Introduction to Differential equation (O.D.E)

A differential equation is an equation containing derivatives such as:

(1) $xy' = y - 5$

(2) $\dfrac{d^2y}{dx^2} - 2\,\dfrac{dy}{dx} - 6x = 0$

(3) $(x-1)\,\dfrac{dy}{dx} = y(x^2 + x - 2)$

(4) $\dfrac{dy}{dx} - \dfrac{y}{x} = x^3$

(5) $y'' = 12x^2 - 4x + 6$

Each equation has an independent variable x and a dependent variable y.
The order of a Differential Equation (D.E.) is the order of the highest ordered derivative in the equation.
Therefore, equations (1), (3) and (4) are first order D.E. while (2) and (5) are second order D.E.

Types of Differential Equations

The objective is to solve a differential equation i.e. to find $y = f(x)$ as a solution to the differential equation.
In this course , we examine two types of D.E. :
Ordinary differential equation (O.D.E.) and First Order O.D.E. (variables separable).
The first order O.D.E. is used in the fields of Social Sciences , Business and Economics applications.
Two kinds of solutions for D.E. :
General solution of $y = f(x)$ where C is an arbitrary constant.
Particular solution when a condition is given in the problem.

Example 1:

Verify that $y = 4x^3 - 5x^2$ is a solution to the differential equation $y' - \dfrac{2}{x}\,y = 4x^2$

need $y' = 12x^2 - 10x$ and replace y and y' into the differential equation

we get: $(12x^2 - 10x) - \dfrac{2}{x}\,(4x^3 - 5x^2) = 4x^2 \rightarrow 12x^2 - 10x - 8x^2 + 10x = 4x^2$

$4x^2 = 4x^2 \quad \surd$ it is a solution.

Example 2:

Verify that $y = 4x^2 + 25x$ is a solution to the differential equation $xy' = y + 6x^2$

need $y' = 8x + 25$ and replace y and y' into the differential equation

we get: $x(8x+25) = (6x^2 + 25x) + 6x^2 \rightarrow 8x^2 + 25x = 6x^2 + 25x + 6x^2$

$8x^2 \neq 12x^2$ X it is not a solution.

Example 3:

Show that $y = 5\,e^{x} - e^{-3x}$ is a solution to the differential equation $y'' + 2y' - 3y = 0$

need $y' = 5\,e^{x} + 3\,e^{-3x}$ and $y'' = 5\,e^{x} - 9\,e^{-3x}$

replace y , y' and y' into the differential equation

we get: $\left(5\,e^{x} - 9\,e^{-3x}\right) + 2\left(5\,e^{x} + 3\,e^{-3x}\right) - 3\left(5\,e^{x} - e^{-3x}\right) = 0$

$5\,e^{x} - 9\,e^{-3x} + 10\,e^{x} + 6\,e^{-3x} - 15\,e^{x} + 3e^{-3x} = 0 \longrightarrow \quad 0 = 0$ $\surd$ it is a solution.

Example 4:

Verify that $y = 5\,e^{-x} + 7x\,e^{-2x}$ is a solution to the differential equation $y'' + 4y' + 4y = 0$

need $y' = -5\,e^{-x} + 7\,e^{-2x} - 14x\,e^{-2x}$ and

$y'' = 5\,e^{-x} - 14\,e^{-2x} + 28x\,e^{-2x}$

replace y , y' and y' into the differential equation

we get: $\left(5\,e^{-x} - 14\,e^{-2x} + 28x\,e^{-2x}\right) + 4\left(-5\,e^{-x} + 7\,e^{-2x} - 14x\,e^{-2x}\right) + 4\left(5\,e^{-x} + 7x\,e^{-2x}\right) = 0$

$5\,e^{-x} - 14\,e^{-2x} + 28x\,e^{-2x} - 20\,e^{-x} + 28\,e^{-2x} - 56x\,e^{-2x} + 20\,e^{-x} + 35x\,e^{-2x} = 0$

$5\,e^{-x} + 14\,e^{-2x} + 7x\,e^{-2x} \neq 0$ X it is not a solution.

Example 5: Ordinary Differential Equation

Solve for y if $y' = 81x^2 - 26x + \dfrac{5}{x}$ with the condition $y(1) = 30$

$$y = \int \left(81x^2 - 26x + \frac{5}{x}\right)\, dx = 81\,\frac{x^3}{3} - 26\,\frac{x^2}{2} + 5\,\ln|x| + C = 27x^3 - 13x^2 + 5\,\ln|x| + C$$

You can always verify that the general solution $y = 27x^3 - 13x^2 + 5\,\ln|x| + C$ is correct by taking the derivative of y and check that it does verify the original differential equation.

replace the condition $x = 1$ and $y = 30$ to solve for C :

$$30 = 27(1)^3 - 13(1)^2 + 5\,\ln 91) + C \rightarrow C = 16$$

Therefore the particular solution is $y = 27x^3 - 13x^2 + 5\,\ln|x| + 16$

Example 6:

Solve for y if $y'' = 6 - 52x + \dfrac{12}{\sqrt{x}}$ with the conditions $y'(4) = 20$ and $y(0) = 6$.

$$y' = \int \left(6 - 52x + 12x^{-1/2}\right)\, dx = 6x - 52\,\frac{x^2}{2} + 12\,\frac{x^{1/2}}{1/2} + C_1 = 6x - 26x^2 + 24x^{1/2} + C_1$$

replace the condition $x = 4$ and $y' = 20$ to solve for C_1 :

$$20 = 6(4) - 26(4)^2 + 24(4)^{1/2} + C_1 \longrightarrow C_1 = 364 \longrightarrow y' = 6x - 26x^2 + 24x^{1/2} + 364$$

$$y = \int \left(6x - 26x^2 + 24x^{1/2} + 364\right)\, dx = 6\,\frac{x^2}{2} - 26\,\frac{x^3}{3} + 24\,\frac{x^{3/2}}{3/2} + 364x + C_2$$

$$y = 3x^2 - \frac{26}{3}\,x^3 + 16x^{3/2} + 364x + C_2$$

replace the condition $x = 0$ and $y = 6$ to solve for C_2 : $6 = 0 - 0 + 0 + 0 + C_2 \longrightarrow C_2 = 6$

$$y = 3x^2 - \frac{26}{3}\,x^3 + 16x^{3/2} + 364x + 6$$

Example 7: First Order Differential Equation (Variables separable)

Solve for y if $(x+1)\,\dfrac{dy}{dx} = \dfrac{x^2-1}{y}$ with the condition $y(2)=3$.

Separate variables: y terms on left side and x terms on right side

we get: $y\,dy = \dfrac{x^2-1}{x+1}\,dx$; reduce the right side: $y\,dy = (x-1)\,dx$

integrate each side of the equation: $\displaystyle\int y\,dy = \int (x-1)\,dx \longrightarrow \frac{y^2}{2} = \frac{x^2}{2} - x + k$

general solution: $y^2 = x^2 - 2x + C$

replace the condition $x=2$ and $y=3$ to solve for C:

$(3)^2 = (2)^2 - 2(2) + C \longrightarrow C = 9 \longrightarrow y^2 = x^2 - 2x + 9 \longrightarrow$ particular solution: $y = \pm\sqrt{x^2-2x+9}$

Example 8: First Order Differential Equation (Variables separable)

Solve for y if $xy' = y(2-x)$ with the condition $y(1)=2$.

Rewrite $x\,\dfrac{dy}{dx} = y(2-x)$; separate variables: y terms on left side and x terms on right side

we get: $\dfrac{1}{y}\,dy = \dfrac{2-x}{x}\,dx$; reduce the right side: $\dfrac{1}{y}\,dy = (2x^{-1}-1)\,dx$

integrate each side of the equation: $\displaystyle\int y^{-1}\,dy = \int (2x^{-1}-1)\,dx \longrightarrow \ln|y| = 2\,\ln|x| - x + k$

general solution: $\ln|y| = 2\,\ln|x| - x + k \longrightarrow y = e^{2\,\ln|x| - x + k} = e^{2\,\ln(x)}\,.\,e^{-x}\,.\,e^{k}$

let $e^{2\,\ln(x)} = e^{\ln(x^2)} = x^2$ and $e^k = C \longrightarrow$ general solution: $y = Cx^2\,e^{-x} = C\,\dfrac{x^2}{e^x}$

replace the condition $x=1$ and $y=2$ to solve for C:

$2 = C\,\dfrac{(1)^2}{e^1} \longrightarrow$ particular solution $y = 2e^1\,\dfrac{x^2}{e^x} \longrightarrow y = 2x^2\,e^{1-x}$

Example 9: First Order Differential Equation (Variables separable)

Solve for y if $y' = y\,\sin(x)$ with the condition $y(\pi) = -3$.

Rewrite $\dfrac{dy}{dx} = y\,\sin(x)$; separate variables: y terms on left side and x terms on right side

we get: $\dfrac{1}{y}\,dy = \sin(x)\,dx$; integrate each side of the equation: $\displaystyle\int y^{-1}\,dy = \int \sin(x)\,dx \longrightarrow \ln|y| = -\cos(x) + k$

general solution: $y = e^{-\cos(x)+k} = e^{-\cos(x)} \,.\, e^{k}$; let $e^k = C \longrightarrow$ general solution: $y = C\,e^{-\cos(x)}$

replace the condition $x = \pi$ and $y = -3$ to solve for C:

$-3 = C\,e^1 \rightarrow C = -3e^{-1} \longrightarrow$ particular solution $y = -3e^{-1}\,e^{-\cos(x)} \longrightarrow y = -3\,e^{-1-\cos(x)}$

Application to First Order Differential Equation

The application in Social Sciences, Business and Economics is a word problem.
The first step is to set up a differential equation based on some words in the problem.

Examples of set up D.E. : k is the constant of proportionality.

(1) If the rate of change of a quantity Q with respect to time t is proportional to the quantity Q has the D.E.

$$\frac{dQ}{dt} = k\,Q$$

(2) If the rate of change of a quantity Q with respect to time t is inversely proportional to the time t has the D.E.

$$\frac{dQ}{dt} = k\,\frac{1}{t}$$

(3) If the rate of change of a quantity Q with respect to time t is proportional to the product of quantity and time has the D.E.

$$\frac{dQ}{dt} = k\,Q\,t$$

(4) If the rate of change of a quantity Q with respect to time t is inversely proportional to the square of quantity Q has the D.E.

$$\frac{dQ}{dt} = k\,\frac{1}{Q^2}$$

(5) If the rate of change of a quantity Q with respect to time t is proportional to the square root of time t has the D.E.

$$\frac{dQ}{dt} = k\,\sqrt{t}$$

(6) If the rate of change of a quantity Q with respect to time t is proportional to the square of time t and inversely proportional to the quantity Q has the D.E.

$$\frac{dQ}{dt} = k\,\frac{t^2}{Q}$$

The second step is to solve each differential equation example by separating the variables.

(1) $\dfrac{dQ}{dt} = k\,Q$ (2) $\dfrac{dQ}{dt} = k\,\dfrac{1}{t}$

(3) $\dfrac{dQ}{dt} = k\,Q\,t$ (4) $\dfrac{dQ}{dt} = k\,\dfrac{1}{Q^2}$

(5) $\dfrac{dQ}{dt} = k\,\sqrt{t}$ (6) $\dfrac{dQ}{dt} = k\,\dfrac{t^2}{Q}$

The third step is to determine the general solution of each differential equation example.

(1) $\ln(Q) = kt + C$ (2) $Q = k\,\ln(t) + C$

(3) $\ln(Q) = k\,\dfrac{t^2}{2} + C$ (4) $\dfrac{Q^3}{3} = k\,t + C$

(5) $Q = \dfrac{2}{3}\,k\,t^{3/2} + C$ (6) $\dfrac{Q^2}{2} = k\,\dfrac{t^3}{3} + C$

The fourth step is to use the given conditions to solve for the value of C and the value of k; with these results, find the particular solution of each D.E.; reduce if possible the particular solution and finally answer to the question(s) specified in the problem .

Note:

At the beginning of each application of differential equation, identify and describe the variables;
write also the given conditions.

Here are some Applications to Differential Equations:

1. The population of a city is increasing at a rate proportional to the difference between the maximum population of 40 000 and the population P at time t (number of years after 1993).
If the population in 1993 was 20 000 and in 1998 was 25 000, what will be the population in 2003?

2. A company believes that the production of units N is increasing at a rate proportional to the square number of units produced at time t in years.
If 200 units are produced presently, and 300 units after 1 year, what is the production in 2 years?

3. In a country of 3 000 000 people, the prime minister suffers a heart attack which the government does not officially publicize. Initially, 50 governmental personnel know of the attack and are spreading this information as a rumor and R is the number of people who heard the rumor at time t in weeks. The spreading of the rumor is increasing at a rate proportional to the number of people who have not heard the rumor at time t.
At the end of one week, 5000 people know the rumor. How many people know the rumor after two weeks?

4. In a city whose population is 100 000 an outbreak of flu occurs. When the city health department begins its record keeping, there are 625 infected people. The number of infected people N is increasing at a rate proportional to the square root number of infected people at time t in weeks. One week later, there are 1600 infected people.
Find the number of infected people two weeks after the record keeping begins.

5. A small town decides to conduct a fund-raising drive for a new park facilities for kids. The cost is \$50 000. The initial amount given by City Hall is \$8 000. The contribution to the fund F is increasing at a rate proportional to the difference between the cost of \$50 000 and the amount F at time t in months. After one month, \$36 000 is in the fund. How much will be in the fund after 3 months?

Word problems on separable differential equations (answers next page)

1. The rate of increase of the population P of a village is proportional to the population size. In 1975 the population was 2500 and in 1977 it was 3000. In what year will the population reach 4320?
2. The rate of decay of a radioactive substance is proportional to the amount $N(t)$ of substance remaining at time t. If initially the amount of substance is 10 g and if 80% of the initial amount remains after 2 hours, find the amount $N(t)$ of radioactive substance at time t.
3. The rate of decomposition of a substance is proportional to the amount N of substance present at time t. If 70% of the initial amount of substance has decomposed after 4 hours, find the amount of substance that remains after 8 hours if initially the amount of substance was 120 g.
4. The rate of increase of the population P of a town is proportional to the time t (the number of years after 1976) and inversely proportional to the population size P. In 1976 the population was 10,000 and in 1986 it was 20,000. In what year will the population be 52,000?
5. The rate of increase of the population P of a city is proportional to the population size P and inversely proportional to the time t (the number of years after 1965). In 1966 the population was 800,000. By 1974 it had grown by 6%. What was the population of the city in 1983?
6. The sales volume S of a company (in millions of dollars) is increasing at a rate inversely proportional to the square root of time t (in years). At present, the sales volume is $40 million. The company predicts that in one year, the sales volume will be $50 million. Find the sales volume in 4 years.
7. The sales volume S of a company (in millions of dollars) is increasing at a rate proportional to the product of the square of both the sales volume S and the time t (in years). The company started with a sales volume of $100 million. Three years later, the sales volume reached $120 million. When will the sales volume reach $165 million?
8. A metal company's production N is increasing at a rate proportional to the product of the number N of units and the time t in years. The company production is presently 400 units. In 2 years, 1200 units are expected. What will be the production in 4 years?
9. A car is bought for $36,000. Its value V is depreciating at a rate proportional to the present value V. After 2 years, the vehicle is worth $18,000. What is the value of the car after 4 years?
10. A piece of machinery is worth $1600. Its value V is depreciating at a rate proportional to the square root of its value V. The piece of machinery will be worth $900 in two years. When will the piece of machinery be worth $625?
11. A rumor starts in a population of 10,000. The rumor spreads at a rate proportional to the number of people who at time t have not heard the rumor. Initially, 25 people have heard the rumor; at the end of 3 weeks, 6675 people have heard it. How many people will have heard the rumor after 6 weeks?
12. The production of N units is increasing at a rate proportional to the product of the number of units N and the time t in years. Initially 100 units are produced and after one year 250 units. When will production reach 500 units?
13. A company's production is increasing at a rate proportional to the number N of units produced and inversely proportional to the square of the time t in years. The maximum production is 5000 units. In the first year, 1000 units are produced. What is the production after 5 years?
14. A piece of furniture is worth $2500. Its value V is depreciating at a rate proportional to the square of its value $V(t)$ at time t in years. The piece of furniture will be worth $1500 in two years. How much will it be worth in 3 years?
15. The rate of increase of the population P of a small city is proportional to the time in years and inversely proportional to the population size P. Initially, the population is 10,000 and in 10 years it will be 20,000. How long will it take for the population to reach 41,000?
16. A rumor starts in a population of 100,000. The rumor spreads at a rate proportional to the time t in weeks and inversely proportional to the number of people N who have heard the rumor. Initially, 100 people have heard the rumor; at the end of 5 weeks, 10,000 people have heard it. How many people will have heard the rumor after 25 weeks?
17. The production of N units is increasing at a rate proportional to the square of the time t in years. Initially, 1300 units are produced; after one year, 2100 units. How long will it take for the production to reach 22,900 units?
18. A company's production is increasing at a rate proportional to the product of the number N of units and the square of the time t in years. Initially, 8 units are produced. In one year, 16 units are expected. After 2 years, what will be the production?
19. A piece of furniture is worth $2500. Its value V is depreciating at a rate proportional to the value and inversely proportional to the square root of the time t in years. The piece of furniture will be worth $1600 in four years. How long will it take for it to be worth $1280?
20. The rate of increase of the population P of a city is proportional to the product of the population size P and the square of time t in years. Initially, the population is 100,000 and in 3 years it is 300,000. How long will it take for the population to be 1,200,000?

Answers

1. $dP/dt = kP$, where t is the number of years after 1975; $P = Ce^{kt}$, $C = 2500$, $k = \frac{1}{2}\ln(1.2)$, so $P = 2500(1.2)^{t/2}$; when $P = 4320$, $t = 2\ln(1.728)/\ln(1.2) \approx 6$ years (1981).
2. $dN/dt = kN$, $N(0) = 10$, $N(2) = 8$; $N = Ce^{kt}$, $C = 10$, $k = \frac{1}{2}\ln(0.8)$, so $N = 10(0.8)^{t/2}$.
3. $dN/dt = kN$, $N(0) = 120$, $N(4) = 36$; $N = Ce^{kt}$, $C = 120$, $k = \frac{1}{4}\ln(0.3)$, so $N = 120(0.3)^{t/4}$; when $t = 8$, $N = 10.8\,\mathrm{g}$.
4. $dP/dt = kt/P$, $P(0) = 10$ (thousands), $P(10) = 20$; $P^2 = kt^2 + C$, $C = 100$, $k = 3$; $P = 52$ when $t^2 = 868$, so $t \approx 29$ years (2005).
5. $dP/dt = kP/t$, $P(1) = 800$ (thousands), $P(9) = 848$; $P = Ct^k$, $C = 800$, $k = \ln(1.06)/\ln 9$; $P(18) \approx 863.732$ (about 863,732).
6. $dS/dt = k/\sqrt{t}$, $S(0) = 40$ (millions), $S(1) = 50$; $S = 2k\sqrt{t} + C$, $C = 40$, $k = 5$; $S(4) = 60$ (\$60 million).
7. $dS/dt = kS^2t^2$; $-1/S = \frac{1}{3}kt^3 + C$, $C = -1/100$, $k = 1/5400$, so $S = 16200/(162 - t^3)$; when $S = 165$, $t^3 = 702/11$, so $t \approx 4$ years.
8. $dN/dt = kNt$, $N(0) = 400$, $N(2) = 1200$; $N = Ce^{kt^2/2}$, $C = 400$, $k = \frac{1}{2}\ln 3$, so $N = 400 \cdot 3^{t^2/4}$; $N(4) = 32{,}400$.
9. $dV/dt = kV$; $V = Ce^{kt}$, $C = 36$, $k = \frac{1}{2}\ln(0.5)$, so $V = 36(0.5)^{t/2}$; $V(4) = 9$ (\$9000).
10. $dV/dt = k\sqrt{t}$, $V(0) = 1600$, $V(2) = 900$; $\sqrt{V} = \frac{1}{2}kt + C$, $C = 40$, $k = -10$; when $V = 625$, $t = 3$ years.
11. Let $N(t)$ be the number of people who have heard the rumor after t weeks. Then $dN/dt = k(10000 - N)$, $N(0) = 25$, $N(3) = 6675$; $N = 10000 - Ce^{-kt}$, $C = 9975$, $k = -\frac{1}{3}\ln\frac{1}{3} = \frac{1}{3}\ln 3$, so $N = 10000 - 9975(\frac{1}{3})^{t/3} = 10000 - 9975(3)^{-t/3}$; $N(6) = 8891\frac{2}{3} \approx 8892$.
12. $dN/dt = kNt$, $N(0) = 100$, $N(1) = 250$; $N = Ce^{kt^2/2}$, $C = 100$, $k = 2\ln(2.5)$, so $N = 100(2.5)^{t^2}$; when $N = 500$, $t^2 = \ln 5/\ln(2.5)$, so $t \approx 1.33$ years.
13. $dN/dt = kN/t^2$; $N = Ce^{-k/t}$, $C = 5$ (thousands), $k = -\ln(0.2) = \ln 5$, so $N = 5(\frac{1}{5})^{1/t} = 5^{1-1/t}$; $N(5) \approx 3.624$ (3624).
14. $dV/dt = kV^2$, $V(0) = 2500$, $V(2) = 1500$; $-1/V = kt + C$, $C = -1/2500$, $k = -1/7500$, so $V = 7500/(t+3)$; $V(3) = 1250$ (\$1250).
15. $dP/dt = kt/P$, $P(0) = 10$ (thousands), $P(10) = 20$; $P^2 = kt^2 + C$, $C = 100$, $k = 3$; $P = 41$ when $t^2 = 527$, so $t \approx 23$ years.
16. $dN/dt = kt/N$, $N(0) = 100$, $N(5) = 10{,}000$; $N^2 = kt^2 + C$, $C = 10{,}000$, $k = 3{,}999{,}600$; when $t = 25$, $N \approx 49{,}998$.
17. $dN/dt = kt^2$, $N(0) = 13$ (hundreds), $N(1) = 21$; $N = \frac{1}{3}kt^3 + C$, $C = 13$, $k = 24$; $N = 229$ when $t^3 = 27$, so $t = 3$ years.
18. $dN/dt = kNt^2$, $N(0) = 8$, $N(1) = 16$; $N = Ce^{kt^3/3}$, $C = 8$, $k = 3\ln 2$, so $N = 8 \cdot 2^{t^3}$; $N(2) = 2048$.
19. $dV/dt = kV/\sqrt{t}$; $V = Ce^{2k\sqrt{t}}$, $C = 2500$, $k = \frac{1}{4}\ln(0.64)$, so $V = 2500(0.64)^{\sqrt{t}/2}$; when $V = 1280$, $\sqrt{t} = 2\ln(0.512)/\ln(0.64) = 3$, so $t = 9$ years.
20. $dP/dt = kPt^2$, $P(0) = 100$ (thousands), $P(3) = 300$; $P = Ce^{kt^3/3}$, $C = 100$, $k = \frac{1}{9}\ln 3$, so $P = 100 \cdot 3^{t^3/27}$; when $N = 1200$, $\frac{1}{27}t^3 = \ln 12/\ln 3$, so $t \approx 3.9$ years.