses the Condorcet candidate (when there is one) as the winner of the election, but this is not true with all methods. Under the plurality method, for example, you can have a Condorcet candidate that does not win the election (see Example 1.9).

Although the method of pairwise comparisons is a pretty good method, in real-life elections it is not used as much as the other three methods we discussed. In the next example we will illustrate one interesting and meaningful (if you are a football fan) application of the method—the selection of draft choices in the National Football League. Because NFL teams are extremely secretive about how they make their draft decisions, we will illustrate the general idea with a made-up example.

EXAMPLE 1.18 THE NFL DRAFT

The Los Angeles LAXers are the newest expansion team in the NFL and are awarded the first pick in the upcoming draft. The draft committee (made up of coaches, scouts, and team executives) has narrowed down the list to five candidates: Allen, Byers, Castillo, Dixon, and Evans. After many meetings, the draft committee is ready to vote for the team's first pick in the draft. The election is to be decided using the method of pairwise comparisons.

Table 1-15 shows the preference schedule obtained after each of the 22 members of the draft committee submits a preference ballot ranking the five candidates. There is a total of 10 separate pairwise comparisons to be looked at, and the results are shown in Table 1-16 (we leave it to the reader to check the details.)

Number of voters	2	6	4	1	1	4	4
1st	*A*	*B*	*B*	*C*	*C*	*D*	*E*
2nd	*D*	*A*	*A*	*B*	*D*	*A*	*C*
3rd	*C*	*C*	*D*	*A*	*A*	*E*	*D*
4th	*B*	*D*	*E*	*D*	*B*	*C*	*B*
5th	*E*	*E*	*C*	*E*	*E*	*B*	*A*

■ **TABLE 1-15** LAXer's draft choice election

We can see from Table 1-16 that the winner of the election is Allen with 3 points. Notice that things are a little trickier here: Allen is the winner of the election even though the draft committee prefers Byers to Allen in a pairwise comparison between the two. We will return to this example in Section 1.6.

Pairwise comparison	Votes	Winner (points)
A v B	A (7); B (15)	B (1)
A v C	A (16); C (6)	A (1)
A v D	A (13); D (9)	A (1)
A v E	A (18); E (4)	A (1)
B v C	B (10); C (12)	C (1)
B v D	B (11); D (11) ← tie	$B\ (\frac{1}{2}); D\ (\frac{1}{2})$
B v E	B (14); E (8)	B (1)
C v D	C (12); D (10)	C (1)
C v E	C (10); E (12)	E (1)
D v E	D (18); E (4)	D (1)
Total points:	$A = 3, B = 2\frac{1}{2}, C = 2, D = 1\frac{1}{2}, E = 1$	

■ **TABLE 1-16** Pairwise comparisons for Example 1.18

You probably noticed in Examples 1-17 and 1-18 that, compared with the other methods, pairwise comparisons takes a lot more work. Each comparison requires a separate calculation, and there seems to be a lot of comparisons that need to be checked. How many? We saw that with 4 candidates there are 6 separate comparisons and with 5 candidates there are 10. As the number of candidates grows, the number of comparisons grows even more. Table 1-17 illustrates the relation

Number of candidates	4	5	6	7	8	9	10	...	N
Number of pairwise comparisons	6	10	15	21	28	36	45	...	$\frac{N(N-1)}{2}$

■ **TABLE 1-17** The number of pairwise comparisons

EXERCISES

WALKING

1.1 Ballots and Preference Schedules

1. Figure 1-8 shows the preference ballots for an election with 21 voters and 5 candidates. Write out the preference schedule for this election.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *B*	1st *A*	1st *C*	1st *D*	1st *A*
2nd *E*	2nd *D*	2nd *E*	2nd *B*	2nd *E*	2nd *C*	2nd *B*
3rd *D*	3rd *B*	3rd *A*	3rd *C*	3rd *D*	3rd *B*	3rd *C*
4th *A*	4th *C*	4th *C*	4th *D*	4th *A*	4th *E*	4th *D*
5th *B*	5th *E*	5th *D*	5th *E*	5th *B*	5th *A*	5th *E*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *B*	1st *A*	1st *D*	1st *D*	1st *A*	1st *C*	1st *A*
2nd *E*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *E*	2nd *D*
3rd *A*	3rd *C*	3rd *B*	3rd *B*	3rd *C*	3rd *D*	3rd *B*
4th *C*	4th *D*	4th *A*	4th *E*	4th *D*	4th *A*	4th *C*
5th *D*	5th *E*	5th *E*	5th *A*	5th *E*	5th *B*	5th *E*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *B*	1st *C*	1st *A*	1st *C*	1st *A*	1st *D*	1st *D*
2nd *E*	2nd *E*	2nd *B*	2nd *E*	2nd *D*	2nd *C*	2nd *C*
3rd *A*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *B*	3rd *B*
4th *C*	4th *A*	4th *D*	4th *A*	4th *C*	4th *A*	4th *E*
5th *D*	5th *B*	5th *E*	5th *B*	5th *E*	5th *E*	5th *A*

FIGURE 1-8

2. Figure 1-9 shows the preference ballots for an election with 17 voters and 4 candidates. Write out the preference schedule for this election.

Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *B*	1st *A*	1st *C*	1st *B*
2nd *A*	2nd *C*	2nd *D*	2nd *A*	2nd *C*
3rd *D*	3rd *D*	3rd *B*	3rd *D*	3rd *D*
4th *B*	4th *A*	4th *C*	4th *B*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *A*	1st *B*	1st *B*	1st *C*	1st *C*
2nd *D*	2nd *C*	2nd *C*	2nd *C*	2nd *A*	2nd *A*
3rd *B*	3rd *D*	3rd *D*	3rd *D*	3rd *D*	3rd *D*
4th *C*	4th *B*	4th *A*	4th *A*	4th *B*	4th *B*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *A*	1st *C*	1st *B*	1st *A*	1st *C*
2nd *C*	2nd *D*	2nd *A*	2nd *C*	2nd *D*	2nd *A*
3rd *D*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *D*
4th *B*	4th *C*	4th *B*	4th *A*	4th *C*	4th *B*

FIGURE 1-9

Exercises 3 through 6 refer to an alternative format for preference ballots in which the names of the candidates appear on the ballot and the voter is asked to put a rank (1, 2, 3, etc.) next to each name [see Fig. 1-1(c)]. (This alternative format makes it easier on the voters and is useful when the names are long or when a misspelled name invalidates the ballot. The main disadvantage is that it tends to favor the candidates that are listed first.)

3. Table 1-25 shows the preference schedule for an election based on the alternative format for the preference ballots. Rewrite Table 1-25 in the conventional preference schedule format used in the text. (Use *A, B, C, D,* and *E* as shorthand for the names of the candidates.)

Number of voters	37	36	24	13	5
Alvarez	3	1	2	4	3
Brownstein	1	2	1	2	5
Clarkson	4	4	5	3	1
Dax	5	3	3	5	4
Easton	2	5	4	1	2

TABLE 1-25

4. Table 1-26 shows the preference schedule for an election based on the alternative format for the preference ballots. Rewrite Table 1-26 in the conventional preference schedule format used in the text. (Use *A, B, C, D,* and *E* as shorthand for the names of the candidates.)

Number of voters	14	10	8	7	4
Andersson	2	3	1	5	3
Broderick	1	1	2	3	2
Clapton	4	5	5	2	4
Dutkiewicz	5	2	4	1	5
Eklundh	3	4	3	4	1

TABLE 1-26

5. Table 1-27 shows the preference schedule for an election. Rewrite Table 1-27 using the alternative preference schedule format.

Number of voters	14	10	8	7	4
1st	*B*	*B*	*A*	*D*	*E*
2nd	*A*	*D*	*B*	*C*	*B*
3rd	*E*	*A*	*E*	*B*	*A*
4th	*D*	*E*	*D*	*E*	*C*
5th	*C*	*C*	*C*	*A*	*D*

TABLE 1-27

6. Table 1-28 shows the preference schedule for an election. Rewrite Table 1-28 using the alternative preference schedule format.

Number of voters	37	36	24	13	5
1st	*A*	*B*	*D*	*C*	*B*
2nd	*C*	*A*	*B*	*A*	*D*
3rd	*B*	*D*	*C*	*E*	*E*
4th	*E*	*C*	*E*	*B*	*A*
5th	*D*	*E*	*A*	*D*	*C*

■ **TABLE 1-28**

7. An election is held to choose the Chair of the Mathematics Department at Tasmania State University. The candidates are Professors Argand, Brandt, Chavez, Dietz, and Epstein (*A*, *B*, *C*, *D*, and *E* for short). Table 1-29 shows the preference schedule for the election.

Number of voters	5	5	3	3	3	2
1st	*A*	*C*	*A*	*B*	*D*	*D*
2nd	*B*	*E*	*D*	*E*	*C*	*C*
3rd	*C*	*D*	*B*	*A*	*B*	*B*
4th	*D*	*A*	*C*	*C*	*E*	*A*
5th	*E*	*B*	*E*	*D*	*A*	*E*

■ **TABLE 1-29**

(a) How many people voted in this election?

(b) How many first-place votes are needed for a majority?

(c) Which candidate had the fewest last-place votes?

8. The student body at Eureka High School is having an election for Homecoming Queen. The candidates are Alicia, Brandy, Cleo, and Dionne (*A*, *B*, *C*, and *D* for short). Table 1-30 shows the preference schedule for the election.

Number of voters	202	160	153	145	125	110	108	102	55
1st	*B*	*C*	*A*	*D*	*D*	*C*	*B*	*A*	*A*
2nd	*D*	*B*	*C*	*B*	*A*	*A*	*C*	*B*	*D*
3rd	*A*	*A*	*B*	*A*	*C*	*D*	*A*	*D*	*C*
4th	*C*	*D*	*D*	*C*	*B*	*B*	*D*	*C*	*B*

■ **TABLE 1-30**

(a) How many students voted in this election?

(b) How many first-place votes are needed for a majority?

(c) Which candidate had the fewest last-place votes?

9. The Demublican Party is holding its annual convention. The 1500 voting delegates are choosing among three possible party platforms: *L* (a liberal platform), *C* (a conservative platform), and *M* (a moderate platform). Seventeen percent of the delegates prefer *L* to *M* and *M* to C. Thirty-two percent of the delegates like *C* the most and *L* the least. The rest of the delegates like *M* the most and *C* the least. Write out the preference schedule for this election.

10. The Epicurean Society is holding its annual election for president. The three candidates are *A*, *B*, and *C*. Twenty percent of the voters like *A* the most and *B* the least. Forty percent of the voters like *B* the most and *A* the least. Of the remaining voters 225 prefer *C* to *B* and *B* to *A*, and 675 prefer *C* to *A* and *A* to *B*. Write out the preference schedule for this election.

1.2 Plurality Method

11. Table 1-31 shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	27	15	11	9	8	1
1st	*C*	*A*	*B*	*D*	*B*	*B*
2nd	*D*	*B*	*D*	*A*	*A*	*A*
3rd	*B*	*D*	*A*	*B*	*C*	*D*
4th	*A*	*C*	*C*	*C*	*D*	*C*

■ **TABLE 1-31**

12. Table 1-32 shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	29	21	18	10	1
1st	*D*	*A*	*B*	*C*	*C*
2nd	*C*	*C*	*A*	*B*	*B*
3rd	*A*	*B*	*C*	*A*	*D*
4th	*B*	*D*	*D*	*D*	*A*

■ **TABLE 1-32**

13. Table 1-33 shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	6	5	4	2	2	2	2
1st	C	A	B	B	C	C	C
2nd	D	D	D	A	B	B	D
3rd	A	C	C	C	A	D	B
4th	B	B	A	D	D	A	A

■ **TABLE 1-33**

14. Table 1-34 shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	6	6	5	4	3	3
1st	A	B	B	D	A	B
2nd	C	C	C	A	C	A
3rd	D	A	D	C	D	C
4th	B	D	A	B	B	D

■ **TABLE 1-34**

15. Table 1-35 shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Percent of voters	24	23	19	14	11	9
1st	C	D	D	B	A	D
2nd	A	A	A	C	C	C
3rd	B	C	E	A	B	A
4th	E	B	C	D	E	E
5th	D	E	B	E	D	B

■ **TABLE 1-35**

16. Table 1-36 shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Percent of voters	25	21	15	12	10	9	8
1st	C	E	B	A	C	C	C
2nd	E	D	D	D	D	B	E
3rd	D	B	E	B	E	A	D
4th	A	A	C	E	A	E	B
5th	B	C	A	C	B	D	A

■ **TABLE 1-36**

17. Table 1-29 (see Exercise 7) shows the preference schedule for an election with five candidates (A, B, C, D, and E). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between candidates, the tie is broken in favor of the candidate with the fewer last-place votes.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

18. Table 1-30 (see Exercise 8) shows the preference schedule for an election with four candidates (A, B, C, and D). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between candidates, the tie is broken in favor of the candidate with the fewer last-place votes.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

19. Table 1-29 (see Exercise 7) shows the preference schedule for an election with five candidates (A, B, C, D, and E). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between two candidates, the tie is broken in favor of the winner of a head-to-head comparison between the candidates.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

20. Table 1-30 (see Exercise 8) shows the preference schedule for an election with four candidates (A, B, C, and D). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between two candidates, the tie is broken in favor of the winner of a head-to-head comparison between the candidates.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

1.3 Borda Count

21. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

22. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

23. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

24. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

25. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the Borda count method to find the complete ranking of the candidates. (*Hint*: The ranking does not depend on the number of voters, so you can pick any convenient number to use for the number of voters.)

26. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the Borda count method to find the complete ranking of the candidates. (*Hint*: The ranking does not depend on the number of voters, so you can pick any convenient number to use for the number of voters.)

27. The 2009 Heisman Award. Table 1-37 shows the results of the balloting for the 2009 Heisman Award. Find the ranking of the top four finalists and the number of points each one received (see Example 1.11).

Player	School	1st	2nd	3rd
Toby Gerhart	Stanford	222	225	160
Mark Ingram	Alabama	227	236	151
Colt McCoy	Texas	203	188	160
Ndamukong Suh	Nebraska	161	105	122

Source: Heisman Award, *www.heisman.com/winners/m-ingram09.php*

TABLE 1-37

28. The 2011 NL Cy Young Award. Table 1-38 shows the top 5 finalists for the 2011 National League Cy Young Award. Find the ranking of the top 5 finalists and the number of points each one received (see Example 1.12).

Pitcher	1st	2nd	3rd	4th	5th
Roy Halladay (PHI)	4	21	7	0	0
Cole Hamels (PHI)	0	0	0	2	13
Ian Kennedy (AZ)	1	3	6	18	3
Clayton Kershaw (LA)	27	3	2	0	0
Cliff Lee (PHI)	0	5	17	9	1

Source: Baseball-Reference.com, *www.baseball-reference.com/awards/awards_2011.shtml*

TABLE 1-38

29. An election was held using the Borda count method. There were four candidates (*A*, *B*, *C*, and *D*) and 110 voters. When the points were tallied (using 4 points for first, 3 points for second, 2 points for third, and 1 point for fourth), *A* had 320 points, *B* had 290 points, and *C* had 180 points. Find how many points *D* had and give the ranking of the candidates. (*Hint*: Figure out how many points are packed in each ballot.)

30. An election was held using the following variation of the Borda count method: 7 points for first-place, 4 points for second, 3 points for third, 2 points for fourth, and 1 point for fifth. There were five candidates (*A*, *B*, *C*, *D*, and *E*) and 30 voters. When the points were tallied *A* had 84 points, *B* had 65 points, *C* had 123 points, and D had 107 points. Find how many points *E* had and give the ranking of the candidates. (*Hint*: Figure out how many points are packed in each ballot.)

1.4 Plurality-with-Elimination

31. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

32. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

33. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

34. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

35. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

36. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

37. Table 1-39 shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). Find the complete ranking of the candidates using the plurality-with-elimination method.

Number of voters	8	7	5	4	3	2
1st	*B*	*C*	*A*	*D*	*A*	*D*
2nd	*E*	*E*	*B*	*C*	*D*	*B*
3rd	*A*	*D*	*C*	*B*	*E*	*C*
4th	*C*	*A*	*D*	*E*	*C*	*A*
5th	*D*	*B*	*E*	*A*	*B*	*E*

■ **TABLE 1-39**

38. Table 1-40 shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). Find the complete ranking of the candidates using the plurality-with-elimination method.

Number of voters	7	6	5	5	5	5	4	2	1
1st	*D*	*C*	*A*	*C*	*D*	*E*	*B*	*A*	*A*
2nd	*B*	*A*	*B*	*A*	*C*	*A*	*E*	*B*	*C*
3rd	*A*	*E*	*E*	*B*	*A*	*D*	*C*	*D*	*E*
4th	*C*	*B*	*C*	*D*	*E*	*B*	*D*	*E*	*B*
5th	*E*	*D*	*D*	*E*	*B*	*C*	*A*	*C*	*D*

■ **TABLE 1-40**

Top-Two IRV. *Exercises 39 and 40 refer to a simple variation of the plurality-with-elimination method called top-two IRV. This method works for winner-only elections. Instead of eliminating candidates one at a time, we eliminate all the candidates except the top two in the first round and transfer their votes to the two remaining candidates.*

39. Find the winner of the election given by Table 1-39 using the *top-two IRV* method.

40. Find the winner of the election given by Table 1-40 using the *top-two IRV* method.

1.5 Pairwise Comparisons

41. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

42. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

43. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

44. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

45. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Find the winner of the election using the method of pairwise comparisons.

46. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (*A*, *B*, *C*, *D*, and *E*). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Find the winner of the election using the method of pairwise comparisons.

47. Table 1-39 (see Exercise 37) shows the preference schedule for an election with 5 candidates. Find the complete ranking of the candidates using the method of pairwise comparisons. (Assume that ties are broken using the results of the pairwise comparisons between the tying candidates.)