

①

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

- Find $P(x)$.
- Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.
- Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.
- Let G be the function given by $G(x) = \int_0^x f(t) dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

②

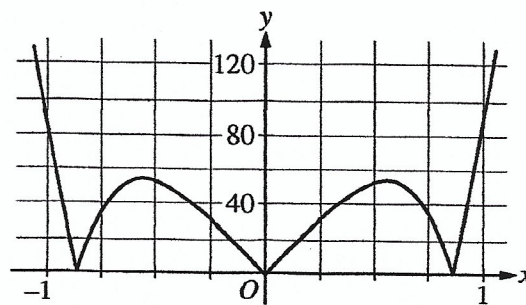
Let $f(x) = \ln(1 + x^3)$.

- The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \cdot \frac{x^n}{n} + \dots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f .
- The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.
- Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2) dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.
- The Maclaurin series for g , evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

③

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = |f^{(5)}(x)|$ is shown above.

- Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.
- Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.
- Find the value of $f^{(6)}(0)$.
- Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = |f^{(5)}(x)|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.



Graph of $y = |f^{(5)}(x)|$

$$f(0) = 0$$

$$f'(0) = 1$$

$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

8. 6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f .

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t) dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the n th-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is

the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

9. 6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n} + \dots$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f .

(b) Determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \dots$$

- (a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g .
- (b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.
- (c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

5 The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the n th derivative of f at $x = 1$ is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.
- (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

6 The Maclaurin series for a function f is given by $\frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \dots + \frac{x^n}{n+2} + \dots$

- (a) Use the ratio test to find the interval of convergence of the Maclaurin series for f .
- (b) Let g be the function given by $g(x) = f(-2x)$. Find the first three terms and the general term of the Maclaurin series for g .
- (c) The first two terms of the Maclaurin series for f are used to approximate $f(0.1)$. Given that $|f'''(x)| \leq 2$ for $0 \leq x \leq 0.1$, use the Lagrange error bound to show that this approximation differs from $f(0.1)$ by at most $\frac{1}{3000}$.

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28. Let f be a function that has derivatives of all orders for all real numbers, and let $P_3(x)$ be the third-degree Taylor polynomial for f about $x = 0$. The Taylor series for f about $x = 0$ converges at $x = 1$, and $|f^{(n)}(x)| \leq \frac{n}{n+1}$ for $1 \leq n \leq 4$ and all values of x . Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \leq k$?

- (E) $\frac{3}{4}$
 (D) $\frac{3}{4}$
 (C) $\frac{4}{5}$
 (B) $\frac{4}{5}$
 (A) $\frac{4}{5}$