4.1 Polynomial Functions and Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Polynomials (Appendix A, Section A.3, pp. A22–A29)
- Using a Graphing Utility to Approximate Local Maxima and Local Minima (Section 2.3, p. 74)
- Intercepts of a Function (Section 2.2, pp. 61–63)
- Graphing Techniques: Transformations (Section 2.5, pp. 90–99)
- Intercepts (Section 1.2, pp. 11–12)

Now Work the **'Are You Prepared?'** problems on page 183.

OBJECTIVES
1. Identify Polynomial Functions and Their Degree (p. 166)
2. Graph Polynomial Functions Using Transformations (p. 170)
3. Identify the Real Zeros of a Polynomial Function and Their Multiplicity (p. 171)
4. Analyze the Graph of a Polynomial Function (p. 178)
5. Build Cubic Models from Data (p. 182)

1 Identify Polynomial Functions and Their Degree

In Chapter 3, we studied the linear function $f(x) = mx + b$, which can be written as

$$f(x) = a_1x + a_0$$

and the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, which can be written as

$$f(x) = a_2x^2 + a_1x + a_0 \qquad a_2 \neq 0$$

Each of these functions is an example of a *polynomial function*.

DEFINITION

A **polynomial function** is a function of the form

$$\boxed{f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \qquad \textbf{(1)}}$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers.

In Words
A polynomial function is a sum of monomials.

A polynomial function is a function whose rule is given by a polynomial in one variable. The **degree** of a polynomial function is the largest power of x that appears. The zero polynomial function $f(x) = 0 + 0x + 0x^2 + \cdots + 0x^n$ is not assigned a degree.

Polynomial functions are among the simplest expressions in algebra. They are easy to evaluate: only addition and repeated multiplication are required. Because of this, they are often used to approximate other, more complicated functions. In this section, we investigate properties of this important class of functions.

EXAMPLE 1 Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

(d) $F(x) = 0$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x - 1)^2$

Solution (a) f is a polynomial function of degree 4.

(b) g is not a polynomial function because $g(x) = \sqrt{x} = x^{\frac{1}{2}}$, so the variable x is raised to the $\frac{1}{2}$ power, which is not a nonnegative integer.

(c) h is not a polynomial function. It is the ratio of two distinct polynomials, and the polynomial in the denominator is of positive degree.

(d) F is the zero polynomial function; it is not assigned a degree.

(e) G is a nonzero constant function. It is a polynomial function of degree 0 since $G(x) = 8 = 8x^0$.

(f) $H(x) = -2x^3(x-1)^2 = -2x^3(x^2 - 2x + 1) = -2x^5 + 4x^4 - 2x^3$. So H is a polynomial function of degree 5. Do you see a way to find the degree of H without multiplying out?

Now Work PROBLEMS 15 AND 19

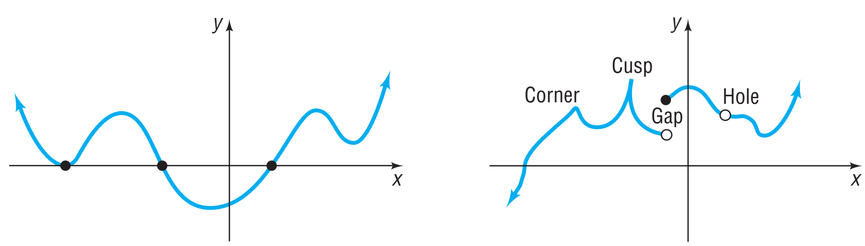
We have already discussed in detail polynomial functions of degrees 0, 1, and 2. See Table 1 for a summary of the properties of the graphs of these polynomial functions.

Table 1

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

One objective of this section is to analyze the graph of a polynomial function. If you take a course in calculus, you will learn that the graph of every polynomial function is both smooth and continuous. By **smooth,** we mean that the graph contains no sharp corners or cusps; by **continuous,** we mean that the graph has no gaps or holes and can be drawn without lifting pencil from paper. See Figures 1(a) and (b).

Figure 1



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

Power Functions

We begin the analysis of the graph of a polynomial function by discussing *power functions,* a special kind of polynomial function.

DEFINITION

In Words

A power function is defined by a single monomial.

A **power function of degree n** is a monomial function of the form

$$f(x) = ax^n \tag{2}$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Examples of power functions are

$$f(x) = 3x \qquad f(x) = -5x^2 \qquad f(x) = 8x^3 \qquad f(x) = -5x^4$$

degree 1 degree 2 degree 3 degree 4

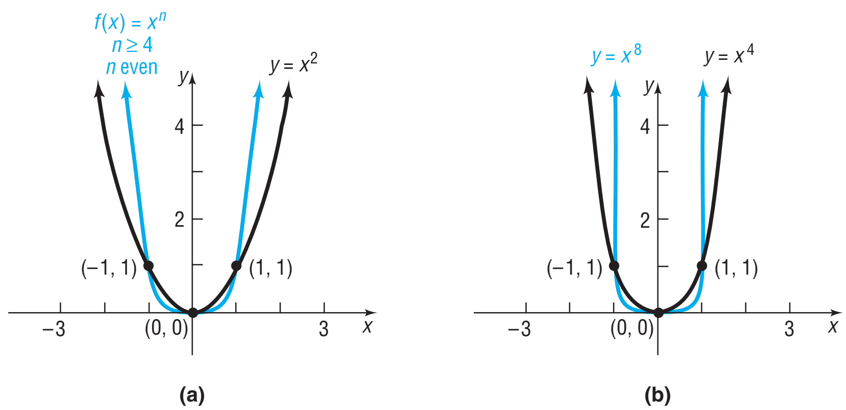
The graph of a power function of degree 1, $f(x) = ax$, is a straight line, with slope a, that passes through the origin. The graph of a power function of degree 2, $f(x) = ax^2$, is a parabola, with vertex at the origin, that opens up if $a > 0$ and down if $a < 0$.

If we know how to graph a power function of the form $f(x) = x^n$, a compression or stretch and, perhaps, a reflection about the x-axis will enable us to obtain the graph of $g(x) = ax^n$. Consequently, we shall concentrate on graphing power functions of the form $f(x) = x^n$.

We begin with power functions of even degree of the form $f(x) = x^n$, $n \geq 2$ and n even. The domain of f is the set of all real numbers, and the range is the set of nonnegative real numbers. Such a power function is an even function (do you see why?), so its graph is symmetric with respect to the y-axis. Its graph always contains the origin and the points $(-1, 1)$ and $(1, 1)$.

If $n = 2$, the graph is the familiar parabola $y = x^2$ that opens up, with vertex at the origin. If $n \geq 4$, the graph of $f(x) = x^n$, n even, will be closer to the x-axis than the parabola $y = x^2$ if $-1 < x < 1$, $x \neq 0$, and farther from the x-axis than the parabola $y = x^2$ if $x < -1$ or if $x > 1$. Figure 2(a) illustrates this conclusion. Figure 2(b) shows the graphs of $y = x^4$ and $y = x^8$ for comparison.

Figure 2



From Figure 2, we can see that as n increases the graph of $f(x) = x^n$, $n \geq 2$ and n even, tends to flatten out near the origin and to increase very rapidly when x is far from 0. For large n, it may appear that the graph coincides with the x-axis near the origin, but it does not; the graph actually touches the x-axis only at the origin (see Table 2). Also, for large n, it may appear that for $x < -1$ or for $x > 1$ the graph is vertical, but it is not; it is only increasing very rapidly in these intervals. If the graphs were enlarged many times, these distinctions would be clear.

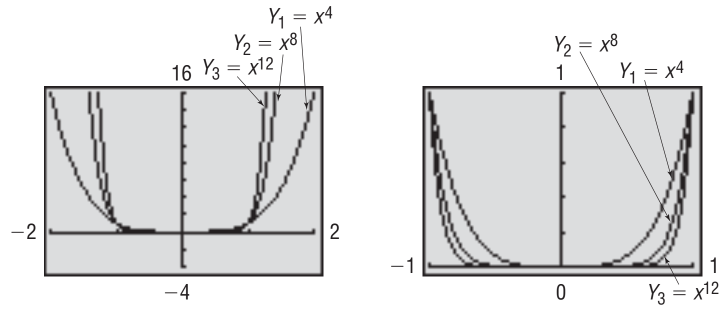
Table 2

	x = 0.1	**x = 0.3**	**x = 0.5**
$f(x) = x^8$	10^{-8}	0.0000656	0.0039063
$f(x) = x^{20}$	10^{-20}	$3.487 \cdot 10^{-11}$	0.000001
$f(x) = x^{40}$	10^{-40}	$1.216 \cdot 10^{-21}$	$9.095 \cdot 10^{-13}$

Seeing the Concept

Graph $Y_1 = x^4$, $Y_2 = x^8$, and $Y_3 = x^{12}$ using the viewing rectangle $-2 \leq x \leq 2$, $-4 \leq y \leq 16$. Then graph each again using the viewing rectangle $-1 \leq x \leq 1$, $0 \leq y \leq 1$. See Figure 3. TRACE along one of the graphs to confirm that for x close to 0 the graph is above the x-axis and that for $x > 0$ the graph is increasing.

Figure 3



(a) (b)

Figure 4

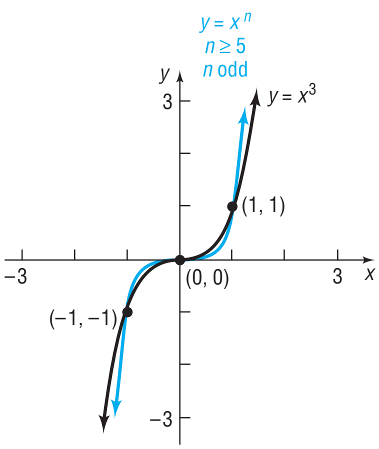
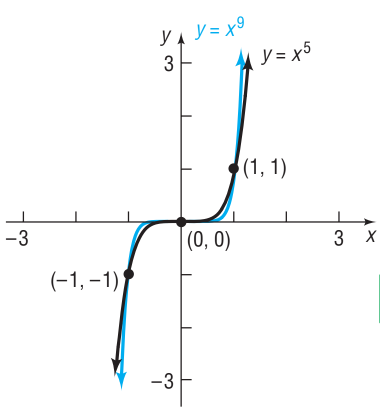


Figure 5



Properties of Power Functions, $f(x) = x^n$, n Is a Positive Even Integer

1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Now we consider power functions of odd degree of the form $f(x) = x^n$, $n \geq 3$ and n odd. The domain and the range of f are the set of real numbers. Such a power function is an odd function (do you see why?), so its graph is symmetric with respect to the origin. Its graph always contains the origin and the points $(-1, -1)$ and $(1, 1)$.

The graph of $f(x) = x^n$ when $n = 3$ has been shown several times and is repeated in Figure 4. If $n \geq 5$, the graph of $f(x) = x^n$, n odd, will be closer to the x-axis than that of $y = x^3$ if $-1 < x < 1$ and farther from the x-axis than that of $y = x^3$ if $x < -1$ or if $x > 1$. Figure 4 also illustrates this conclusion.

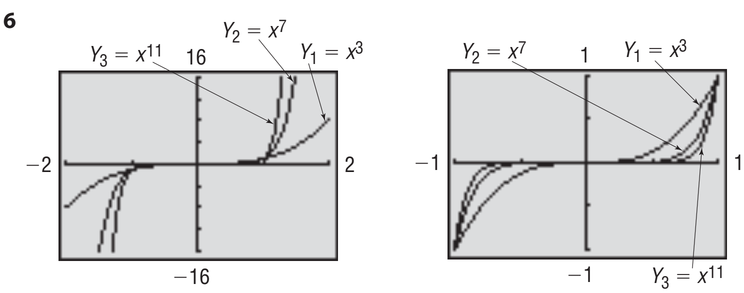
Figure 5 shows the graph of $y = x^5$ and the graph of $y = x^9$ for further comparison.

It appears that each graph coincides with the x-axis near the origin, but it does not; each graph actually crosses the x-axis at the origin. Also, it appears that as x increases the graph becomes vertical, but it does not; each graph is increasing very rapidly.

Seeing the Concept

Graph $Y_1 = x^3$, $Y_2 = x^7$, and $Y_3 = x^{11}$ using the viewing rectangle $-2 \leq x \leq 2$, $-16 \leq y \leq 16$. Then graph each again using the viewing rectangle $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. See Figure 6. TRACE along one of the graphs to confirm that the graph is increasing and crosses the x-axis at the origin.

Figure 6



(a) (b)

To summarize:

Properties of Power Functions, $f(x) = x^n$, n Is a Positive Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

2 Graph Polynomial Functions Using Transformations

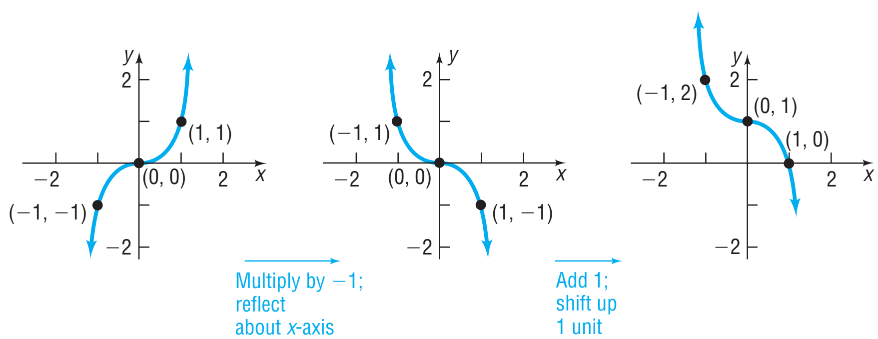
The methods of shifting, compression, stretching, and reflection studied in Section 2.5, when used with the facts just presented, will enable us to graph polynomial functions that are transformations of power functions.

EXAMPLE 2 **Graphing a Polynomial Function Using Transformations**

Graph: $f(x) = 1 - x^5$

Solution It is helpful to rewrite f as $f(x) = -x^5 + 1$. Figure 7 shows the required stages.

Figure 7



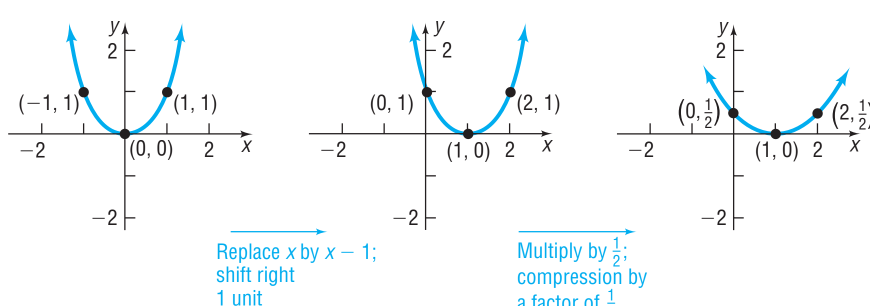
(a) $y = x^5$ (b) $y = -x^5$ (c) $y = -x^5 + 1 = 1 - x^5$

EXAMPLE 3 **Graphing a Polynomial Function Using Transformations**

Graph: $f(x) = \dfrac{1}{2}(x - 1)^4$

Solution Figure 8 shows the required stages.

Figure 8



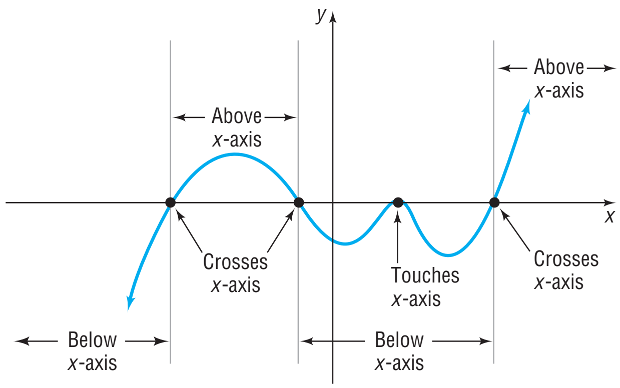
(a) $y = x^4$ (b) $y = (x - 1)^4$ (c) $y = \frac{1}{2}(x - 1)^4$

Now Work PROBLEMS 27 AND 33

3 Identify the Real Zeros of a Polynomial Function and Their Multiplicity

Figure 9 shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or below the x-axis.

Figure 9



If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x) = 0$ and using the Zero-Product Property. For example, if $f(x) = (x - 1)^2(x + 3)$, then the solutions of the equation

$$f(x) = (x - 1)^2(x + 3) = 0$$

are identified as 1 and -3. That is, $f(1) = 0$ and $f(-3) = 0$.

DEFINITION

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

As a consequence of this definition, the following statements are equivalent.

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

So the real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x) = 0$.

EXAMPLE 4 Finding a Polynomial Function from Its Zeros

(a) Find a polynomial function of degree 3 whose zeros are -3, 2, and 5.

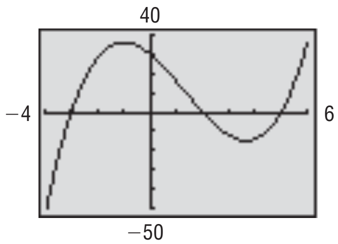
(b) Use a graphing utility to graph the polynomial found in part (a) to verify your result.

Solution (a) If r is a real zero of a polynomial function f, then $x - r$ is a factor of f. This means that $x - (-3) = x + 3$, $x - 2$, and $x - 5$ are factors of f. As a result, any polynomial function of the form

$$f(x) = a(x + 3)(x - 2)(x - 5)$$

where a is a nonzero real number, qualifies. The value of a causes a stretch, compression, or reflection, but does not affect the x-intercepts of the graph. Do you know why?

Figure 10



(b) We choose to graph f with $a = 1$. Then

$$f(x) = (x + 3)(x - 2)(x - 5) = x^3 - 4x^2 - 11x + 30$$

Figure 10 shows the graph of f. Notice that the x-intercepts are -3, 2, and 5.

Seeing the Concept

Graph the function found in Example 4 for $a = 2$ and $a = -1$. Does the value of a affect the zeros of f? How does the value of a affect the graph of f?

Now Work PROBLEM 41

If the same factor $x - r$ occurs more than once, r is called a **repeated,** or **multiple, zero of *f*.** More precisely, we have the following definition.

DEFINITION

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity *m* of *f*.***

EXAMPLE 5 Identifying Zeros and Their Multiplicities

For the polynomial

$$f(x) = 5(x - 2)(x + 3)^2\left(x - \frac{1}{2}\right)^4$$

2 is a zero of multiplicity 1 because the exponent on the factor $x - 2$ is 1.

-3 is a zero of multiplicity 2 because the exponent on the factor $x + 3$ is 2.

$\frac{1}{2}$ is a zero of multiplicity 4 because the exponent on the factor $x - \frac{1}{2}$ is 4.

Now Work PROBLEM 49(a)

Suppose that it is possible to factor completely a polynomial function and, as a result, locate all the x-intercepts of its graph (the real zeros of the function). These x-intercepts then divide the x-axis into open intervals and, on each such interval, the graph of the polynomial will be either above or below the x-axis. Let's look at an example.

EXAMPLE 6 Graphing a Polynomial Using Its x-Intercepts

For the polynomial: $f(x) = x^2(x - 2)$

(a) Find the x- and y-intercepts of the graph of f.

(b) Use the x-intercepts to find the intervals on which the graph of f is above the x-axis and the intervals on which the graph of f is below the x-axis.

(c) Locate other points on the graph and connect all the points plotted with a smooth, continuous curve.

Solution

(a) The y-intercept is $f(0) = 0^2(0 - 2) = 0$. The x-intercepts satisfy the equation

$$f(x) = x^2(x - 2) = 0$$

from which we find

$$\begin{aligned} x^2 &= 0 \quad \text{or} \quad x - 2 = 0 \\ x &= 0 \quad \text{or} \quad \phantom{x - {}} x = 2 \end{aligned}$$

The x-intercepts are 0 and 2.

*Some books use the terms **multiple root** and **root of multiplicity *m*.**

(b) The two x-intercepts divide the x-axis into three intervals:

$$(-\infty, 0) \qquad (0, 2) \qquad (2, \infty)$$

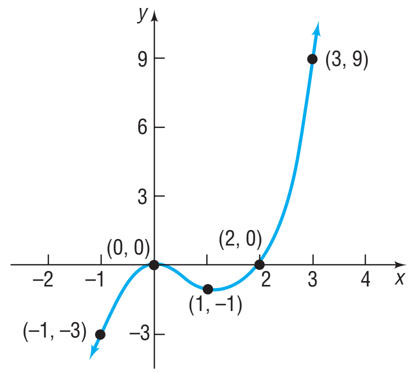
Since the graph of f crosses or touches the x-axis only at $x = 0$ and $x = 2$, it follows that the graph of f is either above the x-axis $[f(x) > 0]$ or below the x-axis $[f(x) < 0]$ on each of these three intervals. To see where the graph lies, we only need to pick one number in each interval, evaluate f there, and see whether the value is positive (above the x-axis) or negative (below the x-axis). See Table 3.

(c) In constructing Table 3, we obtained three additional points on the graph: $(-1, -3)$, $(1, -1)$, and $(3, 9)$. Figure 11 illustrates these points, the intercepts, and a smooth, continuous curve (the graph of f) connecting them.

Table 3

	0	2	x
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Number chosen	-1	1	3
Value of f	$f(-1) = -3$	$f(1) = -1$	$f(3) = 9$
Location of graph	Below x-axis	Below x-axis	Above x-axis
Point on graph	$(-1, -3)$	$(1, -1)$	$(3, 9)$

Figure 11



Look again at Table 3. Since the graph of $f(x) = x^2(x - 2)$ is below the x-axis on both sides of 0, the graph of f *touches* the x-axis at $x = 0$, a *zero of multiplicity 2*. Since the graph of f is below the x-axis for $x < 2$ and above the x-axis for $x > 2$, the graph of f *crosses* the x-axis at $x = 2$, a *zero of multiplicity 1*.

This suggests the following results:

If *r* Is a Zero of Even Multiplicity

The sign of $f(x)$ does not change from one side to the other side of r. The graph of f **touches** the x-axis at r.

If *r* Is a Zero of Odd Multiplicity

The sign of $f(x)$ changes from one side to the other side of r. The graph of f **crosses** the x-axis at r.

Now Work PROBLEM 49(b)

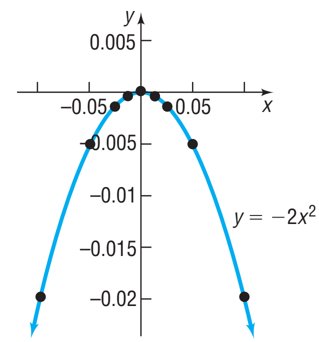
Behavior Near a Zero

The multiplicity of a zero can be used to determine whether the graph of a function touches or crosses the x-axis at the zero. However, we can learn more about the behavior of the graph near its zeros than just whether the graph crosses or touches the x-axis. Consider the function $f(x) = x^2(x - 2)$ whose graph is drawn in Figure 11. The zeros of f are 0 and 2. Table 4 on page 174 shows the values of $f(x) = x^2(x - 2)$ and $y = -2x^2$ for x near 0. Figure 12 shows the points $(-0.1, -0.021)$, $(-0.05, -0.0051)$, and so on, that are on the graph of $f(x) = x^2(x - 2)$ along with the graph of $y = -2x^2$ on the same Cartesian plane. From the table and graph, we can see that the points on the graph of $f(x) = x^2(x - 2)$ and the points on the

Table 4

x	$f(x) = x^2(x-2)$	$y = -2x^2$
−0.1	−0.021	−0.02
−0.05	−0.005125	−0.005
−0.03	−0.001827	−0.0018
−0.01	−0.000201	−0.0002
0	0	0
0.01	−0.000199	−0.0002
0.03	−0.001773	−0.0018
0.05	−0.004875	−0.005
0.1	−0.019	−0.02

Figure 12



graph of $y = -2x^2$ are indistinguishable near $x = 0$. So $y = -2x^2$ describes the behavior of the graph of $f(x) = x^2(x-2)$ near $x = 0$.

But how did we know that the function $f(x) = x^2(x-2)$ behaves like $y = -2x^2$ when x is close to 0? In other words, where did $y = -2x^2$ come from? Because the zero, 0, comes from the factor x^2, we evaluate all factors in the function f at 0 with the exception of x^2.

$$\begin{aligned} f(x) &= x^2(x-2) \\ &\approx x^2(0-2) \\ &= -2x^2 \end{aligned}$$

The factor x^2 gives rise to the zero, so we keep the factor x^2 and let $x = 0$ in the remaining factors to find the behavior near 0.

This tells us that the graph of $f(x) = x^2(x-2)$ will behave like the graph of $y = -2x^2$ near $x = 0$.

Now let's discuss the behavior of $f(x) = x^2(x-2)$ near $x = 2$, the other zero. Because the zero, 2, comes from the factor $x - 2$, we evaluate all factors of the function f at 2 with the exception of $x - 2$.

$$\begin{aligned} f(x) &= x^2(x-2) \\ &\approx 2^2(x-2) \\ &= 4(x-2) \end{aligned}$$

The factor $x - 2$ gives rise to the zero, so we keep the factor $x - 2$ and let $x = 2$ in the remaining factors to find the behavior near 2.

So the graph of $f(x) = x^2(x-2)$ will behave like the graph of $y = 4(x-2)$ near $x = 2$. Table 5 verifies that $f(x) = x^2(x-2)$ and $y = 4(x-2)$ have similar values for x near 2. Figure 13 shows the points $(1.9, -0.361)$, $(1.99, -0.0396)$, and so on, that are on the graph of $f(x) = x^2(x-2)$ along with the graph of $y = 4(x-2)$ on the same Cartesian plane. We can see that the points on the graph of $f(x) = x^2(x-2)$ and the points on the graph of $y = 4(x-2)$ are indistinguishable near $x = 2$. So $y = 4(x-2)$, a line with slope 4, describes the behavior of the graph of $f(x) = x^2(x-2)$ near $x = 2$.

Table 5

x	$f(x) = x^2(x-2)$	$y = 4(x-2)$
1.9	−0.361	−0.4
1.99	−0.0396	−0.04
1.999	−0.003996	−0.004
2	0	0
2.001	0.004004	0.004
2.01	0.0404	0.04
2.1	0.441	0.4

Figure 13

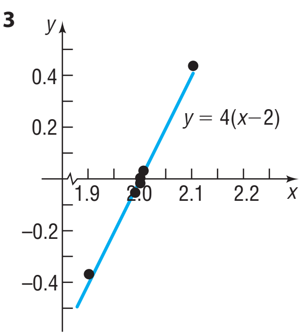


Figure 14

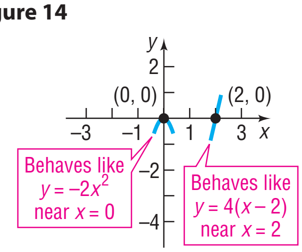


Figure 14 illustrates how we would use this information to begin to graph $f(x) = x^2(x-2)$.

- The multiplicity of a real zero determines whether the graph crosses or touches the x-axis at the zero.
- The behavior of the graph near a real zero determines how the graph touches or crosses the x-axis.

Now Work PROBLEM 49(c)

Turning Points

Look again at Figure 11 on page 173. We cannot be sure just how low the graph actually goes between $x = 0$ and $x = 2$. But we do know that somewhere in the interval $(0, 2)$ the graph of f must change direction (from decreasing to increasing). The points at which a graph changes direction are called **turning points.** In calculus, techniques for locating them are given. So we shall not ask for the location of turning points in our graphs. Instead, we will use the following result from calculus, which tells us the maximum number of turning points that the graph of a polynomial function can have.

THEOREM

Turning Points

If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, the degree of f is at least n.

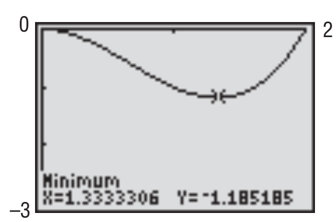
For example, the graph of $f(x) = x^2(x - 2)$ shown in Figure 11 is the graph of a polynomial function of degree 3 and has $3 - 1 = 2$ turning points: one at $(0, 0)$ and the other somewhere between $x = 0$ and $x = 2$.

Based on the theorem, if the graph of a polynomial function has three turning points, then the degree of the function must be at least 4.

Exploration

A graphing utility can be used to locate the turning points of a graph. Graph $Y_1 = x^2(x - 2)$. Use MINIMUM to find the location of the turning point for $0 < x < 2$. See Figure 15.

Figure 15

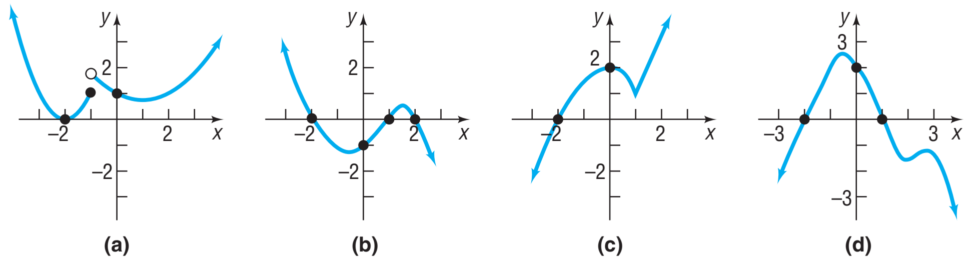


Now Work PROBLEM 49(d)

EXAMPLE 7 **Identifying the Graph of a Polynomial Function**

Which of the graphs in Figure 16 on the next page could be the graph of a polynomial function? For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.

Figure 16



Solution

(a) The graph in Figure 16(a) cannot be the graph of a polynomial function because of the gap that occurs at $x = -1$. Remember, the graph of a polynomial function is continuous—no gaps or holes.

(b) The graph in Figure 16(b) could be the graph of a polynomial function because the graph is smooth and continuous. It has three real zeros, at -2, at 1, and at 2. Since the graph has two turning points, the degree of the polynomial function must be at least 3.

(c) The graph in Figure 16(c) cannot be the graph of a polynomial function because of the cusp at $x = 1$. Remember, the graph of a polynomial function is smooth.

(d) The graph in Figure 16(d) could be the graph of a polynomial function. It has two real zeros, at -2 and at 1. Since the graph has three turning points, the degree of the polynomial function is at least 4.

Now Work PROBLEM 61

End Behavior

One last remark about Figure 11. For very large values of x, either positive or negative, the graph of $f(x) = x^2(x-2)$ looks like the graph of $y = x^3$. To see why, we write f in the form

$$f(x) = x^2(x-2) = x^3 - 2x^2 = x^3\left(1 - \frac{2}{x}\right)$$

Now, for large values of x, either positive or negative, the term $\frac{2}{x}$ is close to 0, so for large values of x

$$f(x) = x^3 - 2x^2 = x^3\left(1 - \frac{2}{x}\right) \approx x^3$$

The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior.**

THEOREM

End Behavior

For large values of x, either positive or negative, the graph of the polynomial function

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$$

resembles the graph of the power function

$$\boxed{y = a_n x^n}$$

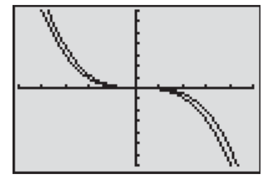
In Words
The end behavior of a polynomial function resembles that of its leading term.

For example, if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative. We can see that the graphs of f and $y = -2x^3$ "behave" the same by considering Table 6 and Figure 17.

Table 6

x	$f(x)$	$y = -2x^3$
10	−1,494	−2,000
100	−1,949,904	−2,000,000
500	−248,749,504	−250,000,000
1,000	−1,994,999,004	−2,000,000,000

Figure 17



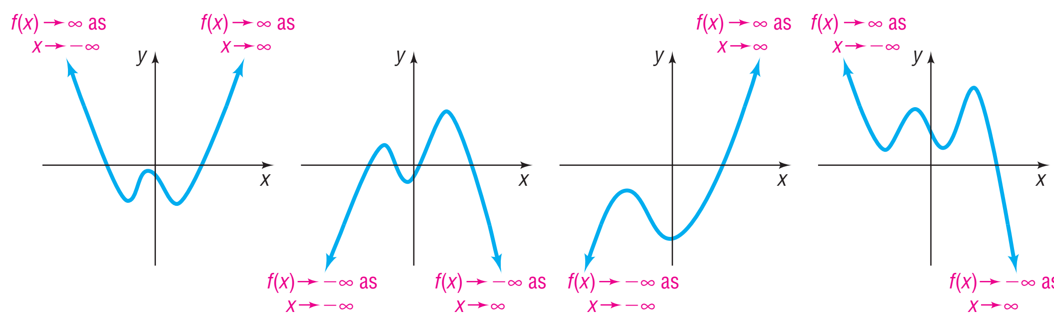
Notice that, as x becomes a larger and larger positive number, the values of f become larger and larger negative numbers. When this happens, we say that f is **unbounded in the negative direction.** Rather than using words to describe the behavior of the graph of the function, we explain its behavior using notation. We can symbolize "the value of f becomes a larger and larger negative number as x becomes a larger and larger positive number" by writing $f(x) \to -\infty$ as $x \to \infty$ (read "the values of f approach negative infinity as x approaches infinity"). In calculus, **limits** are used to convey these ideas. There we use the symbolism $\lim_{x \to \infty} f(x) = -\infty$, read "the limit of $f(x)$ as x approaches infinity equals negative infinity," to mean that $f(x) \to -\infty$ as $x \to \infty$.

When the value of a limit equals infinity, we mean that the values of the function are unbounded in the positive or negative direction and call the limit an **infinite limit.** When we discuss limits as x becomes unbounded in the negative direction or unbounded in the positive direction, we are discussing **limits at infinity.**

Look back at Figures 2 and 4. Based on the preceding theorem and the previous discussion on power functions, the end behavior of a polynomial function can only be of four types. See Figure 18.

Figure 18

End behavior of $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$



(a) $n \ge 2$ even; $a_n > 0$

(b) $n \ge 2$ even; $a_n < 0$

(c) $n \ge 3$ odd; $a_n > 0$

(d) $n \ge 3$ odd; $a_n < 0$

For example, if $f(x) = -2x^4 + x^3 + 4x^2 - 7x + 1$, the graph of f will resemble the graph of the power function $y = -2x^4$ for large $|x|$. The graph of f will behave like Figure 18(b) for large $|x|$.

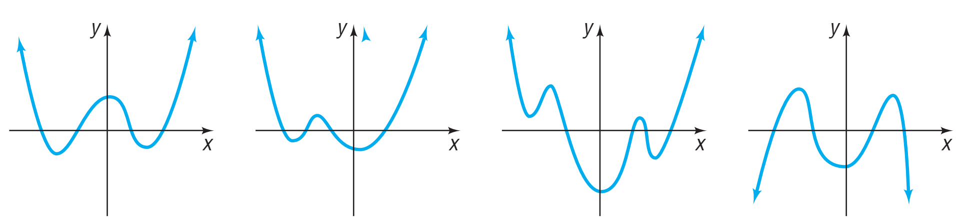
Now Work PROBLEM 49(e)

EXAMPLE 8 Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 19 could be the graph of

$$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6?$$

Figure 19



(a) (b) (c) (d)

Solution The y-intercept of f is $f(0) = -6$. We can eliminate the graph in Figure 19(a), whose y-intercept is positive.

We don't have any methods for finding the x-intercepts of f, so we move on to investigate the turning points of each graph. Since f is of degree 4, the graph of f has at most 3 turning points. We eliminate the graph in Figure 19(c) since that graph has 5 turning points.

Now we look at end behavior. For large values of x, the graph of f will behave like the graph of $y = x^4$. This eliminates the graph in Figure 19(d), whose end behavior is like the graph of $y = -x^4$.

Only the graph in Figure 19(b) could be (and, in fact, is) the graph of $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$.

Now Work PROBLEM 65

SUMMARY Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0$

Degree of the polynomial function f: n
Graph is smooth and continuous.
Maximum number of turning points: $n - 1$
At a zero of even multiplicity: The graph of f touches the x-axis.
At a zero of odd multiplicity: The graph of f crosses the x-axis.
Between zeros, the graph of f is either above or below the x-axis.
End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

4 Analyze the Graph of a Polynomial Function

EXAMPLE 9

How to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function $f(x) = (2x + 1)(x - 3)^2$.

Step-by-Step Solution

Step 1: Determine the end behavior of the graph of the function.

Expand the polynomial to write it in the form

$$
\begin{aligned}
f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
f(x) &= (2x + 1)(x - 3)^2 \\
&= (2x + 1)(x^2 - 6x + 9) \\
&= 2x^3 - 12x^2 + 18x + x^2 - 6x + 9 && \text{Multiply.} \\
&= 2x^3 - 11x^2 + 12x + 9 && \text{Combine like terms.}
\end{aligned}
$$

The polynomial function f is of degree 3. The graph of f behaves like $y = 2x^3$ for large values of $|x|$.

Step 2: Find the x- and y-intercepts of the graph of the function.

The y-intercept is $f(0) = 9$. To find the x-intercepts, we solve $f(x) = 0$.

$$
\begin{aligned}
f(x) &= 0 \\
(2x + 1)(x - 3)^2 &= 0 \\
2x + 1 = 0 \quad &\text{or} \quad (x - 3)^2 = 0 \\
x = -\frac{1}{2} \quad &\text{or} \quad x - 3 = 0 \\
& \qquad\quad x = 3
\end{aligned}
$$

The x-intercepts are $-\frac{1}{2}$ and 3.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

The zeros of f are $-\frac{1}{2}$ and 3. The zero $-\frac{1}{2}$ is a zero of multiplicity 1, so the graph of f crosses the x-axis at $x = -\frac{1}{2}$. The zero 3 is a zero of multiplicity 2, so the graph of f touches the x-axis at $x = 3$.

Step 4: Determine the maximum number of turning points on the graph of the function.

Because the polynomial function is of degree 3 (Step 1), the graph of the function will have at most $3 - 1 = 2$ turning points.

Step 5: Determine the behavior of the graph of f near each x-intercept.

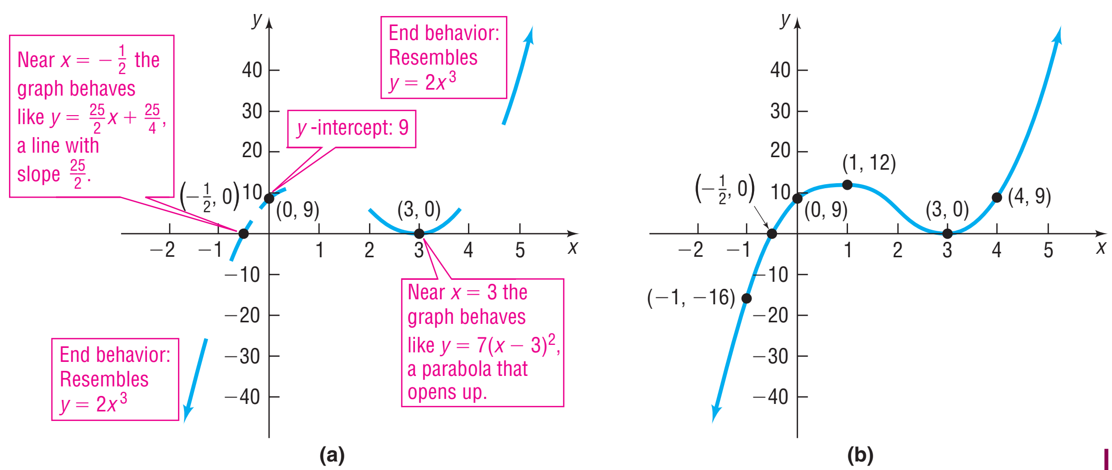
The two x-intercepts are $-\frac{1}{2}$ and 3.

$$\begin{aligned}
\text{Near } -\tfrac{1}{2}:\quad f(x) &= (2x+1)(x-3)^2 \\
&\approx (2x+1)\left(-\frac{1}{2}+3\right)^2 \\
&= (2x+1)\left(\frac{25}{4}\right) \\
&= \frac{25}{2}x + \frac{25}{4} && \text{A line with slope } \frac{25}{2} \\
\text{Near } 3:\quad f(x) &= (2x+1)(x-3)^2 \\
&\approx (2\cdot 3+1)(x-3)^2 \\
&= 7(x-3)^2 && \text{A parabola that opens up}
\end{aligned}$$

Step 6: Put all the information from Steps 1 through 5 together to obtain the graph of f.

Figure 20(a) illustrates the information obtained from Steps 1 through 5. We evaluate f at -1, 1, and 4 to help establish the scale on the y-axis. The graph of f is given in Figure 20(b).

Figure 20



SUMMARY Analyzing the Graph of a Polynomial Function

STEP 1: Determine the end behavior of the graph of the function.

STEP 2: Find the x- and y-intercepts of the graph of the function.

STEP 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

STEP 4: Determine the maximum number of turning points on the graph of the function.

STEP 5: Determine the behavior of the graph near each x-intercept.

STEP 6: Use the information in Steps 1 through 5 to draw a complete graph of the function.

EXAMPLE 10 Analyzing the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = x^2(x-4)(x+1)$$

Solution

STEP 1: End behavior: the graph of f resembles that of the power function $y = x^4$ for large values of $|x|$.

STEP 2: The y-intercept is $f(0) = 0$. The x-intercepts satisfy the equation

$$f(x) = x^2(x-4)(x+1) = 0$$

So

$$x^2 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 0 \quad \text{or} \quad x = 4 \quad \text{or} \quad x = -1$$

The x-intercepts are -1, 0, and 4.

STEP 3: The intercept 0 is a zero of multiplicity 2, so the graph of f will touch the x-axis at 0; 4 and -1 are zeros of multiplicity 1, so the graph of f will cross the x-axis at 4 and -1.

STEP 4: The graph of f will contain at most three turning points.

STEP 5: The three x-intercepts are -1, 0, and 4.

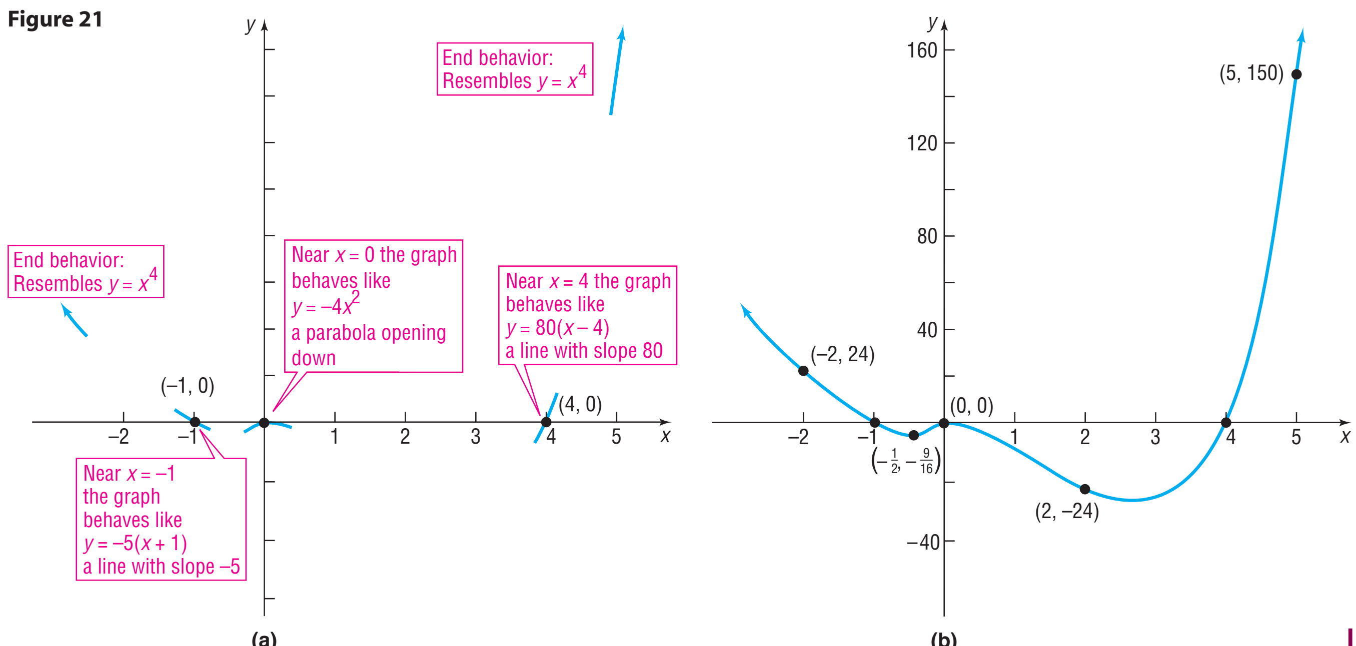
Near -1: $f(x) = x^2(x-4)(x+1) \approx (-1)^2(-1-4)(x+1) = -5(x+1)$ — A line with slope -5

Near 0: $f(x) = x^2(x-4)(x+1) \approx x^2(0-4)(0+1) = -4x^2$ — A parabola opening down

Near 4: $f(x) = x^2(x-4)(x+1) \approx 4^2(x-4)(4+1) = 80(x-4)$ — A line with slope 80

STEP 6: Figure 21(a) illustrates the information obtained from Steps 1–5. The graph of f is given in Figure 21(b). Notice that we evaluated f at $-2, -\frac{1}{2}, 2$, and 5 to help establish the scale on the y-axis.

Figure 21



Exploration

Graph $Y_1 = x^2(x-4)(x+1)$. Compare what you see with Figure 21(b). Use MAXIMUM/MINIMUM to locate the three turning points.

Now Work PROBLEM 69

For polynomial functions that have noninteger coefficients and for polynomials that are not easily factored, we utilize the graphing utility early in the analysis. This is because the amount of information that can be obtained from algebraic analysis is limited.

EXAMPLE 11 How to Use a Graphing Utility to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$$

Step-by-Step Solution

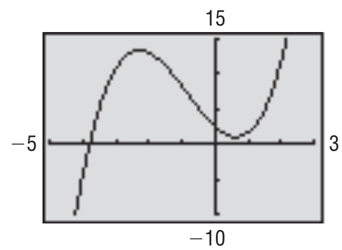
Step 1: Determine the end behavior of the graph of the function.

The polynomial function f is of degree 3. The graph of f behaves like $y = x^3$ for large values of $|x|$.

Step 2: Graph the function using a graphing utility.

See Figure 22 for the graph of f.

Figure 22



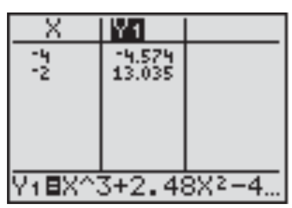
Step 3: Use a graphing utility to approximate the x- and y-intercepts of the graph.

The y-intercept is $f(0) = 2.484406$. In Examples 9 and 10, the polynomial function was factored, so it was easy to find the x-intercepts algebraically. However, it is not readily apparent how to factor f in this example. Therefore, we use a graphing utility's ZERO (or ROOT or SOLVE) feature and find the lone x-intercept to be -3.79, rounded to two decimal places.

Step 4: Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

Table 7 shows values of x on each side of the x-intercept. The points $(-4, -4.57)$ and $(-2, 13.04)$ are on the graph.

Table 7



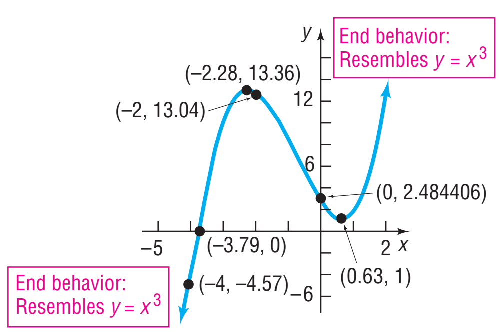
Step 5: Approximate the turning points of the graph.

From the graph of f shown in Figure 22, we can see that f has two turning points. Using MAXIMUM, one turning point is at $(-2.28, 13.36)$, rounded to two decimal places. Using MINIMUM, the other turning point is at $(0.63, 1)$, rounded to two decimal places.

Step 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

Figure 23 shows a graph of f using the information in Steps 1 through 5.

Figure 23



Step 7: Find the domain and the range of the function.

The domain and the range of f are the set of all real numbers.

Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

Based on the graph, f is increasing on the intervals $(-\infty, -2.28)$ and $(0.63, \infty)$. Also, f is decreasing on the interval $(-2.28, 0.63)$.

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

STEP 1: Determine the end behavior of the graph of the function.

STEP 2: Graph the function using a graphing utility.

STEP 3: Use a graphing utility to approximate the x- and y-intercepts of the graph.

STEP 4: Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

STEP 5: Approximate the turning points of the graph.

STEP 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

STEP 7: Find the domain and the range of the function.

STEP 8: Use the graph to determine where the function is increasing and where it is decreasing.

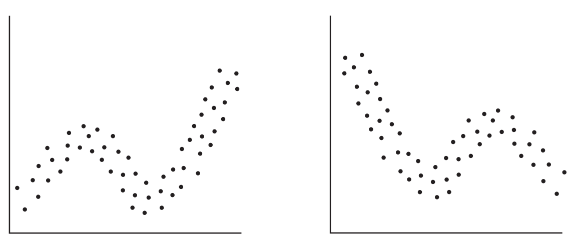
Now Work PROBLEM 87

5 Build Cubic Models from Data

In Section 3.2 we found the line of best fit from data, and in Section 3.4 we found the quadratic function of best fit. It is also possible to find polynomial functions of best fit. However, most statisticians do not recommend finding polynomials of best fit of degree higher than 3.

Data that follow a cubic relation should look like Figure 24(a) or (b).

Figure 24



$y = ax^3 + bx^2 + cx + d, a > 0$ (a) $y = ax^3 + bx^2 + cx + d, a < 0$ (b)

EXAMPLE 12 A Cubic Function of Best Fit

The data in Table 8 represent the weekly cost C (in thousands of dollars) of printing x thousand textbooks.

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables x and C.

(b) Using a graphing utility, find the cubic function of best fit $C = C(x)$ that models the relation between number of texts and cost.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the cost of printing 22 thousand texts per week.

Table 8

Number of Textbooks, x	Cost, C
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

Solution

(a) Figure 25 shows the scatter diagram. A cubic relation may exist between the two variables.

(b) Upon executing the CUBIC REGression program, we obtain the results shown in Figure 26. The output that the utility provides shows us the equation $y = ax^3 + bx^2 + cx + d$. The cubic function of best fit to the data is $C(x) = 0.0155x^3 - 0.5951x^2 + 9.1502x + 98.4327$.

(c) Figure 27 shows the graph of the cubic function of best fit on the scatter diagram. The function fits the data reasonably well.

Figure 25

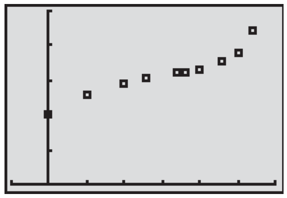


Figure 26

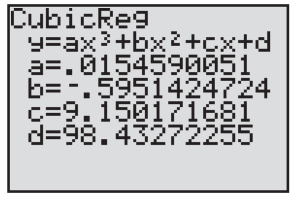
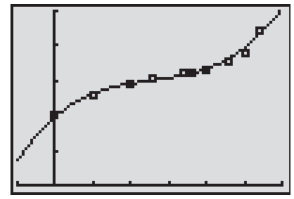


Figure 27



(d) Evaluate the function $C(x)$ at $x = 22$.

$$C(22) = 0.0155(22)^3 - 0.5951(22)^2 + 9.1502(22) + 98.4327 \approx 176.8$$

The model predicts that the cost of printing 22 thousand textbooks in a week will be 176.8 thousand dollars, that is, \$176,800.

4.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The intercepts of the equation $9x^2 + 4y = 36$ are ________. (pp. 11–12)
2. Is the expression $4x^3 - 3.6x^2 - \sqrt{2}$ a polynomial? If so, what is its degree? (pp. A22–A29)
3. To graph $y = x^2 - 4$, you would shift the graph of $y = x^2$ ________ a distance of ______ units. (pp. 90–99)
4. Use a graphing utility to approximate (rounded to two decimal places) the local maximum value and local minimum value of $f(x) = x^3 - 2x^2 - 4x + 5$, for $-3 < x < 3$. (p. 74)
5. ***True or False*** The x-intercepts of the graph of a function $y = f(x)$ are the real solutions of the equation $f(x) = 0$. (pp. 61–63)
6. If $g(5) = 0$, what point is on the graph of g? What is the corresponding x-intercept of the graph of g? (pp. 61–63)

Concepts and Vocabulary

7. The graph of every polynomial function is both ________ and ___________.
8. If r is a real zero of even multiplicity of a function f, then the graph of f ________ (crosses/touches) the x-axis at r.
9. The graphs of power functions of the form $f(x) = x^n$, where n is an even integer, always contain the points ________, _______, and _______.
10. If r is a solution to the equation $f(x) = 0$, name three additional statements that can be made about f and r assuming f is a polynomial function.
11. The points at which a graph changes direction (from increasing to decreasing or decreasing to increasing) are called ______________.
12. The graph of the function $f(x) = 3x^4 - x^3 + 5x^2 - 2x - 7$ will behave like the graph of ________ for large values of $|x|$.
13. If $f(x) = -2x^5 + x^3 - 5x^2 + 7$, then $\lim_{x\to-\infty} f(x) =$ _____ and $\lim_{x\to\infty} f(x) =$ _______.
14. Explain what the notation $\lim_{x\to\infty} f(x) = -\infty$ means.

Skill Building

In Problems 15–26, determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not.

15. $f(x) = 4x + x^3$

16. $f(x) = 5x^2 + 4x^4$

17. $g(x) = \dfrac{1 - x^2}{2}$

18. $h(x) = 3 - \dfrac{1}{2}x$

19. $f(x) = 1 - \dfrac{1}{x}$

20. $f(x) = x(x - 1)$

21. $g(x) = x^{3/2} - x^2 + 2$

22. $h(x) = \sqrt{x}(\sqrt{x} - 1)$

23. $F(x) = 5x^4 - \pi x^3 + \dfrac{1}{2}$

24. $F(x) = \dfrac{x^2 - 5}{x^3}$

25. $G(x) = 2(x - 1)^2(x^2 + 1)$

26. $G(x) = -3x^2(x + 2)^3$

In Problems 27–40, use transformations of the graph of $y = x^4$ or $y = x^5$ to graph each function.

27. $f(x) = (x + 1)^4$

28. $f(x) = (x - 2)^5$

29. $f(x) = x^5 - 3$

30. $f(x) = x^4 + 2$

31. $f(x) = \dfrac{1}{2}x^4$

32. $f(x) = 3x^5$

33. $f(x) = -x^5$

34. $f(x) = -x^4$

35. $f(x) = (x - 1)^5 + 2$

36. $f(x) = (x + 2)^4 - 3$

37. $f(x) = 2(x + 1)^4 + 1$

38. $f(x) = \dfrac{1}{2}(x - 1)^5 - 2$

39. $f(x) = 4 - (x - 2)^5$

40. $f(x) = 3 - (x + 2)^4$

In Problems 41–48, form a polynomial function whose real zeros and degree are given. Answers will vary depending on the choice of a leading coefficient.

41. Zeros: $-1, 1, 3$; degree 3

42. Zeros: $-2, 2, 3$; degree 3

43. Zeros: $-3, 0, 4$; degree 3

44. Zeros: $-4, 0, 2$; degree 3

45. Zeros: $-4, -1, 2, 3$; degree 4

46. Zeros: $-3, -1, 2, 5$; degree 4

47. Zeros: -1, multiplicity 1; 3, multiplicity 2; degree 3

48. Zeros: -2, multiplicity 2; 4, multiplicity 1; degree 3

In Problems 49–60, for each polynomial function:

(a) List each real zero and its multiplicity.
(b) Determine whether the graph crosses or touches the x-axis at each x-intercept.
(c) Determine the behavior of the graph near each x-intercept (zero).
(d) Determine the maximum number of turning points on the graph.
(e) Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

49. $f(x) = 3(x - 7)(x + 3)^2$

50. $f(x) = 4(x + 4)(x + 3)^3$

51. $f(x) = 4(x^2 + 1)(x - 2)^3$

52. $f(x) = 2(x - 3)(x^2 + 4)^3$

53. $f(x) = -2\left(x + \dfrac{1}{2}\right)^2(x + 4)^3$

54. $f(x) = \left(x - \dfrac{1}{3}\right)^2(x - 1)^3$

55. $f(x) = (x - 5)^3(x + 4)^2$

56. $f(x) = (x + \sqrt{3})^2(x - 2)^4$

57. $f(x) = 3(x^2 + 8)(x^2 + 9)^2$

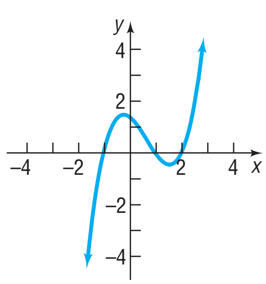
58. $f(x) = -2(x^2 + 3)^3$

59. $f(x) = -2x^2(x^2 - 2)$

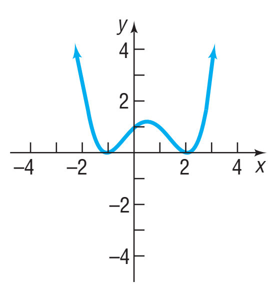
60. $f(x) = 4x(x^2 - 3)$

In Problems 61–64, identify which of the graphs could be the graph of a polynomial function. For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.

61.



62.



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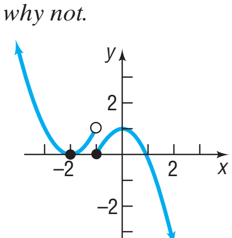

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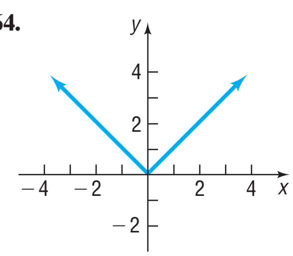
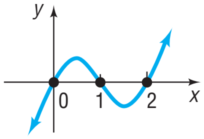
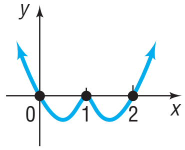


In Problems 65–68, construct a polynomial function that might have the given graph. (More than one answer may be possible.)

65.



66.



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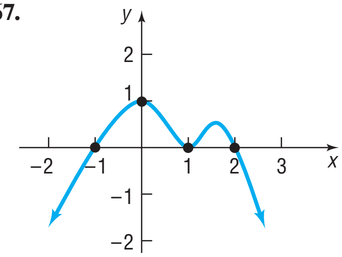

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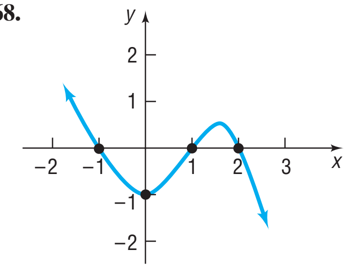


In Problems 69–86, analyze each polynomial function by following Steps 1 through 6 on page 179.

69. $f(x) = x^2(x-3)$

70. $f(x) = x(x+2)^2$

71. $f(x) = (x+4)(x-2)^2$

72. $f(x) = (x-1)(x+3)^2$

73. $f(x) = -2(x+2)(x-2)^3$

74. $f(x) = -\frac{1}{2}(x+4)(x-1)^3$

75. $f(x) = (x+1)(x-2)(x+4)$

76. $f(x) = (x-1)(x+4)(x-3)$

77. $f(x) = x^2(x-2)(x+2)$

78. $f(x) = x^2(x-3)(x+4)$

79. $f(x) = (x+1)^2(x-2)^2$

80. $f(x) = (x+1)^3(x-3)$

81. $f(x) = x^2(x-3)(x+1)$

82. $f(x) = x^2(x-3)(x-1)$

83. $f(x) = (x+2)^2(x-4)^2$

84. $f(x) = (x-2)^2(x+2)(x+4)$

85. $f(x) = x^2(x-2)(x^2+3)$

86. $f(x) = x^2(x^2+1)(x+4)$

In Problems 87–94, analyze each polynomial function f by following Steps 1 through 8 on page 182.

87. $f(x) = x^3 + 0.2x^2 - 1.5876x - 0.31752$

88. $f(x) = x^3 - 0.8x^2 - 4.6656x + 3.73248$

89. $f(x) = x^3 + 2.56x^2 - 3.31x + 0.89$

90. $f(x) = x^3 - 2.91x^2 - 7.668x - 3.8151$

91. $f(x) = x^4 - 2.5x^2 + 0.5625$

92. $f(x) = x^4 - 18.5x^2 + 50.2619$

93. $f(x) = 2x^4 - \pi x^3 + \sqrt{5}x - 4$

94. $f(x) = -1.2x^4 + 0.5x^2 - \sqrt{3}x + 2$

Mixed Practice

In Problems 95–102, analyze each polynomial function by following Steps 1 through 6 on page 179.
[**Hint:** You will need to first factor the polynomial].

95. $f(x) = 4x - x^3$

96. $f(x) = x - x^3$

97. $f(x) = x^3 + x^2 - 12x$

98. $f(x) = x^3 + 2x^2 - 8x$

99. $f(x) = 2x^4 + 12x^3 - 8x^2 - 48x$

100. $f(x) = 4x^3 + 10x^2 - 4x - 10$

101. $f(x) = -x^5 - x^4 + x^3 + x^2$

102. $f(x) = -x^5 + 5x^4 + 4x^3 - 20x^2$

In Problems 103–106, construct a polynomial function f with the given characteristics.

103. Zeros: $-3, 1, 4$; degree 3; y-intercept: 36

104. Zeros: $-4, -1, 2$; degree 3; y-intercept: 16

105. Zeros: -5 (multiplicity 2); 2 (multiplicity 1); 4 (multiplicity 1); degree 4; contains the point $(3, 128)$

106. Zeros: -4 (multiplicity 1); 0 (multiplicity 3); 2 (multiplicity 1); degree 5; contains the point $(-2, 64)$

107. $G(x) = (x+3)^2(x-2)$
(a) Identify the x-intercepts of the graph of G.
(b) What are the x-intercepts of the graph of $y = G(x+3)$?

108. $h(x) = (x+2)(x-4)^3$
(a) Identify the x-intercepts of the graph of h.
(b) What are the x-intercepts of the graph of $y = h(x-2)$?

Applications and Extensions

109. Hurricanes In 2005, Hurricane Katrina struck the Gulf Coast of the United States, killing 1289 people and causing an estimated \$200 billion in damage. The following data represent the number of major hurricane strikes in the United States (category 3, 4, or 5) each decade from 1921 to 2000.

Decade, x	Major Hurricanes Striking United States, H
1921–1930, 1	5
1931–1940, 2	8
1941–1950, 3	10
1951–1960, 4	8
1961–1970, 5	6
1971–1980, 6	4
1981–1990, 7	5
1991–2000, 8	5

Source: National Oceanic & Atmospheric Administration

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the cubic function of best fit that models the relation between decade and number of major hurricanes.
(c) Use the model found in part (b) to predict the number of major hurricanes that struck the United States between 1961 and 1970.
(d) With a graphing utility, draw a scatter diagram of the data and then graph the cubic function of best fit on the scatter diagram.
(e) Concern has risen about the increase in the number and intensity of hurricanes, but some scientists believe this is just a natural fluctuation that could last another decade or two. Use your model to predict the number of major hurricanes that will strike the United States between 2001 and 2010. Does your result appear to agree with what these scientists believe?
(f) From 2001 through 2010, 10 major hurricanes struck the United States. Does this support or contradict your prediction in part (e)?

110. Cost of Manufacturing The following data represent the cost C (in thousands of dollars) of manufacturing Chevy Cobalts and the number x of Cobalts produced.

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables C and x.
(b) Use a graphing utility to find the cubic function of best fit $C = C(x)$.
(c) Graph the cubic function of best fit on the scatter diagram.
(d) Use the function found in part (b) to predict the cost of manufacturing 11 Cobalts.
(e) Interpret the y-intercept.

Number of Cobalts Produced, x	Cost, C
0	10
1	23
2	31
3	38
4	43
5	50
6	59
7	70
8	85
9	105
10	135

111. Temperature The following data represent the temperature T (°Fahrenheit) in Kansas City, Missouri, x hours after midnight on May 15, 2010.

Hours after Midnight, x	Temperature (°F), T
3	45.0
6	44.1
9	51.1
12	57.9
15	63.0
18	63.0
21	59.0
24	54.0

Source: The Weather Underground

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Find the average rate of change in temperature from 9 AM to 12 noon.
(c) What is the average rate of change in temperature from 3 PM to 6 PM?
(d) Decide on a function of best fit to these data (linear, quadratic, or cubic) and use this function to predict the temperature at 5 PM.
(e) With a graphing utility, draw a scatter diagram of the data and then graph the function of best fit on the scatter diagram.
(f) Interpret the y-intercept.

112. Future Value of Money Suppose that you make deposits of \$500 at the beginning of every year into an Individual Retirement Account (IRA) earning interest r. At the beginning of the first year, the value of the account will be \$500; at the beginning of the second year, the value of the account, will be

$$\underbrace{\$500 + \$500r}_{\text{Value of 1st deposit}} + \underbrace{\$500}_{\text{Value of 2nd deposit}} = \$500(1 + r) + \$500 = 500r + 1000$$

(a) Verify that the value of the account at the beginning of the third year is $T(r) = 500r^2 + 1500r + 1500$.

(b) The account value at the beginning of the fourth year is $F(r) = 500r^3 + 2000r^2 + 3000r + 2000$. If the annual rate of interest is $5\% = 0.05$, what will be the value of the account at the beginning of the fourth year?

113. A Geometric Series In calculus, you will learn that certain functions can be approximated by polynomial functions. We will explore one such function now.

(a) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_2 = g_2(x) = 1 + x + x^2 + x^3$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(b) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_3 = g_3(x) = 1 + x + x^2 + x^3 + x^4$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(c) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_4 = g_4(x) = 1 + x + x^2 + x^3 + x^4 + x^5$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(d) What do you notice about the values of the function as more terms are added to the polynomial? Are there some values of x for which the approximations are better?

Explaining Concepts: Discussion and Writing

114. Can the graph of a polynomial function have no y-intercept? Can it have no x-intercepts? Explain.

115. Write a few paragraphs that provide a general strategy for graphing a polynomial function. Be sure to mention the following: degree, intercepts, end behavior, and turning points.

116. Make up a polynomial that has the following characteristics: crosses the x-axis at -1 and 4, touches the x-axis at 0 and 2, and is above the x-axis between 0 and 2. Give your polynomial to a fellow classmate and ask for a written critique.

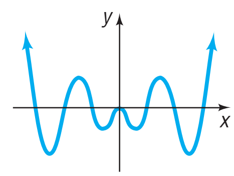
117. Make up two polynomials, not of the same degree, with the following characteristics: crosses the x-axis at -2, touches the x-axis at 1, and is above the x-axis between -2 and 1. Give your polynomials to a fellow classmate and ask for a written critique.

118. The graph of a polynomial function is always smooth and continuous. Name a function studied earlier that is smooth and not continuous. Name one that is continuous, but not smooth.

119. Which of the following statements are true regarding the graph of the cubic polynomial $f(x) = x^3 + bx^2 + cx + d$? (Give reasons for your conclusions.)

(a) It intersects the y-axis in one and only one point.
(b) It intersects the x-axis in at most three points.
(c) It intersects the x-axis at least once.
(d) For $|x|$ very large, it behaves like the graph of $y = x^3$.
(e) It is symmetric with respect to the origin.
(f) It passes through the origin.

120. The illustration shows the graph of a polynomial function.



(a) Is the degree of the polynomial even or odd?
(b) Is the leading coefficient positive or negative?
(c) Is the function even, odd, or neither?
(d) Why is x^2 necessarily a factor of the polynomial?
(e) What is the minimum degree of the polynomial?
(f) Formulate five different polynomials whose graphs could look like the one shown. Compare yours to those of other students. What similarities do you see? What differences?

121. Design a polynomial function with the following characteristics: degree 6; four distinct real zeros, one of multiplicity 3; y-intercept 3; behaves like $y = -5x^6$ for large values of $|x|$. Is this polynomial unique? Compare your polynomial with those of other students. What terms will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the polynomial?

Interactive Exercises

Ask your instructor if the applet exercise below is of interest to you.

Multiplicity and Turning Points *Open the Multiplicity applet. On the screen you will see the graph of* $f(x) = (x+2)^a x^b (x-2)^c$ *where* $a = \{1, 2, 3\}, b = \{1, 2, 3\}$, and $c = \{1, 2, 3, 4\}$.

1. Grab the slider for the exponent a and move it from 1 to 2 to 3. What happens to the graph as the value of a changes? In particular, describe the behavior of the graph around the zero -2.

2. On the same graph, grab the slider for the exponent a and move it to 1. Grab the slider for the exponent b and move it from 1 to 2 to 3. What happens to the graph as the value of b changes? In particular, describe the behavior of the graph around the zero 0.

3. On the same graph, grab the slider for the exponent b and move it to 1. Grab the slider for the exponent c and move it from 1 to 2 to 3 to 4. What happens to the graph as the value of c changes? In particular, describe the behavior of the graph around the zero 2.

4. Experiment with the graph by adjusting a, b, and c. Based on your experiences conjecture the role the exponent plays in the behavior of the graph around each zero of the function.

5. Obtain a graph of the function for the values of a, b, and c in the following table. Conjecture a relation between the degree of a polynomial and the number of turning points after completing the table. In the table, a can be 1, 2, or 3; b can be 1, 2, or 3; and c can be 1, 2, 3, or 4.

Values of $a, b,$ and c	Degree of Polynomial	Number of Turning Points
$a = 1, b = 1, c = 1$	3	
$a = 1, b = 1, c = 2$	4	
$a = 1, b = 1, c = 3$	5	
$a = 1, b = 1, c = 4$		
$a = 1, b = 2, c = 1$		
$a = 1, b = 2, c = 2$		
$a = 1, b = 2, c = 3$		
$a = 1, b = 2, c = 4$		
$a = 1, b = 3, c = 1$		
$a = 1, b = 3, c = 2$		
$a = 1, b = 3, c = 3$		
$a = 1, b = 3, c = 4$		
$a = 2, b = 1, c = 1$		
$a = 2, b = 1, c = 2$		
$a = 2, b = 1, c = 3$		
$a = 2, b = 1, c = 4$		
$a = 2, b = 2, c = 1$		
$a = 2, b = 2, c = 2$		
$a = 3, b = 3, c = 4$		

'Are You Prepared?' Answers

1. $(-2, 0), (2, 0), (0, 9)$ **2.** Yes; 3 **3.** Down; 4 **4.** Local maximum value 6.48 at $x = -0.67$; local minimum value -3 at $x = 2$
5. True **6.** $(5, 0)$; 5

4.2 Properties of Rational Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Rational Expressions (Appendix A, Section A.5, pp. A36–A42)
- Polynomial Division (Appendix A, Section A.3, pp. A25–A28)
- Graph of $f(x) = \frac{1}{x}$ (Section 1.2, Example 12, p. 16)
- Graphing Techniques: Transformations (Section 2.5, pp. 90–99)

Now Work the **'Are You Prepared?'** problems on page 196.

OBJECTIVES
1. Find the Domain of a Rational Function (p. 189)
2. Find the Vertical Asymptotes of a Rational Function (p. 192)
3. Find the Horizontal or Oblique Asymptote of a Rational Function (p. 193)

Ratios of integers are called *rational numbers*. Similarly, ratios of polynomial functions are called *rational functions*. Examples of rational functions are

$$R(x) = \frac{x^2 - 4}{x^2 + x + 1} \qquad F(x) = \frac{x^3}{x^2 - 4} \qquad G(x) = \frac{3x^2}{x^4 - 1}$$

DEFINITION

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

1 Find the Domain of a Rational Function

EXAMPLE 1 Finding the Domain of a Rational Function

(a) The domain of $R(x) = \dfrac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, the domain is $\{x | x \neq -5\}$.

(b) The domain of $R(x) = \dfrac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2; that is, the domain is $\{x | x \neq -2, x \neq 2\}$.

(c) The domain of $R(x) = \dfrac{x^3}{x^2 + 1}$ is the set of all real numbers.

(d) The domain of $R(x) = \dfrac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1; that is, the domain is $\{x | x \neq 1\}$.

Although $\dfrac{x^2 - 1}{x - 1}$ reduces to $x + 1$, it is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of R is $\{x | x \neq 1\}$ and the domain of f is the set of all real numbers.

Now Work PROBLEM 15

If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and if p and q have no common factors, then the rational function R is said to be in **lowest terms.** For a rational function $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms, the real zeros, if any, of the numerator in the domain of R are the x-intercepts of the graph of R and so will play a major role in the graph of R. The real zeros of the denominator of R [that is, the numbers x, if any, for which $q(x) = 0$], although not in the domain of R, also play a major role in the graph of R.

We have already discussed the properties of the rational function $y = \dfrac{1}{x}$. (Refer to Example 12, page 16). The next rational function that we take up is $H(x) = \dfrac{1}{x^2}$.

EXAMPLE 2 Graphing $y = \dfrac{1}{x^2}$

Analyze the graph of $H(x) = \dfrac{1}{x^2}$.

Solution The domain of $H(x) = \dfrac{1}{x^2}$ is the set of all real numbers x except 0. The graph has no y-intercept, because x can never equal 0. The graph has no x-intercept because the equation $H(x) = 0$ has no solution. Therefore, the graph of H will not cross or touch either of the coordinate axes. Because

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

H is an even function, so its graph is symmetric with respect to the y-axis.

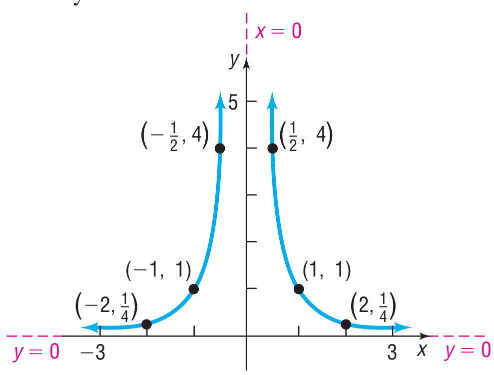
Table 9 shows the behavior of $H(x) = \dfrac{1}{x^2}$ for selected positive numbers x. (We will use symmetry to obtain the graph of H when $x < 0$.) From the first three rows of Table 9, we see that, as the values of x approach (get closer to) 0, the values of $H(x)$ become larger and larger positive numbers, so H is unbounded in the positive direction. We use limit notation, $\lim\limits_{x \to 0} H(x) = \infty$, read "the limit of $H(x)$ as x approaches zero equals infinity," to mean that $H(x) \to \infty$ as $x \to 0$.

Look at the last four rows of Table 9. As $x \to \infty$, the values of $H(x)$ approach 0 (the end behavior of the graph). In calculus, this is symbolized by writing $\lim\limits_{x \to \infty} H(x) = 0$. Figure 28 shows the graph. Notice the use of red dashed lines to convey the ideas discussed above.

Table 9

x	$H(x) = \dfrac{1}{x^2}$
$\frac{1}{2}$	4
$\frac{1}{100}$	10,000
$\frac{1}{10,000}$	100,000,000
1	1
2	$\frac{1}{4}$
100	$\frac{1}{10,000}$
10,000	$\frac{1}{100,000,000}$

Figure 28 $H(x) = \dfrac{1}{x^2}$

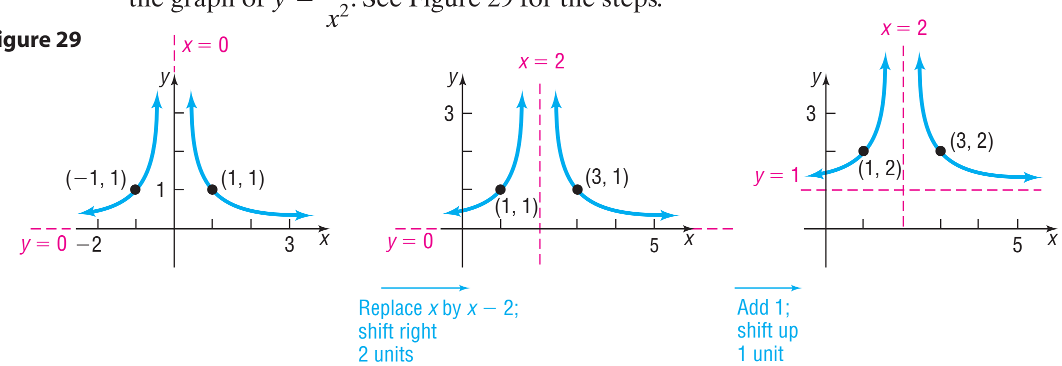


EXAMPLE 3 Using Transformations to Graph a Rational Function

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$

Solution The domain of R is the set of all real numbers except $x = 2$. To graph R, start with the graph of $y = \dfrac{1}{x^2}$. See Figure 29 for the steps.

Figure 29



(a) $y = \dfrac{1}{x^2}$ **(b)** $y = \dfrac{1}{(x-2)^2}$ **(c)** $y = \dfrac{1}{(x-2)^2} + 1$

Now Work PROBLEM 33

Asymptotes

Let's investigate the roles of the vertical line $x = 2$ and the horizontal line $y = 1$ in Figure 29(c).

First, we look at the end behavior of $R(x) = \dfrac{1}{(x-2)^2} + 1$. Table 10(a) shows the values of R at $x = 10, 100, 1000, 10{,}000$. Notice that, as x becomes unbounded in the positive direction, the values of R approach 1, so $\lim_{x\to\infty} R(x) = 1$. From Table 10(b) we see that, as x becomes unbounded in the negative direction, the values of R also approach 1, so $\lim_{x\to-\infty} R(x) = 1$.

Even though $x = 2$ is not in the domain of R, the behavior of the graph of R near $x = 2$ is important. Table 10(c) shows the values of R at $x = 1.5, 1.9, 1.99, 1.999$, and 1.9999. We see that, as x approaches 2 for $x < 2$, denoted $x \to 2^-$, the values of R are increasing without bound, so $\lim_{x\to 2^-} R(x) = \infty$. From Table 10(d), we see that, as x approaches 2 for $x > 2$, denoted $x \to 2^+$, the values of R are also increasing without bound, so $\lim_{x\to 2^+} R(x) = \infty$.

Table 10

x	$R(x)$
10	1.0156
100	1.0001
1000	1.000001
10,000	1.00000001

(a)

x	$R(x)$
−10	1.0069
−100	1.0001
−1000	1.000001
−10,000	1.00000001

(b)

x	$R(x)$
1.5	5
1.9	101
1.99	10,001
1.999	1,000,001
1.9999	100,000,001

(c)

x	$R(x)$
2.5	5
2.1	101
2.01	10,001
2.001	1,000,001
2.0001	100,000,001

(d)

The vertical line $x = 2$ and the horizontal line $y = 1$ are called *asymptotes* of the graph of R.

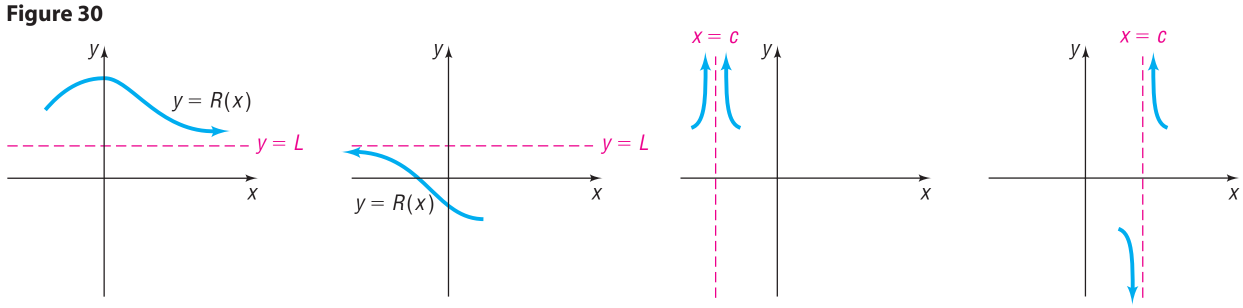
DEFINITION

Let R denote a function.

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R. [Refer to Figures 30(a) and (b).]

If, as x approaches some number c, the values $|R(x)| \to \infty$ [$R(x) \to -\infty$ or $R(x) \to \infty$], then the line $x = c$ is a **vertical asymptote** of the graph of R. [Refer to Figures 30(c) and (d).]

Figure 30



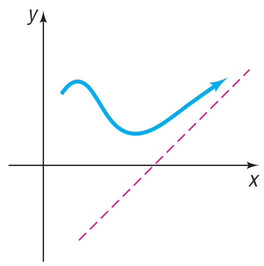
(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x\to-\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = \infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = -\infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Figure 31



A horizontal asymptote, when it occurs, describes the **end behavior** of the graph as $x \to \infty$ or as $x \to -\infty$. **The graph of a function may intersect a horizontal asymptote.**

A vertical asymptote, when it occurs, describes the behavior of the graph when x is close to some number c. **The graph of a rational function will never intersect a vertical asymptote.**

There is a third possibility. If, as $x \to -\infty$ or as $x \to \infty$, the value of a rational function $R(x)$ approaches a linear expression $ax + b$, $a \neq 0$, then the line $y = ax + b$, $a \neq 0$, is an **oblique asymptote** of R. Figure 31 shows an oblique asymptote. An oblique asymptote, when it occurs, describes the end behavior of the graph. **The graph of a function may intersect an oblique asymptote.**

Now Work PROBLEM 25

2 Find the Vertical Asymptotes of a Rational Function

The vertical asymptotes of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, are located at the real zeros of the denominator of $q(x)$. Suppose that r is a real zero of q, so $x - r$ is a factor of q. As x approaches r, symbolized as $x \to r$, the values of $x - r$ approach 0, causing the ratio to become unbounded, that is, $|R(x)| \to \infty$. Based on the definition, we conclude that the line $x = r$ is a vertical asymptote.

THEOREM

Locating Vertical Asymptotes

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x = r$.

WARNING If a rational function is not in lowest terms, an application of this theorem may result in an incorrect listing of vertical asymptotes. ■

EXAMPLE 4 Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x + 3}{x - 1}$ (b) $R(x) = \dfrac{x}{x^2 - 4}$

(c) $H(x) = \dfrac{x^2}{x^2 + 1}$ (d) $G(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$

Solution

(a) F is in lowest terms and the only zero of the denominator is 1. The line $x = 1$ is the vertical asymptote of the graph of F.

(b) R is in lowest terms and the zeros of the denominator $x^2 - 4$ are -2 and 2. The lines $x = -2$ and $x = 2$ are the vertical asymptotes of the graph of R.

(c) H is in lowest terms and the denominator has no real zeros, because the equation $x^2 + 1 = 0$ has no real solutions. The graph of H has no vertical asymptotes.

(d) Factor the numerator and denominator of $G(x)$ to determine if it is in lowest terms.

$$G(x) = \frac{x^2 - 9}{x^2 + 4x - 21} = \frac{(x + 3)(x - 3)}{(x + 7)(x - 3)} = \frac{x + 3}{x + 7} \qquad x \neq 3$$

The only zero of the denominator of $G(x)$ in lowest terms is -7. The line $x = -7$ is the only vertical asymptote of the graph of G.

WARNING In Example 4(a), the vertical asymptote is $x = 1$. Do not say that the vertical asymptote is 1. ■

As Example 4 points out, rational functions can have no vertical asymptotes, one vertical asymptote, or more than one vertical asymptote.

Exploration

Graph each of the following rational functions:

$$R(x) = \frac{1}{x-1} \quad R(x) = \frac{1}{(x-1)^2} \quad R(x) = \frac{1}{(x-1)^3} \quad R(x) = \frac{1}{(x-1)^4}$$

Each has the vertical asymptote $x = 1$. What happens to the value of $R(x)$ as x approaches 1 from the right side of the vertical asymptote; that is, what is $\lim_{x\to 1^+} R(x)$? What happens to the value of $R(x)$ as x approaches 1 from the left side of the vertical asymptote; that is, what is $\lim_{x\to 1^-} R(x)$? How does the multiplicity of the zero in the denominator affect the graph of R?

Now Work PROBLEM 47 (FIND THE VERTICAL ASYMPTOTES, IF ANY.)

3 Find the Horizontal or Oblique Asymptote of a Rational Function

The procedure for finding horizontal and oblique asymptotes is somewhat more involved. To find such asymptotes, we need to know how the values of a function behave as $x \to -\infty$ or as $x \to \infty$. That is, we need to find the end behavior of the rational function.

If a rational function $R(x)$ is **proper,** that is, if the degree of the numerator is less than the degree of the denominator, then as $x \to -\infty$ or as $x \to \infty$ the value of $R(x)$ approaches 0. Consequently, the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph.

THEOREM

If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

EXAMPLE 5 **Finding a Horizontal Asymptote**

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

Solution Since the degree of the numerator, 1, is less than the degree of the denominator, 2, the rational function R is proper. The line $y = 0$ is a horizontal asymptote of the graph of R.

To see why $y = 0$ is a horizontal asymptote of the function R in Example 5, we investigate the behavior of R as $x \to -\infty$ and $x \to \infty$. When $|x|$ is very large, the numerator of R, which is $x - 12$, can be approximated by the power function $y = x$, while the denominator of R, which is $4x^2 + x + 1$, can be approximated by the power function $y = 4x^2$. Applying these ideas to $R(x)$, we find

$$R(x) = \frac{x - 12}{4x^2 + x + 1} \underset{\substack{\uparrow \\ \text{For } |x| \text{ very large}}}{\approx} \frac{x}{4x^2} = \frac{1}{4x} \underset{\substack{\uparrow \\ \text{As } x \to -\infty \text{ or } x \to \infty}}{\longrightarrow} 0$$

This shows that the line $y = 0$ is a horizontal asymptote of the graph of R.

If a rational function $R(x) = \frac{p(x)}{q(x)}$ is **improper,** that is, if the degree of the numerator is greater than or equal to the degree of the denominator, we use long division to write the rational function as the sum of a polynomial $f(x)$ (the quotient) plus a proper rational function $\frac{r(x)}{q(x)}$ ($r(x)$ is the remainder). That is, we write

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\dfrac{r(x)}{q(x)}$ is a proper rational function. Since $\dfrac{r(x)}{q(x)}$ is proper, $\dfrac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result,

$$R(x) = \frac{p(x)}{q(x)} \to f(x) \qquad \text{as } x \to -\infty \text{ or as } x \to \infty$$

The possibilities are listed next.

1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.
2. If $f(x) = ax + b$, $a \neq 0$, the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

   We illustrate each of the possibilities in Examples 6, 7, and 8.

EXAMPLE 6 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Solution Since the degree of the numerator, 4, is greater than the degree of the denominator, 3, the rational function H is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r}
3x \quad + 3 \qquad\qquad\qquad \\
x^3 - x^2 + 1 \overline{\big) 3x^4 \qquad\quad - x^2 \qquad\quad} \\
\underline{3x^4 - 3x^3 \qquad\quad + 3x} \qquad \\
3x^3 - x^2 - 3x \qquad \\
\underline{3x^3 - 3x^2 \qquad + 3} \\
2x^2 - 3x - 3
\end{array}$$

As a result,

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1} = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $H(x) \to 3x + 3$. We conclude that the graph of the rational function H has an oblique asymptote $y = 3x + 3$.

EXAMPLE 7 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Solution Since the degree of the numerator, 2, equals the degree of the denominator, 2, the rational function R is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r} 2 \\ 4x^2 - 1 \overline{)8x^2 - x + 2} \\ \underline{8x^2 - 2} \\ -x + 4 \end{array}$$

As a result,

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1} = 2 + \frac{-x + 4}{4x^2 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{-x + 4}{4x^2 - 1} \approx \frac{-x}{4x^2} = \frac{-1}{4x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $R(x) \to 2$. We conclude that $y = 2$ is a horizontal asymptote of the graph.

In Example 7, notice that the quotient 2 obtained by long division is the quotient of the leading coefficients of the numerator polynomial and the denominator polynomial $\left(\frac{8}{4}\right)$. This means that we can avoid the long division process for rational functions where the numerator and denominator *are of the same degree* and conclude that the quotient of the leading coefficients will give us the horizontal asymptote.

Now Work PROBLEMS 43 AND 45

EXAMPLE 8 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

Solution Since the degree of the numerator, 5, is greater than the degree of the denominator, 3, the rational function G is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r} 2x^2 - 1 \\ x^3 - 1 \overline{)2x^5 - x^3 + 2} \\ \underline{2x^5 - 2x^2 } \\ -x^3 + 2x^2 + 2 \\ \underline{-x^3 + 1} \\ 2x^2 + 1 \end{array}$$

As a result,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} = 2x^2 - 1 + \frac{2x^2 + 1}{x^3 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 + 1}{x^3 - 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $G(x) \to 2x^2 - 1$. We conclude that, for large values of $|x|$, the graph of G approaches the graph of $y = 2x^2 - 1$. That is, the graph of G will look like the graph of $y = 2x^2 - 1$ as $x \to -\infty$ or $x \to \infty$. Since $y = 2x^2 - 1$ is not a linear function, G has no horizontal or oblique asymptote.

SUMMARY Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.
   (a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

Note: The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote.

4.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. ***True or False*** The quotient of two polynomial expressions is a rational expression. (pp. A36–A42)
2. What are the quotient and remainder when $3x^4 - x^2$ is divided by $x^3 - x^2 + 1$. (pp. A25–A28)
3. Graph $y = \frac{1}{x}$. (p. 16)
4. Graph $y = 2(x + 1)^2 - 3$ using transformations. (pp. 90–99)

Concepts and Vocabulary

5. ***True or False*** The domain of every rational function is the set of all real numbers.
6. If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a __________ __________ of the graph of R.
7. If, as x approaches some number c, the values of $|R(x)| \to \infty$, then the line $x = c$ is a __________ __________ of the graph of R.
8. For a rational function R, if the degree of the numerator is less than the degree of the denominator, then R is ________.
9. ***True or False*** The graph of a rational function may intersect a horizontal asymptote.
10. ***True or False*** The graph of a rational function may intersect a vertical asymptote.
11. If a rational function is proper, then __________ is a horizontal asymptote.
12. ***True or False*** If the degree of the numerator of a rational function equals the degree of the denominator, then the ratio of the leading coefficients gives rise to the horizontal asymptote.

Skill Building

In Problems 13–24, find the domain of each rational function.

13. $R(x) = \dfrac{4x}{x-3}$

14. $R(x) = \dfrac{5x^2}{3+x}$

15. $H(x) = \dfrac{-4x^2}{(x-2)(x+4)}$

16. $G(x) = \dfrac{6}{(x+3)(4-x)}$

17. $F(x) = \dfrac{3x(x-1)}{2x^2-5x-3}$

18. $Q(x) = \dfrac{-x(1-x)}{3x^2+5x-2}$

19. $R(x) = \dfrac{x}{x^3-8}$

20. $R(x) = \dfrac{x}{x^4-1}$

21. $H(x) = \dfrac{3x^2+x}{x^2+4}$

22. $G(x) = \dfrac{x-3}{x^4+1}$

23. $R(x) = \dfrac{3(x^2-x-6)}{4(x^2-9)}$

24. $F(x) = \dfrac{-2(x^2-4)}{3(x^2+4x+4)}$

In Problems 25–30, use the graph shown to find

(a) The domain and range of each function
(b) The intercepts, if any
(c) Horizontal asymptotes, if any
(d) Vertical asymptotes, if any
(e) Oblique asymptotes, if any

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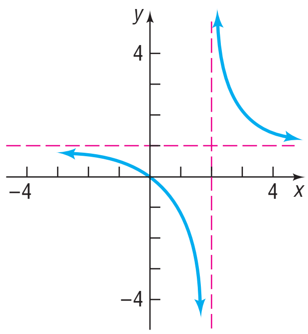

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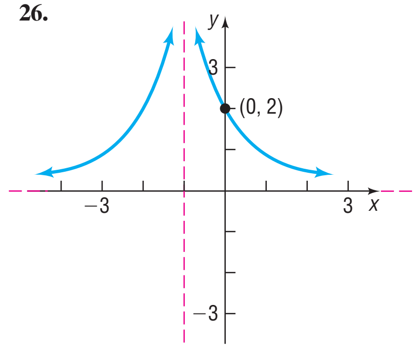

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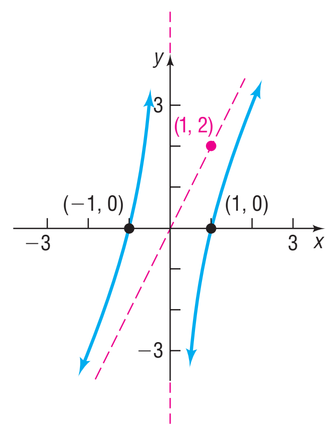

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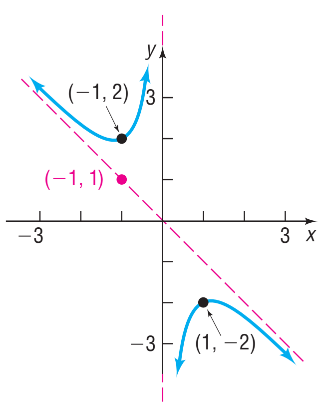

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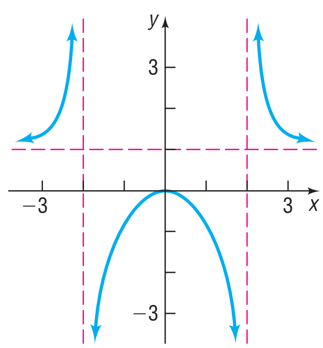

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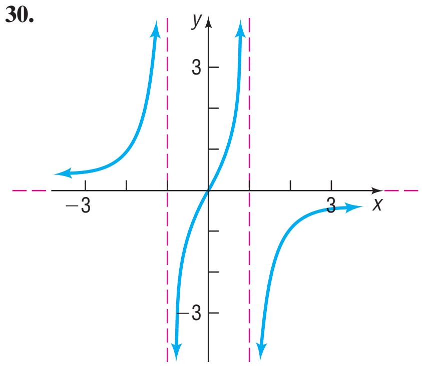


In Problems 31–42, graph each rational function using transformations.

31. $F(x) = 2 + \dfrac{1}{x}$

32. $Q(x) = 3 + \dfrac{1}{x^2}$

33. $R(x) = \dfrac{1}{(x-1)^2}$

34. $R(x) = \dfrac{3}{x}$

35. $H(x) = \dfrac{-2}{x+1}$

36. $G(x) = \dfrac{2}{(x+2)^2}$

37. $R(x) = \dfrac{-1}{x^2+4x+4}$

38. $R(x) = \dfrac{1}{x-1} + 1$

39. $G(x) = 1 + \dfrac{2}{(x-3)^2}$

40. $F(x) = 2 - \dfrac{1}{x+1}$

41. $R(x) = \dfrac{x^2-4}{x^2}$

42. $R(x) = \dfrac{x-4}{x}$

In Problems 43–54, find the vertical, horizontal, and oblique asymptotes, if any, of each rational function.

43. $R(x) = \dfrac{3x}{x+4}$

44. $R(x) = \dfrac{3x+5}{x-6}$

45. $H(x) = \dfrac{x^3-8}{x^2-5x+6}$

46. $G(x) = \dfrac{x^3+1}{x^2-5x-14}$

47. $T(x) = \dfrac{x^3}{x^4-1}$

48. $P(x) = \dfrac{4x^2}{x^3-1}$

49. $Q(x) = \dfrac{2x^2-5x-12}{3x^2-11x-4}$

50. $F(x) = \dfrac{x^2+6x+5}{2x^2+7x+5}$

51. $R(x) = \dfrac{6x^2+7x-5}{3x+5}$

52. $R(x) = \dfrac{8x^2+26x-7}{4x-1}$

53. $G(x) = \dfrac{x^4-1}{x^2-x}$

54. $F(x) = \dfrac{x^4-16}{x^2-2x}$

Applications and Extensions

55. Gravity In physics, it is established that the acceleration due to gravity, g (in m/sec^2), at a height h meters above sea level is given by

$$g(h) = \frac{3.99 \times 10^{14}}{(6.374 \times 10^6 + h)^2}$$

where 6.374×10^6 is the radius of Earth in meters.

(a) What is the acceleration due to gravity at sea level?

(b) The Willis Tower in Chicago, Illinois, is 443 meters tall. What is the acceleration due to gravity at the top of the Willis Tower?

(c) The peak of Mount Everest is 8848 meters above sea level. What is the acceleration due to gravity on the peak of Mount Everest?

(d) Find the horizontal asymptote of $g(h)$.

(e) Solve $g(h) = 0$. How do you interpret your answer?

56. Population Model A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declared the insect endangered and transplanted the insect into a protected area. The population P of the insect t months after being transplanted is

$$P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t}$$

(a) How many insects were discovered? In other words, what was the population when $t = 0$?

(b) What will the population be after 5 years?

(c) Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

57. Resistance in Parallel Circuits From Ohm's law for circuits, it follows that the total resistance R_{tot} of two components hooked in parallel is given by the equation

$$R_{tot} = \frac{R_1 R_2}{R_1 + R_2}$$

where R_1 and R_2 are the individual resistances.

(a) Let $R_1 = 10$ ohms, and graph R_{tot} as a function of R_2.

(b) Find and interpret any asymptotes of the graph obtained in part (a).

(c) If $R_2 = 2\sqrt{R_1}$, what value of R_1 will yield an R_{tot} of 17 ohms?

Source: *en.wikipedia.org/wiki/Series_and_parallel_circuits*

58. Newton's Method In calculus you will learn that, if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function, then the *derivative* of $p(x)$ is

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

Newton's Method is an efficient method for approximating the x-intercepts (or real zeros) of a function, such as $p(x)$. The following steps outline Newton's Method.

STEP 1: Select an initial value x_0 that is somewhat close to the x-intercept being sought.

STEP 2: Find values for x using the relation

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} \qquad n = 1, 2, \ldots$$

until you get two consecutive values x_n and x_{n+1} that agree to whatever decimal place accuracy you desire.

STEP 3: The approximate zero will be x_{n+1}.

Consider the polynomial $p(x) = x^3 - 7x - 40$.

(a) Evaluate $p(5)$ and $p(-3)$.

(b) What might we conclude about a zero of p? Explain.

(c) Use Newton's Method to approximate an x-intercept, r, $-3 < r < 5$, of $p(x)$ to four decimal places.

(d) Use a graphing utility to graph $p(x)$ and verify your answer in part (c).

(e) Using a graphing utility, evaluate $p(r)$ to verify your result.

Explaining Concepts: Discussion and Writing

59. If the graph of a rational function R has the vertical asymptote $x = 4$, the factor $x - 4$ must be present in the denominator of R. Explain why.

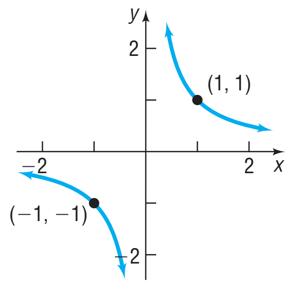
60. If the graph of a rational function R has the horizontal asymptote $y = 2$, the degree of the numerator of R equals the degree of the denominator of R. Explain why.

61. Can the graph of a rational function have both a horizontal and an oblique asymptote? Explain.

62. Make up a rational function that has $y = 2x + 1$ as an oblique asymptote. Explain the methodology that you used.

'Are You Prepared?' Answers

1. True **2.** Quotient: $3x + 3$; remainder: $2x^2 - 3x - 3$ **3.**



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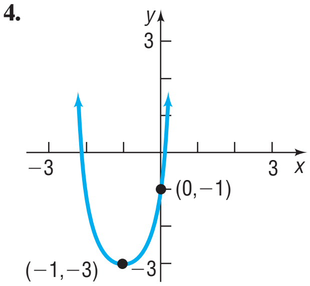


4.3 The Graph of a Rational Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Section 1.2, pp. 11–12)

Now Work the **'Are You Prepared?'** problem on page 211.

OBJECTIVES 1 Analyze the Graph of a Rational Function (p. 199)
2 Solve Applied Problems Involving Rational Functions (p. 210)

1 Analyze the Graph of a Rational Function

We commented earlier that calculus provides the tools required to graph a polynomial function accurately. The same holds true for rational functions. However, we can gather together quite a bit of information about their graphs to get an idea of the general shape and position of the graph.

EXAMPLE 1 **How to Analyze the Graph of a Rational Function**

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step-by-Step Solution

Step 1: Factor the numerator and denominator of R. Find the domain of the rational function.

$$R(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: Write R in lowest terms.

Because there are no common factors between the numerator and denominator, R is in lowest terms.

Step 3: Locate the intercepts of the graph. Determine the behavior of the graph of R near each x-intercept using the same procedure as for polynomial functions. Plot each x-intercept and indicate the behavior of the graph near it.

Since 0 is in the domain of R, the y-intercept is $R(0) = \dfrac{1}{4}$. The x-intercepts are found by determining the real zeros of the numerator of R that are in the domain of R. By solving $x - 1 = 0$, the only real zero of the numerator is 1, so the only x-intercept of the graph of R is 1. We analyze the behavior of the graph of R near $x = 1$:

Near 1: $R(x) = \dfrac{x-1}{(x+2)(x-2)} \approx \dfrac{x-1}{(1+2)(1-2)} = -\dfrac{1}{3}(x-1)$

Plot the point $(1, 0)$ and draw a line through $(1, 0)$ with a negative slope. See Figure 32(a) on page 201.

Step 4: Locate the vertical asymptotes. Graph each vertical asymptote using a dashed line.

The vertical asymptotes are the zeros of the denominator with the rational function in lowest terms. With R written in lowest terms, we find that the graph of R has two vertical asymptotes: the lines $x = -2$ and $x = 2$.

Step 5: Locate the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptotes using a dashed line. Plot any points at which the graph of R intersects the asymptote.

Because the degree of the numerator is less than the degree of the denominator, R is proper and the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph. To determine if the graph of R intersects the horizontal asymptote, solve the equation $R(x) = 0$:

$$\begin{aligned} \frac{x-1}{x^2-4} &= 0 \\ x - 1 &= 0 \\ x &= 1 \end{aligned}$$

The only solution is $x = 1$, so the graph of R intersects the horizontal asymptote at $(1, 0)$.

Step 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

The zero of the numerator, 1, and the zeros of the denominator, -2 and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \qquad (-2, 1) \qquad (1, 2) \qquad (2, \infty)$$

Now construct Table 11.

Table 11

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(2, \infty)$
Number chosen	-3	0	$\frac{3}{2}$	3
Value of R	$R(-3) = -0.8$	$R(0) = \frac{1}{4}$	$R\left(\frac{3}{2}\right) = -\frac{2}{7}$	$R(3) = 0.4$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-3, -0.8)$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{3}{2}, -\frac{2}{7}\right)$	$(3, 0.4)$

Figure 32(a) shows the asymptotes, the points from Table 11, the y-intercept, the x-intercept, and the behavior of the graph near the x-intercept, 1.

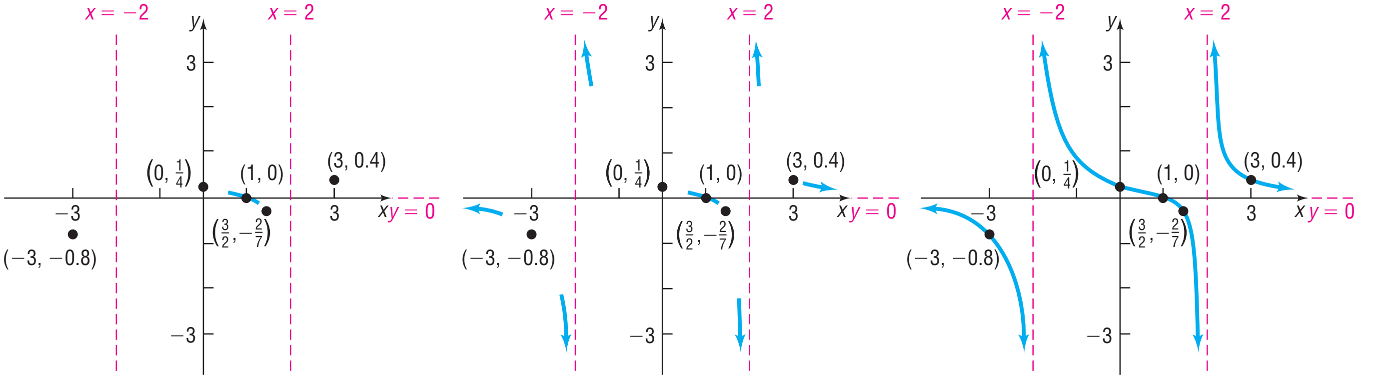
Step 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.

- Since $y = 0$ (the x-axis) is a horizontal asymptote and the graph lies below the x-axis for $x < -2$, we can sketch a portion of the graph by placing a small arrow to the far left and under the x-axis.
- Since the line $x = -2$ is a vertical asymptote and the graph lies below the x-axis for $x < -2$, we place an arrow well below the x-axis and approaching the line $x = -2$ from the left $\left(\lim_{x \to -2^-} R(x) = -\infty\right)$.
- Since the graph is above the x-axis for $-2 < x < 1$ and $x = -2$ is a vertical asymptote, the graph will continue on the right of $x = -2$ at the top $\left(\lim_{x \to -2^+} R(x) = +\infty\right)$. Similar explanations account for the other arrows shown in Figure 32(b).

Step 8: Use the results obtained in Steps 1 through 7 to graph R.

Figure 32(c) shows the graph of R.

Figure 32



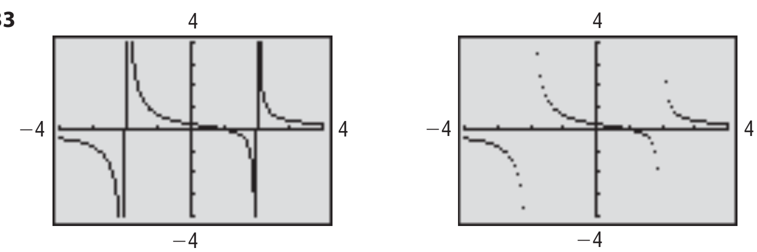
(a) (b) (c)

Exploration

Graph $R(x) = \dfrac{x-1}{x^2-4}$

Result The analysis just completed in Example 1 helps us to set the viewing rectangle to obtain a complete graph. Figure 33(a) shows the graph of $R(x) = \dfrac{x-1}{x^2-4}$ in connected mode, and Figure 33(b) shows it in dot mode. Notice in Figure 33(a) that the graph has vertical lines at $x = -2$ and $x = 2$. This is due to the fact that, when the graphing utility is in connected mode, it will connect the dots between consecutive pixels and vertical lines may occur. We know that the graph of R does not cross the lines $x = -2$ and $x = 2$, since R is not defined at $x = -2$ or $x = 2$. So, when graphing rational functions, dot mode should be used to avoid extraneous vertical lines that are not part of the graph. See Figure 33(b).

Figure 33



Connected mode
(a)

Dot mode
(b)

Now Work PROBLEM 7

SUMMARY Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. The x-intercepts are the zeros of the numerator of R that are in the domain of R. Determine the behavior of the graph of R near each x-intercept.

STEP 4: Determine the vertical asymptotes. Graph each vertical asymptote using a dashed line.

STEP 5: Determine the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

(Continued)

Step 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

Step 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.

Step 8: Use the results obtained in Steps 1 through 7 to graph R.

EXAMPLE 2 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

Solution

Step 1: $R(x) = \dfrac{(x+1)(x-1)}{x}$. The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: Because x cannot equal 0, there is no y-intercept. The graph has two x-intercepts: -1 and 1.

$$\text{Near } -1: \quad R(x) = \frac{(x+1)(x-1)}{x} \approx \frac{(x+1)(-1-1)}{-1} = 2(x+1)$$

$$\text{Near } 1: \quad R(x) = \frac{(x+1)(x-1)}{x} \approx \frac{(1+1)(x-1)}{1} = 2(x-1)$$

Plot the point $(-1, 0)$ and indicate a line with positive slope there. Plot the point $(1, 0)$ and indicate a line with positive slope there.

Step 4: The real zero of the denominator with R in lowest terms is 0, so the graph of R has the line $x = 0$ (the y-axis) as a vertical asymptote. Graph $x = 0$ using a dashed line.

Step 5: Since the degree of the numerator, 2, is one greater than the degree of the denominator, 1, the rational function will have an oblique asymptote. To find the oblique asymptote, we use long division.

$$\begin{array}{r} x \\ x \overline{)\, x^2 - 1} \\ \underline{x^2 } \\ -1 \end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. Graph $y = x$ using a dashed line.

To determine whether the graph of R intersects the asymptote $y = x$, we solve the equation $R(x) = x$.

$$R(x) = \frac{x^2 - 1}{x} = x$$
$$x^2 - 1 = x^2$$
$$-1 = 0 \quad \text{Impossible}$$

We conclude that the equation $\dfrac{x^2 - 1}{x} = x$ has no solution, so the graph of R does not intersect the line $y = x$.

Step 6: The zeros of the numerator are -1 and 1; the zero of the denominator is 0. Use these values to divide the x-axis into four intervals:

$$(-\infty, -1) \qquad (-1, 0) \qquad (0, 1) \qquad (1, \infty)$$

Now construct Table 12. Plot the points from Table 12. You should now have Figure 34(a).

COMMENT Because the denominator of the rational function is a monomial, we can also find the oblique asymptote as follows:

$$\frac{x^2 - 1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$$

Since $\dfrac{1}{x} \rightarrow 0$ as $x \rightarrow \infty$, $y = x$ is the oblique asymptote. ■

COMMENT Notice that R is an odd function and so its graph is symmetric with respect to the origin. This observation reduces the work involved in graphing R. ■

Table 12

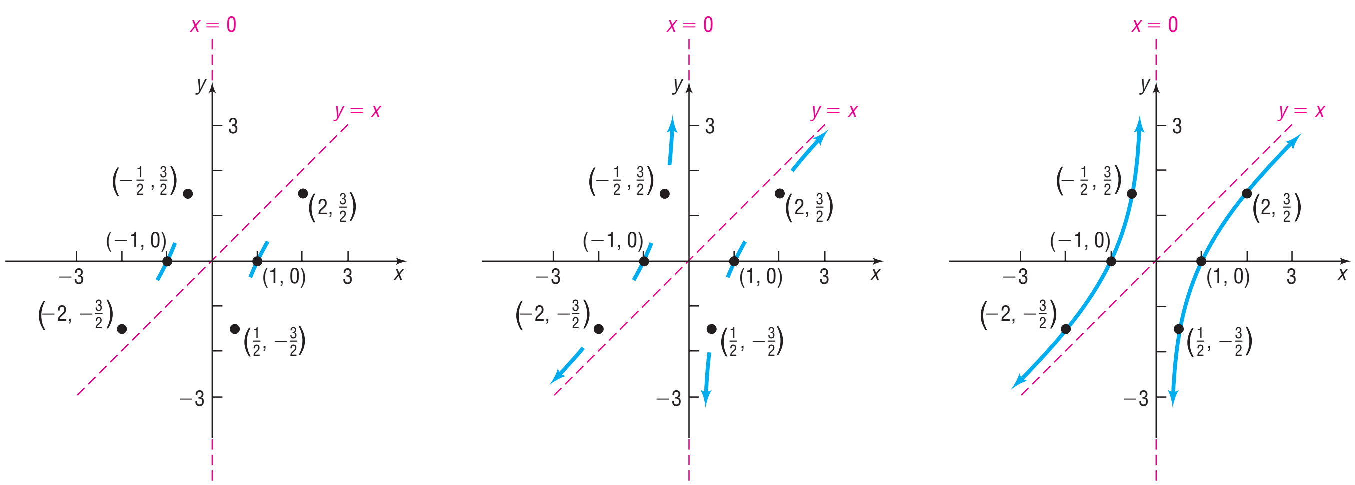
	-1	0	1	x
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of *R*	$R(-2) = -\frac{3}{2}$	$R\left(-\frac{1}{2}\right) = \frac{3}{2}$	$R\left(\frac{1}{2}\right) = -\frac{3}{2}$	$R(2) = \frac{3}{2}$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$\left(-2, -\frac{3}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$\left(\frac{1}{2}, -\frac{3}{2}\right)$	$\left(2, \frac{3}{2}\right)$

Step 7: Since the graph of R is below the x-axis for $x < -1$ and is above the x-axis for $x > 1$, and since the graph of R does not intersect the oblique asymptote $y = x$, the graph of R will approach the line $y = x$ as shown in Figure 34(b).

Since the graph of R is above the x-axis for $-1 < x < 0$, the graph of R will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ $\left[\lim_{x \to 0^-} R(x) = \infty\right]$; since the graph of R is below the x-axis for $0 < x < 1$, the graph of R will approach the vertical asymptote $x = 0$ at the bottom to the right of $x = 0$ $\left[\lim_{x \to 0^+} R(x) = -\infty\right]$. See Figure 34(b).

Step 8: The complete graph is given in Figure 34(c).

Figure 34



(a) (b) (c)

Seeing the Concept

Graph $R(x) = \dfrac{x^2 - 1}{x}$ and compare what you see with Figure 34(c). Could you have predicted from the graph that $y = x$ is an oblique asymptote? Graph $y = x$ and ZOOM-OUT. What do you observe?

Now Work PROBLEM 15

EXAMPLE 3 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

Solution

Step 1: R is completely factored. The domain of R is $\{x | x \neq 0\}$.

Step 2: R is in lowest terms.

STEP 3: There is no y-intercept. Since $x^4 + 1 = 0$ has no real solutions, there are no x-intercepts.

STEP 4: R is in lowest terms, so $x = 0$ (the y-axis) is a vertical asymptote of R. Graph the line $x = 0$ using dashes.

STEP 5: Since the degree of the numerator, 4, is two more than the degree of the denominator, 2, the rational function will not have a horizontal or oblique asymptote. We use long division to find the end behavior of R.

$$\begin{array}{r} x^2 \\ x^2 \overline{\smash{)}\, x^4 + 1} \\ \underline{x^4 } \\ 1 \end{array}$$

The quotient is x^2, so the graph of R will approach the graph of $y = x^2$ as $x \to -\infty$ and as $x \to \infty$. The graph of R does not intersect $y = x^2$. Do you know why? Graph $y = x^2$ using dashes.

STEP 6: The numerator has no real zeros, and the denominator has one real zero at 0. We divide the x-axis into the two intervals

$$(-\infty, 0) \qquad (0, \infty)$$

and construct Table 13.

Table 13

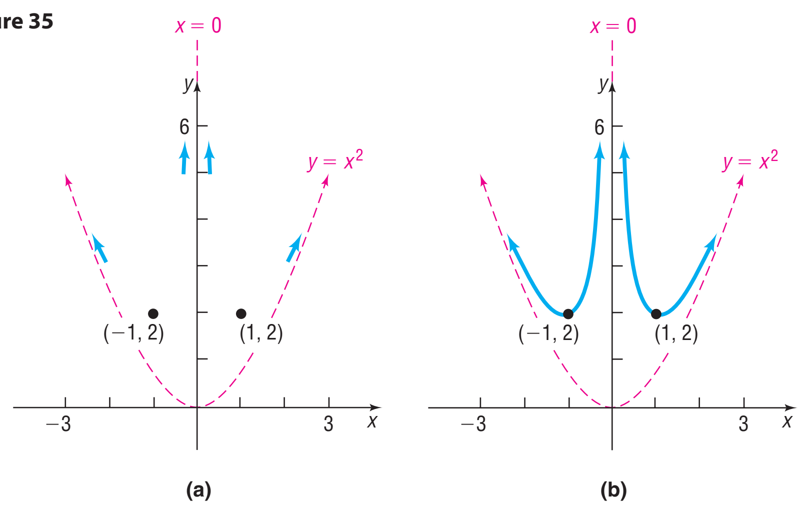
Interval	$(-\infty, 0)$	$(0, \infty)$
Number chosen	-1	1
Value of R	$R(-1) = 2$	$R(1) = 2$
Location of graph	Above x-axis	Above x-axis
Point on graph	$(-1, 2)$	$(1, 2)$

Plot the points $(-1, 2)$ and $(1, 2)$.

STEP 7: Since the graph of R is above the x-axis and does not intersect $y = x^2$, we place arrows above $y = x^2$ as shown in Figure 35(a). Also, since the graph of R is above the x-axis, it will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ and at the top to the right of $x = 0$. See Figure 35(a).

STEP 8: Figure 35(b) shows the complete graph.

Figure 35



COMMENT Notice that R in Example 3 is an even function. Do you see the symmetry about the y-axis in the graph of R?

Seeing the Concept

Graph $R(x) = \dfrac{x^4 + 1}{x^2}$ and compare what you see with Figure 35(b). Use MINIMUM to find the two turning points. Enter $Y_2 = x^2$ and ZOOM-OUT. What do you see?

Now Work PROBLEM 13

EXAMPLE 4 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Solution

Step 1: Factor R to get

$$R(x) = \frac{3x(x-1)}{(x+4)(x-3)}$$

The domain of R is $\{x \mid x \neq -4, x \neq 3\}$.

Step 2: R is in lowest terms.

Step 3: The y-intercept is $R(0) = 0$. Plot the point $(0, 0)$. Since the real solutions of the equation $3x(x - 1) = 0$ are $x = 0$ and $x = 1$, the graph has two x-intercepts, 0 and 1. We determine the behavior of the graph of R near each x-intercept.

$$\text{Near } 0: \quad R(x) = \frac{3x(x-1)}{(x+4)(x-3)} \approx \frac{3x(0-1)}{(0+4)(0-3)} = \frac{1}{4}x$$

$$\text{Near } 1: \quad R(x) = \frac{3x(x-1)}{(x+4)(x-3)} \approx \frac{3(1)(x-1)}{(1+4)(1-3)} = -\frac{3}{10}(x-1)$$

Plot the point $(0, 0)$ and show a line with positive slope there. Plot the point $(1, 0)$ and show a line with negative slope there.

Step 4: R is in lowest terms. The real solutions of the equation $(x + 4)(x - 3) = 0$ are $x = -4$ and $x = 3$, so the graph of R has two vertical asymptotes, the lines $x = -4$ and $x = 3$. Graph these lines using dashes.

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$.

To find out whether the graph of R intersects the asymptote, solve the equation $R(x) = 3$.

$$\begin{aligned} R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} &= 3 \\ 3x^2 - 3x &= 3x^2 + 3x - 36 \\ -6x &= -36 \\ x &= 6 \end{aligned}$$

The graph intersects the line $y = 3$ at $x = 6$, and $(6, 3)$ is a point on the graph of R. Plot the point $(6, 3)$ and graph the line $y = 3$ using dashes.

Step 6: The real zeros of the numerator, 0 and 1, and the real zeros of the denominator, -4 and 3, divide the x-axis into five intervals:

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, 3) \quad (3, \infty)$$

Construct Table 14. Plot the points from Table 14. Figure 36(a) shows the graph we have so far.

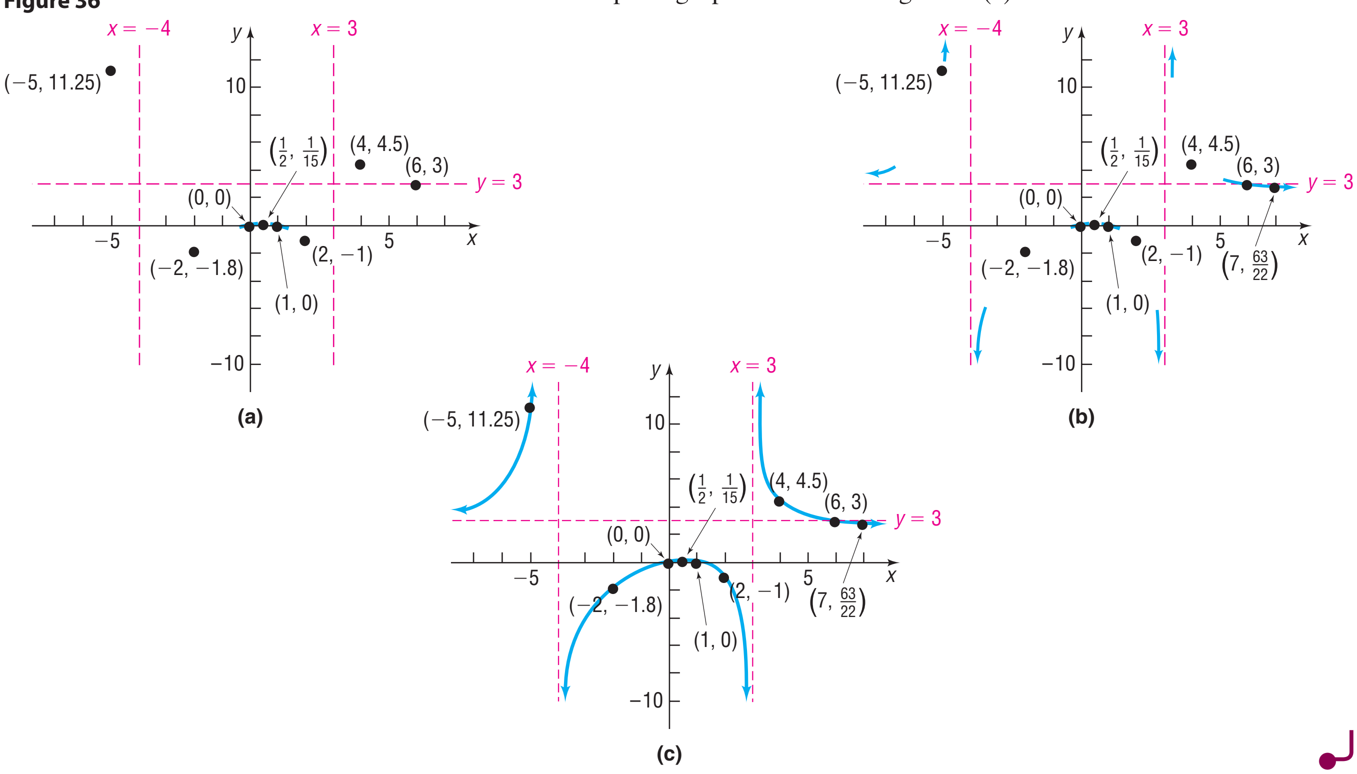
Table 14

	-4	0	1	3	x
Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Number chosen	-5	-2	$\frac{1}{2}$	2	4
Value of *R*	$R(-5) = 11.25$	$R(-2) = -1.8$	$R\left(\frac{1}{2}\right) = \frac{1}{15}$	$R(2) = -1$	$R(4) = 4.5$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-5, 11.25)$	$(-2, -1.8)$	$\left(\frac{1}{2}, \frac{1}{15}\right)$	$(2, -1)$	$(4, 4.5)$

STEP 7:
- Since the graph of R is above the x-axis for $x < -4$ and only crosses the line $y = 3$ at $(6, 3)$, as x approaches $-\infty$ the graph of R will approach the horizontal asymptote $y = 3$ from above $\left(\lim_{x\to-\infty} R(x) = 3\right)$.
- The graph of R will approach the vertical asymptote $x = -4$ at the top to the left of $x = -4$ $\left(\lim_{x\to-4^-} R(x) = +\infty\right)$ and at the bottom to the right of $x = -4$ $\left(\lim_{x\to-4^+} R(x) = -\infty\right)$.
- The graph of R will approach the vertical asymptote $x = 3$ at the bottom to the left of $x = 3$ $\left(\lim_{x\to3^-} R(x) = -\infty\right)$ and at the top to the right of $x = 3$ $\left(\lim_{x\to3^+} R(x) = +\infty\right)$.
- We do not know whether the graph of R crosses or touches the line $y = 3$ at $(6, 3)$. To see whether the graph, in fact, crosses or touches the line $y = 3$, we plot an additional point to the right of $(6, 3)$. We use $x = 7$ to find $R(7) = \frac{63}{22} < 3$. The graph crosses $y = 3$ at $x = 6$. Because $(6, 3)$ is the only point where the graph of R intersects the asymptote $y = 3$, the graph must approach the line $y = 3$ from below as $x \to \infty$ $\left(\lim_{x\to\infty} R(x) = 3\right)$. See Figure 36(b).

STEP 8: The complete graph is shown in Figure 36(c).

Figure 36

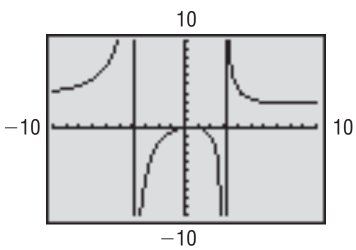


Exploration

Graph $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

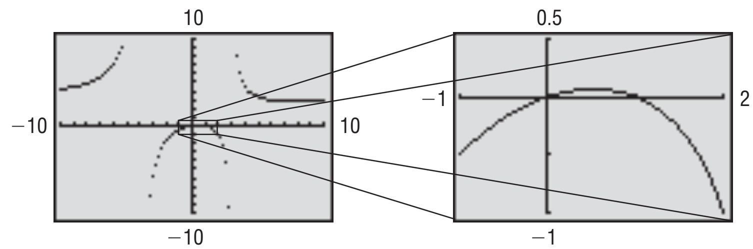
Result Figure 37 shows the graph in connected mode, and Figure 38(a) shows it in dot mode. Neither graph displays clearly the behavior of the function between the two x-intercepts, 0 and 1. Nor do they clearly display the fact that the graph crosses the horizontal asymptote at $(6, 3)$. To see these parts better, we graph R for $-1 \le x \le 2$ [Figure 38(b)] and for $4 \le x \le 60$ [Figure 39(b)].

Figure 37



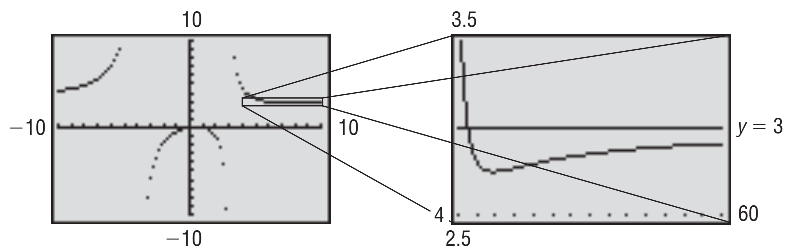
Connected mode

Figure 38



Dot mode
(a)
(b)

Figure 39



Dot mode
(a)
(b)

The new graphs reflect the behavior produced by the analysis. Furthermore, we observe two turning points, one between 0 and 1 and the other to the right of 6. Rounded to two decimal places, these turning points are $(0.52, 0.07)$ and $(11.48, 2.75)$.

EXAMPLE 5 Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution

Step 1: Factor R and obtain

$$R(x) = \frac{(2x - 1)(x - 2)}{(x + 2)(x - 2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: In lowest terms,

$$R(x) = \frac{2x - 1}{x + 2} \qquad x \neq -2, x \neq 2$$

Step 3: The y-intercept is $R(0) = -\dfrac{1}{2}$. Plot the point $\left(0, -\dfrac{1}{2}\right)$.

The graph has one x-intercept: $\dfrac{1}{2}$.

$$\text{Near } \frac{1}{2}: \quad R(x) = \frac{2x - 1}{x + 2} \approx \frac{2x - 1}{\frac{1}{2} + 2} = \frac{2}{5}(2x - 1)$$

Plot the point $\left(\dfrac{1}{2}, 0\right)$ showing a line with positive slope.

Step 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms*, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. Graph the line $x = -2$ using dashes.

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. Graph the line $y = 2$ using dashes.

To find out whether the graph of R intersects the horizontal asymptote $y = 2$, we solve the equation $R(x) = 2$.

$$\begin{aligned} R(x) = \frac{2x-1}{x+2} &= 2 \\ 2x - 1 &= 2(x+2) \\ 2x - 1 &= 2x + 4 \\ -1 &= 4 \qquad \text{Impossible} \end{aligned}$$

The graph does not intersect the line $y = 2$.

Step 6: Look at the factored expression for R in Step 1. The real zeros of the numerator and denominator, $-2, \frac{1}{2}$, and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \qquad \left(-2, \frac{1}{2}\right) \qquad \left(\frac{1}{2}, 2\right) \qquad (2, \infty)$$

Construct Table 15. Plot the points in Table 15.

Table 15

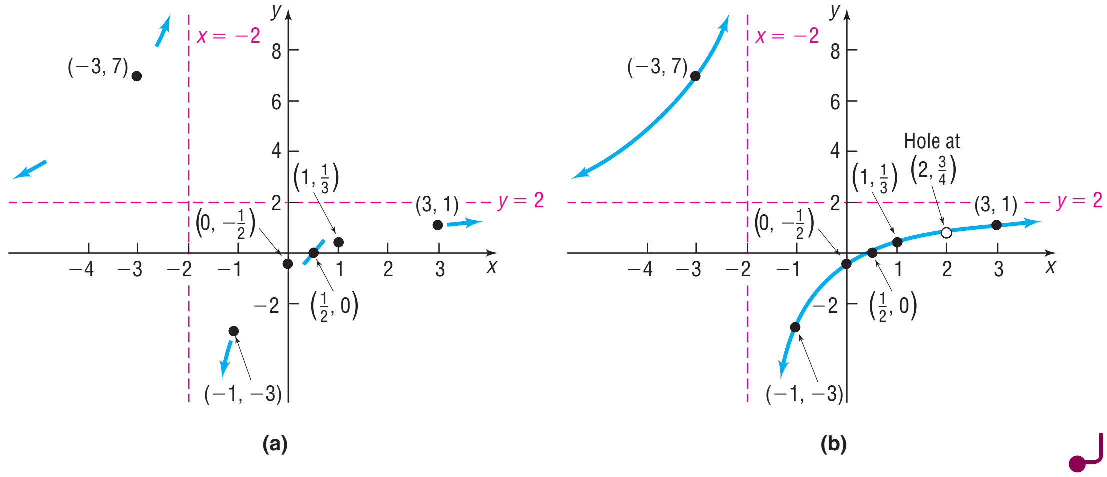
	-2	$1/2$	2	x
Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of *R*	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Above x-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

Step 7:
- From Table 15 we know that the graph of R is above the x-axis for $x < -2$.
- From Step 5 we know that the graph of R does not intersect the asymptote $y = 2$. Therefore, the graph of R will approach $y = 2$ from above as $x \to -\infty$ and will approach the vertical asymptote $x = -2$ at the top from the left.
- Since the graph of R is below the x-axis for $-2 < x < \frac{1}{2}$, the graph of R will approach $x = -2$ at the bottom from the right.
- Finally, since the graph of R is above the x-axis for $x > \frac{1}{2}$ and does not intersect the horizontal asymptote $y = 2$, the graph of R will approach $y = 2$ from below as $x \to \infty$. See Figure 40(a).

Step 8: See Figure 40(b) for the complete graph. Since R is not defined at 2, there is a hole at the point $\left(2, \frac{3}{4}\right)$.

COMMENT The coordinates of the hole were obtained by evaluating R in lowest terms at 2. R in lowest terms is $\frac{2x-1}{x+2}$, which, at $x = 2$, is $\frac{2(2)-1}{2+2} = \frac{3}{4}$. ■

Figure 40



Exploration

Graph $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$. Do you see the hole at $\left(2, \dfrac{3}{4}\right)$? TRACE along the graph. Did you obtain an ERROR at $x = 2$? Are you convinced that an algebraic analysis of a rational function is required in order to accurately interpret the graph obtained with a graphing utility?

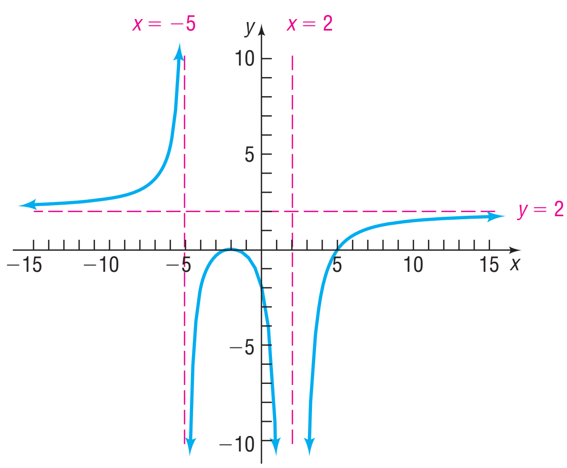
As Example 5 shows, **the zeros of the denominator of a rational function give rise to either vertical asymptotes or holes in the graph.**

Now Work PROBLEM 33

EXAMPLE 6 Constructing a Rational Function from Its Graph

Find a rational function that might have the graph shown in Figure 41.

Figure 41



Solution The numerator of a rational function $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms determines the x-intercepts of its graph. The graph shown in Figure 41 has x-intercepts -2 (even multiplicity; graph touches the x-axis) and 5 (odd multiplicity; graph crosses the x-axis). So one possibility for the numerator is $p(x) = (x + 2)^2(x - 5)$.

The denominator of a rational function in lowest terms determines the vertical asymptotes of its graph. The vertical asymptotes of the graph are $x = -5$ and $x = 2$. Since $R(x)$ approaches ∞ to the left of $x = -5$ and $R(x)$ approaches $-\infty$ to the right of $x = -5$, we know that $(x + 5)$ is a factor of odd multiplicity in $q(x)$. Also, $R(x)$ approaches $-\infty$ on both sides of $x = 2$, so $(x - 2)$ is a factor of even multiplicity in

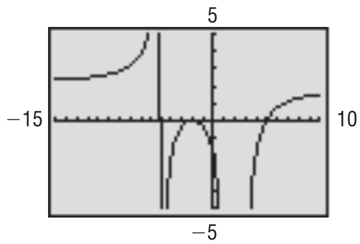
$q(x)$. A possibility for the denominator is $q(x) = (x + 5)(x - 2)^2$. So far we have

$$R(x) = \frac{(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}.$$

The horizontal asymptote of the graph given in Figure 41 is $y = 2$, so we know that the degree of the numerator must equal the degree of the denominator and the quotient of leading coefficients must be $\frac{2}{1}$. This leads to

$$R(x) = \frac{2(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}$$

Figure 42



✓Check: Figure 42 shows the graph of R on a graphing utility. Since Figure 42 looks similar to Figure 41, we have found a rational function R for the graph in Figure 41.

Now Work PROBLEM 45

2 Solve Applied Problems Involving Rational Functions

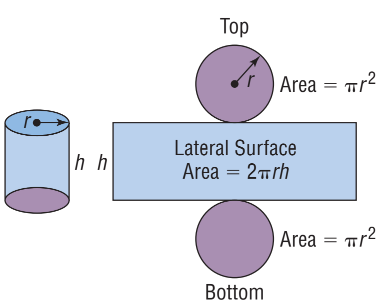
EXAMPLE 7 Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

Solution (a) Figure 43 illustrates the components of a can in the shape of a right circular cylinder. Notice that the material required to produce a cylindrical can of height h and radius r consists of a rectangle of area $2\pi rh$ and two circles, each of area πr^2. The total cost C (in cents) of manufacturing the can is therefore

Figure 43



$$\begin{aligned} C &= \text{Cost of the top and bottom} + \text{Cost of the side} \\ &= \underbrace{2(\pi r^2)}_{\text{Total area of top and bottom}} \underbrace{(0.05)}_{\text{Cost/unit area}} + \underbrace{(2\pi rh)}_{\text{Total area of side}} \underbrace{(0.02)}_{\text{Cost/unit area}} \\ &= 0.10\pi r^2 + 0.04\pi rh \end{aligned}$$

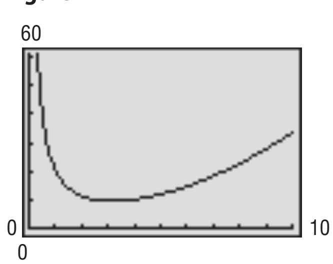
But we have the additional restriction that the height h and radius r must be chosen so that the volume V of the can is 500 cubic centimeters. Since $V = \pi r^2 h$, we have

$$500 = \pi r^2 h \quad \text{so} \quad h = \frac{500}{\pi r^2}$$

Substituting this expression for h, the cost C, in cents, as a function of the radius r is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \cdot \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

Figure 44



(b) See Figure 44 for the graph of $C = C(r)$.
(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 centimeters.
(d) The least cost is $C(3.17) \approx 9.47$¢.

Now Work PROBLEM 55

4.3 Assess Your Understanding

'Are You Prepared?'

The answer is given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the intercepts of the graph of the equation $y = \dfrac{x^2 - 1}{x^2 - 4}$. (pp. 11–12)

Concepts and Vocabulary

2. If the numerator and the denominator of a rational function have no common factors, the rational function is ___ ________ ______.
3. The graph of a rational function never intersects a ________ asymptote.
4. ***True or False*** The graph of a rational function sometimes intersects an oblique asymptote.
5. ***True or False*** The graph of a rational function sometimes has a hole.
6. $R(x) = \dfrac{x(x-2)^2}{x-2}$
   (a) Find the domain of R.
   (b) Find the x-intercepts of R.

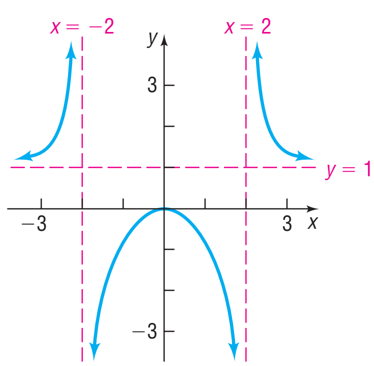
Skill Building

In Problems 7–44, follow Steps 1 through 8 on pages 201–202 to analyze the graph of each function.

7. $R(x) = \dfrac{x+1}{x(x+4)}$
8. $R(x) = \dfrac{x}{(x-1)(x+2)}$
9. $R(x) = \dfrac{3x+3}{2x+4}$
10. $R(x) = \dfrac{2x+4}{x-1}$
11. $R(x) = \dfrac{3}{x^2-4}$
12. $R(x) = \dfrac{6}{x^2-x-6}$
13. $P(x) = \dfrac{x^4+x^2+1}{x^2-1}$
14. $Q(x) = \dfrac{x^4-1}{x^2-4}$
15. $H(x) = \dfrac{x^3-1}{x^2-9}$
16. $G(x) = \dfrac{x^3+1}{x^2+2x}$
17. $R(x) = \dfrac{x^2}{x^2+x-6}$
18. $R(x) = \dfrac{x^2+x-12}{x^2-4}$
19. $G(x) = \dfrac{x}{x^2-4}$
20. $G(x) = \dfrac{3x}{x^2-1}$
21. $R(x) = \dfrac{3}{(x-1)(x^2-4)}$
22. $R(x) = \dfrac{-4}{(x+1)(x^2-9)}$
23. $H(x) = \dfrac{x^2-1}{x^4-16}$
24. $H(x) = \dfrac{x^2+4}{x^4-1}$
25. $F(x) = \dfrac{x^2-3x-4}{x+2}$
26. $F(x) = \dfrac{x^2+3x+2}{x-1}$
27. $R(x) = \dfrac{x^2+x-12}{x-4}$
28. $R(x) = \dfrac{x^2-x-12}{x+5}$
29. $F(x) = \dfrac{x^2+x-12}{x+2}$
30. $G(x) = \dfrac{x^2-x-12}{x+1}$
31. $R(x) = \dfrac{x(x-1)^2}{(x+3)^3}$
32. $R(x) = \dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$
33. $R(x) = \dfrac{x^2+x-12}{x^2-x-6}$
34. $R(x) = \dfrac{x^2+3x-10}{x^2+8x+15}$
35. $R(x) = \dfrac{6x^2-7x-3}{2x^2-7x+6}$
36. $R(x) = \dfrac{8x^2+26x+15}{2x^2-x-15}$
37. $R(x) = \dfrac{x^2+5x+6}{x+3}$
38. $R(x) = \dfrac{x^2+x-30}{x+6}$
39. $f(x) = x + \dfrac{1}{x}$
40. $f(x) = 2x + \dfrac{9}{x}$
41. $f(x) = x^2 + \dfrac{1}{x}$
42. $f(x) = 2x^2 + \dfrac{16}{x}$
43. $f(x) = x + \dfrac{1}{x^3}$
44. $f(x) = 2x + \dfrac{9}{x^3}$

In Problems 45–48, find a rational function that might have the given graph. (More than one answer might be possible.)

45.



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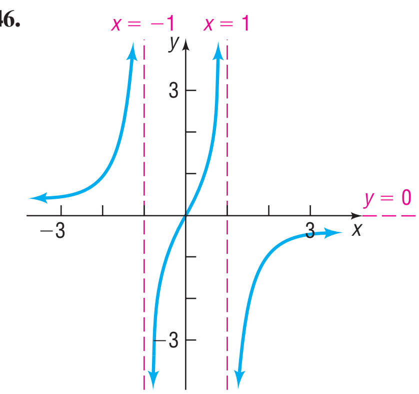
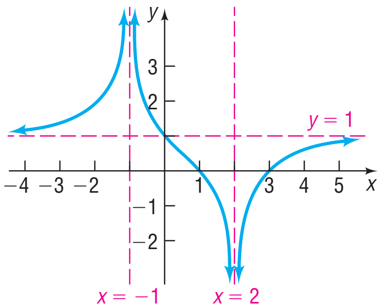
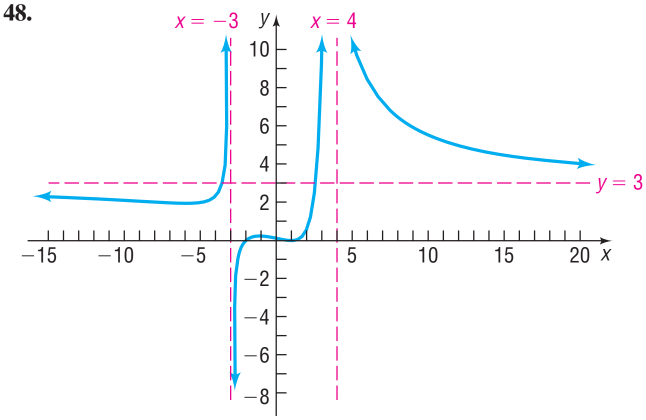


47.



48.



Applications and Extensions

49. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{t}{2t^2 + 1}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?
(b) Using your graphing utility, graph $C = C(t)$.
(c) Determine the time at which the concentration is highest.

50. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t minutes after injection is given by

$$C(t) = \frac{50t}{t^2 + 25}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?
(b) Using your graphing utility, graph $C = C(t)$.
(c) Determine the time at which the concentration is highest.

51. Minimum Cost A rectangular area adjacent to a river is to be fenced in; no fence is needed on the river side. The enclosed area is to be 1000 square feet. Fencing for the side parallel to the river is \$5 per linear foot, and fencing for the other two sides is \$8 per linear foot; the four corner posts are \$25 apiece. Let x be the length of one of the sides perpendicular to the river.
(a) Write a function $C(x)$ that describes the cost of the project.
(b) What is the domain of C?
(c) Use a graphing utility to graph $C = C(x)$.
(d) Find the dimensions of the cheapest enclosure.

Source: *www.uncwil.edu/courses/math111hb/PandR/rational/rational.html*

52. Doppler Effect The Doppler effect (named after Christian Doppler) is the change in the pitch (frequency) of the sound from a source (s) as heard by an observer (o) when one or both are in motion. If we assume both the source and the observer are moving in the same direction, the relationship is

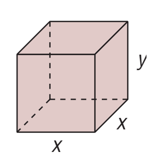
$$f' = f_a\left(\frac{v - v_o}{v - v_s}\right)$$

where f' = perceived pitch by the observer
f_a = actual pitch of the source
v = speed of sound in air (assume 772.4 mph)
v_o = speed of the observer
v_s = speed of the source

Suppose that you are traveling down the road at 45 mph and you hear an ambulance (with siren) coming toward you from the rear. The actual pitch of the siren is 600 hertz (Hz).
(a) Write a function $f'(v_s)$ that describes this scenario.
(b) If $f' = 620$ Hz, find the speed of the ambulance.
(c) Use a graphing utility to graph the function.
(d) Verify your answer from part (b).

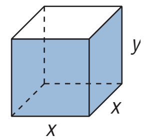
Source: *www.kettering.edu/~drussell/*

53. Minimizing Surface Area United Parcel Service has contracted you to design a closed box with a square base that has a volume of 10,000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

54. Minimizing Surface Area United Parcel Service has contracted you to design an open box with a square base that has a volume of 5000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

55. Cost of a Can A can in the shape of a right circular cylinder is required to have a volume of 500 cubic centimeters. The top and bottom are made of material that costs 6¢ per square centimeter, while the sides are made of material that costs 4¢ per square centimeter.
(a) Express the total cost C of the material as a function of the radius r of the cylinder. (Refer to Figure 43.)
(b) Graph $C = C(r)$. For what value of r is the cost C a minimum?

56. Material Needed to Make a Drum A steel drum in the shape of a right circular cylinder is required to have a volume of 100 cubic feet.

(a) Express the amount A of material required to make the drum as a function of the radius r of the cylinder.
(b) How much material is required if the drum's radius is 3 feet?
(c) How much material is required if the drum's radius is 4 feet?
(d) How much material is required if the drum's radius is 5 feet?
(e) Graph $A = A(r)$. For what value of r is A smallest?

Explaining Concepts: Discussion and Writing

57. Graph each of the following functions:

$$y = \frac{x^2 - 1}{x - 1} \qquad y = \frac{x^3 - 1}{x - 1}$$
$$y = \frac{x^4 - 1}{x - 1} \qquad y = \frac{x^5 - 1}{x - 1}$$

Is $x = 1$ a vertical asymptote? Why not? What is happening for $x = 1$? What do you conjecture about $y = \frac{x^n - 1}{x - 1}$, $n \geq 1$ an integer, for $x = 1$?

58. Graph each of the following functions:

$$y = \frac{x^2}{x - 1} \qquad y = \frac{x^4}{x - 1} \qquad y = \frac{x^6}{x - 1} \qquad y = \frac{x^8}{x - 1}$$

What similarities do you see? What differences?

59. Write a few paragraphs that provide a general strategy for graphing a rational function. Be sure to mention the following: proper, improper, intercepts, and asymptotes.

60. Create a rational function that has the following characteristics: crosses the x-axis at 2; touches the x-axis at -1; one vertical asymptote at $x = -5$ and another at $x = 6$; and one horizontal asymptote, $y = 3$. Compare your function to a fellow classmate's. How do they differ? What are their similarities?

61. Create a rational function that has the following characteristics: crosses the x-axis at 3; touches the x-axis at -2; one vertical asymptote, $x = 1$; and one horizontal asymptote, $y = 2$. Give your rational function to a fellow classmate and ask for a written critique of your rational function.

62. Create a rational function with the following characteristics: three real zeros, one of multiplicity 2; y-intercept 1; vertical asymptotes, $x = -2$ and $x = 3$; oblique asymptote, $y = 2x + 1$. Is this rational function unique? Compare your function with those of other students. What will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the rational function?

63. Explain the circumstances under which the graph of a rational function will have a hole.

'Are You Prepared?' Answer

1. $\left(0, \frac{1}{4}\right), (1, 0), (-1, 0)$

4.4 Polynomial and Rational Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Inequalities (Appendix A, Section A.9, pp. A75–A78)
- Solving Quadratic Inequalities (Section 3.5, pp. 155–157)

Now Work the **'Are You Prepared?'** problems on page 217.

OBJECTIVES
1 Solve Polynomial Inequalities (p. 214)
2 Solve Rational Inequalities (p. 215)

1 Solve Polynomial Inequalities

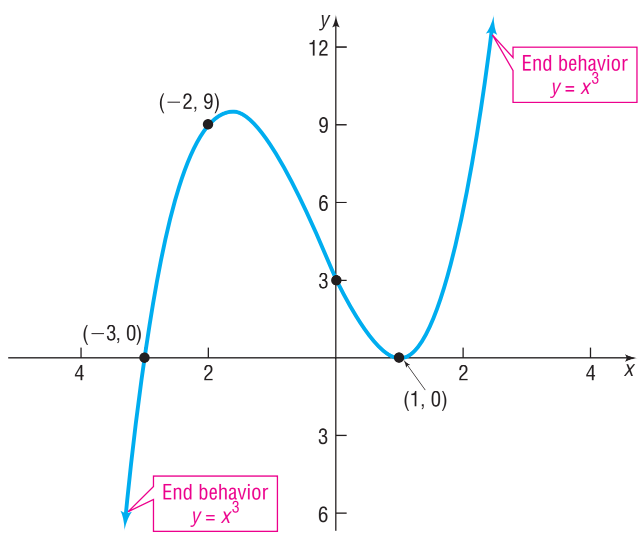
In this section we solve inequalities that involve polynomials of degree 3 and higher, along with inequalities that involve rational functions. To help understand the algebraic procedure for solving such inequalities, we use the information obtained in the previous three sections about the graphs of polynomial and rational functions. The approach follows the same methodology that we used to solve inequalities involving quadratic functions.

EXAMPLE 1 Solving a Polynomial Inequality Using Its Graph

Solve $(x+3)(x-1)^2 > 0$ by graphing $f(x) = (x+3)(x-1)^2$.

Solution Graph $f(x) = (x+3)(x-1)^2$ and determine the intervals of x for which the graph is above the x-axis. These values of x result in $f(x)$ being positive. Using Steps 1 through 6 on page 179, we obtain the graph shown in Figure 45.

Figure 45



From the graph, we can see that $f(x) > 0$ for $-3 < x < 1$ or $x > 1$. The solution set is $\{x \mid -3 < x < 1 \text{ or } x > 1\}$ or, using interval notation, $(-3, 1) \cup (1, \infty)$.

Now Work PROBLEM 9

The results of Example 1 lead to the following approach to solving polynomial and rational inequalities algebraically. Suppose that the polynomial or rational inequality is in one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \le 0 \qquad f(x) \ge 0$$

Locate the zeros of f if f is a polynomial function, and locate the zeros of the numerator and the denominator if f is a rational function. If we use these zeros to divide the real number line into intervals, we know that on each interval the graph of f is either above the x-axis $[f(x) > 0]$ or below the x-axis $[f(x) < 0]$. In other words, we have found the solution of the inequality.

EXAMPLE 2

How to Solve a Polynomial Inequality Algebraically

Solve the inequality $x^4 > x$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a polynomial expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$x^4 > x$$
$$x^4 - x > 0 \quad \text{Subtract } x \text{ from both sides of the inequality.}$$

This inequality is equivalent to the one we wish to solve.

Step 2: Determine the real zeros (x-intercepts of the graph) of f.

Find the real zeros of $f(x) = x^4 - x$ by solving $x^4 - x = 0$.

$$x^4 - x = 0$$
$$x(x^3 - 1) = 0 \quad \text{Factor out } x.$$
$$x(x-1)(x^2 + x + 1) = 0 \quad \text{Factor the difference of two cubes.}$$
$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0 \quad \text{Set each factor equal to zero and solve.}$$
$$x = 0 \quad \text{or} \quad x = 1$$

The equation $x^2 + x + 1 = 0$ has no real solutions. Do you see why?

Step 3: Use the zeros found in Step 2 to divide the real number line into intervals.

Use the real zeros to separate the real number line into three intervals:

$$(-\infty, 0) \qquad (0, 1) \qquad (1, \infty)$$

Step 4: Select a number in each interval, evaluate f at the number, and determine whether f is positive or negative. If f is positive, all values of f in the interval are positive. If f is negative, all values of f in the interval are negative.

Select a test number in each interval found in Step 3 and evaluate $f(x) = x^4 - x$ at each number to determine if $f(x)$ is positive or negative. See Table 16.

Table 16

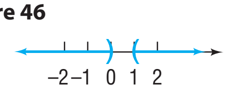
	0		1 x
Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-1	$\frac{1}{2}$	2
Value of f	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive, we conclude that $f(x) > 0$ for all numbers x for which $x < 0$ or $x > 1$. Because the original inequality is strict, numbers x that satisfy the equation $x^4 = x$ are not solutions. The solution set of the inequality $x^4 > x$ is $\{x \mid x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

Figure 46 shows the graph of the solution set.

NOTE If the inequality is not strict ($\le$ or $\ge$), include the solutions of $f(x) = 0$ in the solution set.

Figure 46



Now Work PROBLEM 21

2 Solve Rational Inequalities

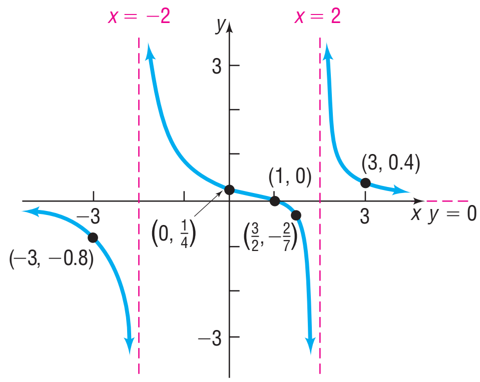
Just as we presented a graphical approach to help us understand the algebraic procedure for solving inequalities involving polynomials, we present a graphical approach to help us understand the algebraic procedure for solving inequalities involving rational expressions.

EXAMPLE 3 Solving a Rational Inequality Using Its Graph

Solve $\frac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \frac{x-1}{x^2-4}$.

Solution Graph $R(x) = \frac{x-1}{x^2-4}$ and determine the intervals of x such that the graph is above or on the x-axis. Do you see why? These values of x result in $R(x)$ being positive or zero. We graphed $R(x) = \frac{x-1}{x^2-4}$ in Example 1, Section 4.3 (pp. 199–201). We reproduce the graph in Figure 47.

Figure 47



From the graph, we can see that $R(x) \geq 0$ for $-2 < x \leq 1$ or $x > 2$. The solution set is $\{x \mid -2 < x \leq 1 \text{ or } x > 2\}$ or, using interval notation, $(-2, 1] \cup (2, \infty)$.

Now Work PROBLEM 33

To solve a rational inequality algebraically, we follow the same approach that we used to solve a polynomial inequality algebraically. However, we must also identify the zeros of the denominator of the rational function, because the sign of a rational function may change on either side of a vertical asymptote. Convince yourself of this by looking at Figure 47. Notice that the function values are negative for $x < -2$ and are positive for $x > -2$ (but less than 1).

EXAMPLE 4 How to Solve a Rational Inequality Algebraically

Solve the inequality $\frac{4x+5}{x+2} \geq 3$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$\frac{4x+5}{x+2} \geq 3$$

$$\frac{4x+5}{x+2} - 3 \geq 0 \quad \text{Subtract 3 from both sides of the inequality.}$$

$$\frac{4x+5}{x+2} - 3 \cdot \frac{x+2}{x+2} \geq 0 \quad \text{Multiply 3 by } \frac{x+2}{x+2}.$$

$$\frac{4x+5-3x-6}{x+2} \geq 0 \quad \text{Write as a single quotient.}$$

$$\frac{x-1}{x+2} \geq 0 \quad \text{Combine like terms.}$$

Step 2: *Determine the real zeros (x-intercepts of the graph) of f and the real numbers for which f is undefined.*

The zero of $f(x) = \dfrac{x-1}{x+2}$ is 1. Also, f is undefined for $x = -2$.

Step 3: *Use the zeros and undefined values found in Step 2 to divide the real number line into intervals.*

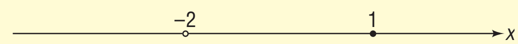
Use the zero and undefined value to separate the real number line into three intervals:

$$(-\infty, -2) \qquad (-2, 1) \qquad (1, \infty)$$

Step 4: *Select a number in each interval, evaluate f at the number, and determine whether f is positive or negative. If f is positive, all values of f in the interval are positive. If f is negative, all values of f in the interval are negative.*

Select a test number in each interval found in Step 3 and evaluate $f(x) = \dfrac{x-1}{x+2}$ at each number to determine if $f(x)$ is positive or negative. See Table 17.

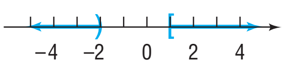
Table 17



Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number chosen	-3	0	2
Value of *f*	$f(-3) = 4$	$f(0) = -\dfrac{1}{2}$	$f(2) = \dfrac{1}{4}$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive or zero, we conclude that $f(x) \geq 0$ for all numbers x for which $x < -2$ or $x \geq 1$. Notice we do not include -2 in the solution because -2 is not in the domain of f. The solution set of the inequality $\dfrac{4x+5}{x+2} \geq 3$ is $\{x \mid x < -2 \text{ or } x \geq 1\}$ or, using interval notation, $(-\infty, -2) \cup [1, \infty)$. Figure 48 shows the graph of the solution set.

Figure 48



Now Work PROBLEM 39

SUMMARY Steps for Solving Polynomial and Rational Inequalities Algebraically

STEP 1: Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient and find the domain of f.

STEP 2: Determine the real numbers at which the expression f equals zero and, if the expression is rational, the real numbers at which the expression f is undefined.

STEP 3: Use the numbers found in Step 2 to separate the real number line into intervals.

STEP 4: Select a number in each interval and evaluate f at the number.

(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.

(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.

If the inequality is not strict ($\geq$ or $\leq$), include the solutions of $f(x) = 0$ that are in the domain of f in the solution set. Be careful to exclude values of x where f is undefined.

4.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve the inequality $3 - 4x > 5$. Graph the solution set. (pp. A75–A78)

2. Solve the inequality $x^2 - 5x \leq 24$. Graph the solution set. (pp. 155–157)

Concepts and Vocabulary

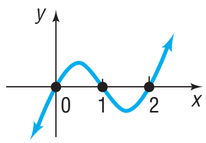
3. ***True or False*** A test number for the interval $-2 < x < 5$ could be 4.

4. ***True or False*** The graph of $f(x) = \dfrac{x}{x-3}$ is above the x-axis for $x < 0$ or $x > 3$, so the solution set of the inequality $\dfrac{x}{x-3} \geq 0$ is $\{x|x \leq 0 \text{ or } x \geq 3\}$.

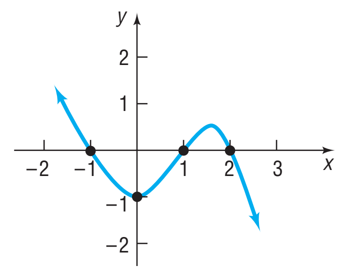
Skill Building

In Problems 5–8, use the graph of the function f to solve the inequality.

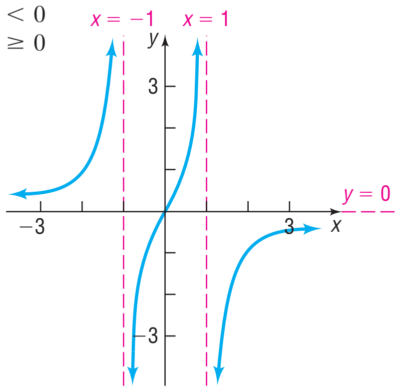
5. (a) $f(x) > 0$
(b) $f(x) \leq 0$



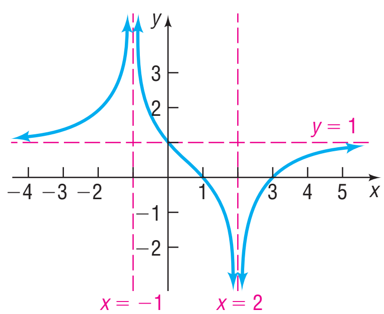
6. (a) $f(x) < 0$
(b) $f(x) \geq 0$



7. (a) $f(x) < 0$
(b) $f(x) \geq 0$



8. (a) $f(x) > 0$
(b) $f(x) \leq 0$



In Problems 9–14, solve the inequality by using the graph of the function.
[**Hint:** The graphs were drawn in Problems 69–74 of Section 4.1.]

9. Solve $f(x) < 0$, where $f(x) = x^2(x-3)$.

10. Solve $f(x) \leq 0$, where $f(x) = x(x+2)^2$.

11. Solve $f(x) \geq 0$, where $f(x) = (x+4)(x-2)^2$.

12. Solve $f(x) > 0$, where $f(x) = (x-1)(x+3)^2$.

13. Solve $f(x) \leq 0$, where $f(x) = -2(x+2)(x-2)^3$.

14. Solve $f(x) < 0$, where $f(x) = -\dfrac{1}{2}(x+4)(x-1)^3$.

In Problems 15–18, solve the inequality by using the graph of the function.
[**Hint:** The graphs were drawn in Problems 7–10 of Section 4.3.]

15. Solve $R(x) > 0$, where $R(x) = \dfrac{x+1}{x(x+4)}$.

16. Solve $R(x) < 0$, where $R(x) = \dfrac{x}{(x-1)(x+2)}$.

17. Solve $R(x) \leq 0$, where $R(x) = \dfrac{3x+3}{2x+4}$.

18. Solve $R(x) \geq 0$, where $R(x) = \dfrac{2x+4}{x-1}$.

In Problems 19–48, solve each inequality algebraically.

19. $(x-5)^2(x+2) < 0$

20. $(x-5)(x+2)^2 > 0$

21. $x^3 - 4x^2 > 0$

22. $x^3 + 8x^2 < 0$

23. $2x^3 > -8x^2$

24. $3x^3 < -15x^2$

25. $(x-1)(x-2)(x-3) \leq 0$

26. $(x+1)(x+2)(x+3) \leq 0$

27. $x^3 - 2x^2 - 3x > 0$

28. $x^3 + 2x^2 - 3x > 0$

29. $x^4 > x^2$

30. $x^4 < 9x^2$

31. $x^4 > 1$

32. $x^3 > 1$

33. $\dfrac{x+1}{x-1} > 0$

34. $\dfrac{x-3}{x+1} > 0$

35. $\dfrac{(x-1)(x+1)}{x} \leq 0$

36. $\dfrac{(x-3)(x+2)}{x-1} \leq 0$

37. $\dfrac{(x-2)^2}{x^2-1} \geq 0$

38. $\dfrac{(x+5)^2}{x^2-4} \geq 0$

39. $\dfrac{x+4}{x-2} \leq 1$

40. $\dfrac{x+2}{x-4} \geq 1$

41. $\dfrac{3x-5}{x+2} \leq 2$

42. $\dfrac{x-4}{2x+4} \geq 1$

43. $\dfrac{1}{x-2} < \dfrac{2}{3x-9}$

44. $\dfrac{5}{x-3} > \dfrac{3}{x+1}$

45. $\dfrac{x^2(3+x)(x+4)}{(x+5)(x-1)} \geq 0$

46. $\dfrac{x(x^2+1)(x-2)}{(x-1)(x+1)} \geq 0$

47. $\dfrac{(3-x)^3(2x+1)}{x^3-1} < 0$

48. $\dfrac{(2-x)^3(3x-2)}{x^3+1} < 0$

Mixed Practice

In Problems 49–60, solve each inequality algebraically.

49. $(x+1)(x-3)(x-5) > 0$

50. $(2x-1)(x+2)(x+5) < 0$

51. $7x - 4 \geq -2x^2$

52. $x^2 + 3x \geq 10$

53. $\dfrac{x+1}{x-3} \leq 2$

54. $\dfrac{x-1}{x+2} \geq -2$

55. $3(x^2-2) < 2(x-1)^2 + x^2$

56. $(x-3)(x+2) < x^2 + 3x + 5$

57. $6x - 5 < \dfrac{6}{x}$

58. $x + \dfrac{12}{x} < 7$

59. $x^3 - 9x \leq 0$

60. $x^3 - x \geq 0$

Applications and Extensions

61. For what positive numbers will the cube of a number exceed four times its square?

62. For what positive numbers will the cube of a number be less than the number?

63. What is the domain of the function $f(x) = \sqrt{x^4 - 16}$?

64. What is the domain of the function $f(x) = \sqrt{x^3 - 3x^2}$?

65. What is the domain of the function $f(x) = \sqrt{\dfrac{x-2}{x+4}}$?

66. What is the domain of the function $f(x) = \sqrt{\dfrac{x-1}{x+4}}$?

In Problems 67–70, determine where the graph of f is below the graph of g by solving the inequality $f(x) \leq g(x)$. Graph f and g together.

67. $f(x) = x^4 - 1$
$g(x) = -2x^2 + 2$

68. $f(x) = x^4 - 1$
$g(x) = x - 1$

69. $f(x) = x^4 - 4$
$g(x) = 3x^2$

70. $f(x) = x^4$
$g(x) = 2 - x^2$

71. Average Cost Suppose that the daily cost C of manufacturing bicycles is given by $C(x) = 80x + 5000$. Then the average daily cost $\overline{C}$ is given by $\overline{C}(x) = \dfrac{80x + 5000}{x}$. How many bicycles must be produced each day for the average cost to be no more than \$100?

72. Average Cost See Problem 71. Suppose that the government imposes a \$1000 per day tax on the bicycle manufacturer so that the daily cost C of manufacturing x bicycles is now given by $C(x) = 80x + 6000$. Now the average daily cost $\overline{C}$ is given by $\overline{C}(x) = \dfrac{80x + 6000}{x}$. How many bicycles must be produced each day for the average cost to be no more than \$100?

73. Bungee Jumping Originating on Pentecost Island in the Pacific, the practice of a person jumping from a high place harnessed to a flexible attachment was introduced to western culture in 1979 by the Oxford University Dangerous Sport Club. One important parameter to know before attempting a bungee jump is the amount the cord will stretch at the bottom of the fall. The stiffness of the cord is related to the amount of stretch by the equation

$$K = \frac{2W(S+L)}{S^2}$$

where W = weight of the jumper (pounds)
K = cord's stiffness (pounds per foot)
L = free length of the cord (feet)
S = stretch (feet)

(a) A 150-pound person plans to jump off a ledge attached to a cord of length 42 feet. If the stiffness of the cord is no less than 16 pounds per foot, how much will the cord stretch?

(b) If safety requirements will not permit the jumper to get any closer than 3 feet to the ground, what is the minimum height required for the ledge in part (a)?

Source: American Institute of Physics, Physics News Update, No. 150, November 5, 1993

74. Gravitational Force According to Newton's Law of universal gravitation, the attractive force F between two bodies is given by

$$F = G\frac{m_1 m_2}{r^2}$$

where m_1, m_2 = the masses of the two bodies
r = distance between the two bodies
G = gravitational constant = 6.6742×10^{-11} newtons meter^2 kilogram^{-2}

Suppose an object is traveling directly from Earth to the moon. The mass of Earth is 5.9742×10^{24} kilograms, the mass of the moon is 7.349×10^{22} kilograms, and the mean distance from Earth to the moon is 384,400 kilometers. For an object between Earth and the moon, how far from Earth is the force on the object due to the moon greater than the force on the object due to Earth?

Source: *www.solarviews.com;en.wikipedia.org*

75. Field Trip Mrs. West has decided to take her fifth grade class to a play. The manager of the theater agreed to discount the regular \$40 price of the ticket by \$0.20 for each ticket sold. The cost of the bus, \$500, will be split equally among each of the students. How many students must attend to keep the cost per student at or below \$40?

Explaining Concepts: Discussion and Writing

76. Make up an inequality that has no solution. Make up one that has exactly one solution.

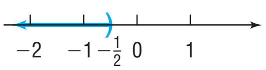
77. The inequality $x^4 + 1 < -5$ has no solution. Explain why.

78. A student attempted to solve the inequality $\frac{x+4}{x-3} \le 0$ by multiplying both sides of the inequality by $x - 3$ to get $x + 4 \le 0$. This led to a solution of $\{x \mid x \le -4\}$. Is the student correct? Explain.

79. Write a rational inequality whose solution set is $\{x \mid -3 < x \le 5\}$.

'Are You Prepared?' Answers

1. $\left\{x \,\middle|\, x < -\frac{1}{2}\right\}$ or $\left(-\infty, -\frac{1}{2}\right)$



2. $\{x \mid -3 \le x \le 8\}$ or $[-3, 8]$

4.5 The Real Zeros of a Polynomial Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Evaluating Functions (Section 2.1, pp. 49–52)
- Factoring Polynomials (Appendix A, Section A.3, pp. A28–A29)
- Synthetic Division (Appendix A, Section A.4, pp. A32–A35)
- Polynomial Division (Appendix A, Section A.3, pp. A25–A28)
- Intercepts of a Quadratic Function (Section 3.3, pp. 138–139)

Now Work the **'Are You Prepared?'** problems on page 230.

OBJECTIVES
1. Use the Remainder and Factor Theorems (p. 221)
2. Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function (p. 224)
3. Find the Real Zeros of a Polynomial Function (p. 224)
4. Solve Polynomial Equations (p. 227)
5. Use the Theorem for Bounds on Zeros (p. 227)
6. Use the Intermediate Value Theorem (p. 228)

In Section 4.1, we were able to identify the real zeros of a polynomial function because either the polynomial function was in factored form or it could be easily factored. But how do we find the real zeros of a polynomial function if it is not factored or cannot be easily factored?

Recall that if r is a real zero of a polynomial function f then $f(r) = 0$, r is an x-intercept of the graph of f, $x - r$ is a factor of f, and r is a solution of the equation $f(x) = 0$. For example, if $x - 4$ is a factor of f, then 4 is a real zero of f and 4 is a solution to the equation $f(x) = 0$. For polynomial and rational functions, we have seen the importance of the real zeros for graphing. In most cases, however, the real zeros of a polynomial function are difficult to find using algebraic methods. No nice formulas like the quadratic formula are available to help us find zeros for polynomials of degree 3 or higher. Formulas do exist for solving any third- or fourth-degree polynomial equation, but they are somewhat complicated. No general formulas exist for polynomial equations of degree 5 or higher. Refer to the Historical Feature at the end of this section for more information.

1 Use the Remainder and Factor Theorems

When we divide one polynomial (the dividend) by another (the divisor), we obtain a quotient polynomial and a remainder, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check our work, we verify that

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

This checking routine is the basis for a famous theorem called the **division algorithm* for polynomials,** which we now state without proof.

THEOREM

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\boxed{\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\text{dividend}}{\underset{\uparrow}{f(x)}} = \underset{\text{quotient}}{\underset{\uparrow}{q(x)}}\,\underset{\text{divisor}}{\underset{\uparrow}{g(x)}} + \underset{\text{remainder}}{\underset{\uparrow}{r(x)}} \qquad \textbf{(1)}}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

In equation (1), $f(x)$ is the **dividend,** $g(x)$ is the **divisor,** $q(x)$ is the **quotient,** and $r(x)$ is the **remainder.**

If the divisor $g(x)$ is a first-degree polynomial of the form

$$g(x) = x - c \qquad c \text{ a real number}$$

then the remainder $r(x)$ is either the zero polynomial or a polynomial of degree 0. As a result, for such divisors, the remainder is some number, say R, and

$$f(x) = (x - c)q(x) + R \qquad \textbf{(2)}$$

This equation is an identity in x and is true for all real numbers x. Suppose that $x = c$. Then equation (2) becomes

$$f(c) = (c - c)q(c) + R$$
$$f(c) = R$$

*A systematic process in which certain steps are repeated a finite number of times is called an **algorithm.** For example, long division is an algorithm.

Substitute $f(c)$ for R in equation (2) to obtain

$$f(x) = (x - c)q(x) + f(c) \tag{3}$$

We have now proved the **Remainder Theorem.**

REMAINDER THEOREM

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE 1

Using the Remainder Theorem

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ (b) $x + 2$

Solution

(a) We could use long division or synthetic division, but it is easier to use the Remainder Theorem, which says that the remainder is $f(3)$.

$$f(3) = (3)^3 - 4(3)^2 - 5 = 27 - 36 - 5 = -14$$

The remainder is -14.

(b) To find the remainder when $f(x)$ is divided by $x + 2 = x - (-2)$, evaluate $f(-2)$.

$$f(-2) = (-2)^3 - 4(-2)^2 - 5 = -8 - 16 - 5 = -29$$

The remainder is -29.

Compare the method used in Example 1(a) with the method used in Example 1 of Appendix A, Section A.4. Which method do you prefer? Give reasons.

COMMENT A graphing utility provides another way to find the value of a function using the eVALUEate feature. Consult your manual for details. Then check the results of Example 1.

An important and useful consequence of the Remainder Theorem is the **Factor Theorem.**

FACTOR THEOREM

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

The Factor Theorem actually consists of two separate statements:

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

The proof requires two parts.

Proof

1. Suppose that $f(c) = 0$. Then, by equation (3), we have

$$f(x) = (x - c)q(x)$$

for some polynomial $q(x)$. That is, $x - c$ is a factor of $f(x)$.

2. Suppose that $x - c$ is a factor of $f(x)$. Then there is a polynomial function q such that

$$f(x) = (x - c)q(x)$$

Replacing x by c, we find that

$$f(c) = (c - c)q(c) = 0 \cdot q(c) = 0$$

This completes the proof. ■

EXAMPLE 2

Using the Factor Theorem

Use the Factor Theorem to determine whether the function

$$f(x) = 2x^3 - x^2 + 2x - 3$$

has the factor

(a) $x - 1$ (b) $x + 3$

Solution The Factor Theorem states that if $f(c) = 0$ then $x - c$ is a factor.

(a) Because $x - 1$ is of the form $x - c$ with $c = 1$, we find the value of $f(1)$. We choose to use substitution.

$$f(1) = 2(1)^3 - (1)^2 + 2(1) - 3 = 2 - 1 + 2 - 3 = 0$$

By the Factor Theorem, $x - 1$ is a factor of $f(x)$.

(b) To test the factor $x + 3$, we first need to write it in the form $x - c$. Since $x + 3 = x - (-3)$, we find the value of $f(-3)$. We choose to use synthetic division.

$$\begin{array}{r|rrrr} -3 & 2 & -1 & 2 & -3 \\ & & -6 & 21 & -69 \\ \hline & 2 & -7 & 23 & -72 \end{array}$$

Because $f(-3) = -72 \neq 0$, we conclude from the Factor Theorem that $x - (-3) = x + 3$ is not a factor of $f(x)$.

Now Work PROBLEM 11

In Example 2(a), we found that $x - 1$ is a factor of f. To write f in factored form, use long division or synthetic division. Using synthetic division,

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -3 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & 0 \end{array}$$

The quotient is $q(x) = 2x^2 + x + 3$ with a remainder of 0, as expected. We can write f in factored form as

$$f(x) = 2x^3 - x^2 + 2x - 3 = (x - 1)(2x^2 + x + 3)$$

The next theorem concerns the number of real zeros that a polynomial function may have. In counting the zeros of a polynomial, we count each zero as many times as its multiplicity.

THEOREM

Number of Real Zeros

A polynomial function cannot have more real zeros than its degree.

Proof The proof is based on the Factor Theorem. If r is a real zero of a polynomial function f, then $f(r) = 0$ and, hence, $x - r$ is a factor of $f(x)$. Each real zero corresponds to a factor of degree 1. Because f cannot have more first-degree factors than its degree, the result follows. ■

2 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

The next result, called the **Rational Zeros Theorem,** provides information about the rational zeros of a polynomial *with integer coefficients*.

THEOREM

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0, \quad a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

EXAMPLE 3 **Listing Potential Rational Zeros**

List the potential rational zeros of

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Solution Because f has integer coefficients, we may use the Rational Zeros Theorem. First, list all the integers p that are factors of the constant term $a_0 = -6$ and all the integers q that are factors of the leading coefficient $a_3 = 2$.

$$\begin{array}{lll} p: & \pm1, \pm2, \pm3, \pm6 & \text{Factors of } -6 \\ q: & \pm1, \pm2 & \text{Factors of } 2 \end{array}$$

Now form all possible ratios $\frac{p}{q}$.

$$\frac{p}{q}: \quad \pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

If f has a rational zero, it will be found in this list, which contains 12 possibilities.

COMMENT For the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$, we know 5 is not a zero, because 5 is not in the list of potential rational zeros. However, -1 may or may not be a zero.

Now Work PROBLEM 33

Be sure that you understand what the Rational Zeros Theorem says: For a polynomial with integer coefficients, *if* there is a rational zero, it is one of those listed. It may be the case that the function does not have any rational zeros.

Long division, synthetic division, or substitution can be used to test each potential rational zero to determine whether it is indeed a zero. To make the work easier, integers are usually tested first.

3 Find the Real Zeros of a Polynomial Function

EXAMPLE 4 **How to Find the Real Zeros of a Polynomial Function**

Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$. Write f in factored form.

Step-by-Step Solution

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Since f is a polynomial of degree 3, there are at most three real zeros.

Step 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros. Use the Factor Theorem to determine if each potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 2 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

List the potential rational zeros obtained in Example 3:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

From our list of potential rational zeros, we will test 6 to determine if it is a zero of f. Because $f(6) = 780 \neq 0$, we know that 6 is not a zero of f. Now, let's test if -6 is a zero. Because $f(-6) = 0$, we know that -6 is a zero and $x - (-6) = x + 6$ is a factor of f. Use long division or synthetic division to factor f. (We will not show the division here, but you are encouraged to verify the results shown.) After dividing f by $x + 6$, the quotient is $2x^2 - x - 1$, so

$$\begin{aligned} f(x) &= 2x^3 + 11x^2 - 7x - 6 \\ &= (x + 6)(2x^2 - x - 1) \end{aligned}$$

Now any solution of the equation $2x^2 - x - 1 = 0$ will be a zero of f. We call the equation $2x^2 - x - 1 = 0$ a **depressed equation** of f. Because any solution to the equation $2x^2 - x - 1 = 0$ is a zero of f, we work with the depressed equation to find the remaining zeros of f.

The depressed equation $2x^2 - x - 1 = 0$ is a quadratic equation with discriminant $b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$. The equation has two real solutions, which can be found by factoring.

$$\begin{aligned} 2x^2 - x - 1 = (2x + 1)(x - 1) &= 0 \\ 2x + 1 = 0 \quad \text{or} \quad x - 1 &= 0 \\ x = -\frac{1}{2} \quad \text{or} \quad x &= 1 \end{aligned}$$

The zeros of f are $-6, -\frac{1}{2}$, and 1.

We completely factor f as follows:

$$\begin{aligned} f(x) = 2x^3 + 11x^2 - 7x - 6 &= (x + 6)(2x^2 - x - 1) \\ &= (x + 6)(2x + 1)(x - 1) \end{aligned}$$

Notice that the three zeros of f are in the list of potential rational zeros.

SUMMARY Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 2 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE 5 Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$. Write f in factored form.

Solution

STEP 1: There are at most five real zeros.

STEP 2: Because the leading coefficient is $a_5 = 1$, the potential rational zeros are the integers $\pm 1, \pm 2, \pm 4, \pm 8$, and ± 16, the factors of the constant term, 16.

We test the potential rational zero 1 first, using synthetic division.

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & & 1 & -4 & 8 & -16 & 16 \\ \hline & 1 & -4 & 8 & -16 & 16 & 0 \end{array}$$

The remainder is $f(1) = 0$, so 1 is a zero and $x - 1$ is a factor of f. Using the entries in the bottom row of the synthetic division, we can begin to factor f.

$$\begin{aligned} f(x) &= x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 \\ &= (x - 1)(x^4 - 4x^3 + 8x^2 - 16x + 16) \end{aligned}$$

We now work with the first depressed equation:

$$q_1(x) = x^4 - 4x^3 + 8x^2 - 16x + 16 = 0$$

Repeat Step 2: The potential rational zeros of q_1 are still $\pm 1, \pm 2, \pm 4, \pm 8$, and ± 16. We test 1 first, since it may be a repeated zero of f.

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 8 & -16 & 16 \\ & & 1 & -3 & 5 & -11 \\ \hline & 1 & -3 & 5 & -11 & 5 \end{array}$$

Since the remainder is 5, 1 is not a repeated zero. Try 2 next.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

The remainder is $f(2) = 0$, so 2 is a zero and $x - 2$ is a factor of f. Again using the bottom row, we find

$$\begin{aligned} f(x) &= x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 \\ &= (x - 1)(x - 2)(x^3 - 2x^2 + 4x - 8) \end{aligned}$$

The remaining zeros satisfy the new depressed equation

$$q_2(x) = x^3 - 2x^2 + 4x - 8 = 0$$

Notice that $q_2(x)$ can be factored using grouping. (Alternatively, you could repeat Step 2 and check the potential rational zero 2.) Then

$$\begin{aligned} x^3 - 2x^2 + 4x - 8 &= 0 \\ x^2(x - 2) + 4(x - 2) &= 0 \\ (x^2 + 4)(x - 2) &= 0 \\ x^2 + 4 = 0 \quad \text{or} \quad x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Since $x^2 + 4 = 0$ has no real solutions, the real zeros of f are 1 and 2, with 2 being a zero of multiplicity 2. The factored form of f is

$$\begin{aligned} f(x) &= x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 \\ &= (x - 1)(x - 2)^2(x^2 + 4) \end{aligned}$$

Now Work PROBLEM 45

4 Solve Polynomial Equations

EXAMPLE 6

Solving a Polynomial Equation

Find the real solutions of the equation: $x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 = 0$

Solution The real solutions of this equation are the real zeros of the polynomial function

$$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

Using the result of Example 5, the real zeros of f are 1 and 2. So, $\{1, 2\}$ is the solution set of the equation

$$x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 = 0$$

Now Work PROBLEM 57

In Example 5, the quadratic factor $x^2 + 4$ that appears in the factored form of f is called *irreducible*, because the polynomial $x^2 + 4$ cannot be factored over the real numbers. In general, a quadratic factor $ax^2 + bx + c$ is **irreducible** if it cannot be factored over the real numbers, that is, if it is prime over the real numbers.

Refer to Examples 4 and 5. The polynomial function of Example 4 has three real zeros, and its factored form contains three linear factors. The polynomial function of Example 5 has two distinct real zeros, and its factored form contains two distinct linear factors and one irreducible quadratic factor.

THEOREM

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible (prime) quadratic factors.

We prove this result in Section 4.6, and, in fact, shall draw several additional conclusions about the zeros of a polynomial function. One conclusion is worth noting now. If a polynomial with real coefficients is of odd degree, it must contain at least one linear factor. (Do you see why? Consider the end behavior of polynomial functions of odd degree.) This means that it must have at least one real zero.

THEOREM

A polynomial function of odd degree that has real coefficients has at least one real zero.

5 Use the Theorem for Bounds on Zeros

The search for the real zeros of a polynomial function can be reduced somewhat if *bounds* on the zeros are found. A number M is a **bound** on the zeros of a polynomial if every zero lies between $-M$ and M, inclusive. That is, M is a bound on the zeros of a polynomial f if

$$-M \leq \text{any real zero of } f \leq M$$

THEOREM

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

A bound M on the real zeros of f is the smaller of the two numbers

$$\boxed{\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, 1 + \text{Max}\{|a_0|, |a_1|, \cdots, |a_{n-1}|\}} \quad \textbf{(4)}$$

where Max { } means "choose the largest entry in { }."

EXAMPLE 7 **Using the Theorem for Finding Bounds on Zeros**

Find a bound on the real zeros of each polynomial function.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

Solution (a) The leading coefficient of f is 1.

$$f(x) = x^5 + 3x^3 - 9x^2 + 5 \quad a_4 = 0, a_3 = 3, a_2 = -9, a_1 = 0, a_0 = 5$$

Evaluate the two expressions in (4).

$$\begin{aligned}\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\{1, |5| + |0| + |-9| + |3| + |0|\}\\ &= \text{Max}\{1, 17\} = 17\\ 1 + \text{Max}\{|a_0|, |a_1|, \cdots, |a_{n-1}|\} &= 1 + \text{Max}\{|5|, |0|, |-9|, |3|, |0|\}\\ &= 1 + 9 = 10\end{aligned}$$

The smaller of the two numbers, 10, is the bound. Every real zero of f lies between -10 and 10.

(b) First write g so that it is the product of a constant times a polynomial whose leading coefficient is 1 by factoring out the leading coefficient of g, 4.

$$g(x) = 4x^5 - 2x^3 + 2x^2 + 1 = 4\left(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4}\right)$$

Next evaluate the two expressions in (4) with $a_4 = 0$, $a_3 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$, $a_1 = 0$, and $a_0 = \frac{1}{4}$.

$$\begin{aligned}\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\left\{1, \left|\frac{1}{4}\right| + |0| + \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + |0|\right\}\\ &= \text{Max}\left\{1, \frac{5}{4}\right\} = \frac{5}{4}\\ 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} &= 1 + \text{Max}\left\{\left|\frac{1}{4}\right|, |0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, |0|\right\}\\ &= 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

COMMENT The bounds on the zeros of a polynomial provide good choices for setting Xmin and Xmax of the viewing rectangle. With these choices, all the x-intercepts of the graph can be seen. ■

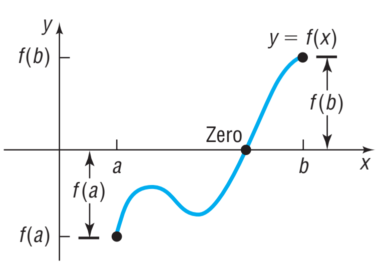
The smaller of the two numbers, $\frac{5}{4}$, is the bound. Every real zero of g lies between $-\frac{5}{4}$ and $\frac{5}{4}$.

Now Work PROBLEM 69

6 Use the Intermediate Value Theorem

The next result, called the **Intermediate Value Theorem,** is based on the fact that the graph of a polynomial function is continuous; that is, it contains no "holes" or "gaps." Although the proof of this result requires advanced methods in calculus, it is easy to "see" why the result is true. Look at Figure 49.

Figure 49 If $f(a) < 0$ and $f(b) > 0$, there is a zero between a and b.



THEOREM **Intermediate Value Theorem**

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.

EXAMPLE 8 Using the Intermediate Value Theorem to Locate a Real Zero

Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2.

Solution Evaluate f at 1 and at 2.

$$f(1) = -1 \quad \text{and} \quad f(2) = 23$$

Because $f(1) < 0$ and $f(2) > 0$, it follows from the Intermediate Value Theorem that the polynomial function f has at least one zero between 1 and 2.

Now Work PROBLEM 77

Let's look at the polynomial f of Example 8 more closely. Based on the Rational Zeros Theorem, ± 1 are the only potential rational zeros. Since $f(1) \neq 0$, we conclude that the zero between 1 and 2 is irrational. We can use the Intermediate Value Theorem to approximate it.

Approximating the Real Zeros of a Polynomial Function

STEP 1: Find two consecutive integers a and $a + 1$ such that f has a zero between them.
STEP 2: Divide the interval $[a, a + 1]$ into 10 equal subintervals.
STEP 3: Evaluate f at each endpoint of the subintervals until the Intermediate Value Theorem applies; this subinterval then contains a zero.
STEP 4: Repeat the process starting at Step 2 until the desired accuracy is achieved.

EXAMPLE 9 Approximating a Real Zero of a Polynomial Function

$f(x) = x^5 - x^3 - 1$ has exactly one zero between 1 and 2. Approximate it correct to two decimal places.

Solution Divide the interval $[1, 2]$ into 10 equal subintervals: $[1, 1.1]$, $[1.1, 1.2]$, $[1.2, 1.3]$, $[1.3, 1.4]$, $[1.4, 1.5]$, $[1.5, 1.6]$, $[1.6, 1.7]$, $[1.7, 1.8]$, $[1.8, 1.9]$, $[1.9, 2]$. Now find the value of f at each endpoint until the Intermediate Value Theorem applies.

$$f(x) = x^5 - x^3 - 1$$
$$f(1.0) = -1 \qquad f(1.2) = -0.23968$$
$$f(1.1) = -0.72049 \qquad f(1.3) = 0.51593$$

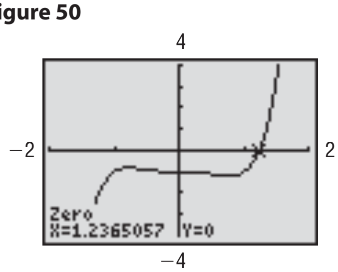
We can stop here and conclude that the zero is between 1.2 and 1.3. Now divide the interval $[1.2, 1.3]$ into 10 equal subintervals and proceed to evaluate f at each endpoint.

$$f(1.20) = -0.23968 \qquad f(1.23) \approx -0.0455613$$
$$f(1.21) \approx -0.1778185 \qquad f(1.24) \approx 0.025001$$
$$f(1.22) \approx -0.1131398$$

We conclude that the zero lies between 1.23 and 1.24, and so, correct to two decimal places, the zero is 1.23.

COMMENT The TABLE feature of a graphing calculator makes the computations in the solution to Example 9 a lot easier.

Figure 50



Exploration

We examine the polynomial f given in Example 9. The Theorem on Bounds of Zeros tells us that every zero is between -2 and 2. If we graph f using $-2 \le x \le 2$ (see Figure 50), we see that f has exactly one x-intercept. Using ZERO or ROOT, we find this zero to be 1.24 rounded to two decimal places. Correct to two decimal places, the zero is 1.23.

Now Work PROBLEM 89

There are many other numerical techniques for approximating the zeros of a polynomial. The one outlined in Example 9 (a variation of the *bisection method*) has the advantages that it will always work, it can be programmed rather easily on a computer, and each time it is used another decimal place of accuracy is achieved. See Problem 115 for the bisection method, which places the zero in a succession of intervals, with each new interval being half the length of the preceding one.

Historical Feature

Formulas for the solution of third- and fourth-degree polynomial equations exist, and, while not very practical, they do have an interesting history.

In the 1500s in Italy, mathematical contests were a popular pastime, and persons possessing methods for solving problems kept them secret. (Solutions that were published were already common knowledge.) Niccolo of Brescia (1499–1557), commonly referred to as Tartaglia ("the stammerer"), had the secret for solving cubic (third-degree) equations, which gave him a decided advantage in the contests. Girolamo Cardano (1501–1576) found out that Tartaglia had the secret, and, being interested in cubics, he requested it from Tartaglia. The reluctant Tartaglia hesitated for some time, but finally, swearing Cardano to secrecy with midnight oaths by candlelight, told him the secret. Cardano then published the solution in his book *Ars Magna* (1545), giving Tartaglia the credit but rather compromising the secrecy. Tartaglia exploded into bitter recriminations, and each wrote pamphlets that reflected on the other's mathematics, moral character, and ancestry.

The quartic (fourth-degree) equation was solved by Cardano's student Lodovico Ferrari, and this solution also was included, with credit and this time with permission, in the *Ars Magna*.

Attempts were made to solve the fifth-degree equation in similar ways, all of which failed. In the early 1800s, P. Ruffini, Niels Abel, and Evariste Galois all found ways to show that it is not possible to solve fifth-degree equations by formula, but the proofs required the introduction of new methods. Galois's methods eventually developed into a large part of modern algebra.

Historical Problems

Problems 1–8 develop the Tartaglia–Cardano solution of the cubic equation and show why it is not altogether practical.

1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - \frac{b}{3}$.
2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.
3. Based on Problem 2, we have the two equations
$$3HK = -p \quad \text{and} \quad H^3 + K^3 = -q$$
Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that
$$H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
[**Hint:** Look for an equation that is quadratic in form.]
4. Use the solution for H from Problem 3 and the equation $H^3 + K^3 = -q$ to show that
$$K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
5. Use the results from Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is
$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
6. Use the result of Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.
7. Use a calculator and the result of Problem 5 to solve the equation $x^3 + 3x - 14 = 0$.
8. Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

4.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find $f(-1)$ if $f(x) = 2x^2 - x$. (pp. 49–52)
2. Factor the expression $6x^2 + x - 2$. (pp. A28–A29)
3. Find the quotient and remainder if $3x^4 - 5x^3 + 7x - 4$ is divided by $x - 3$. (pp. A25–A28 or A32–A35)
4. Find the intercepts of $f(x) = x^2 + x - 3$. (pp. 138–139)

Concepts and Vocabulary

5. In the process of polynomial division, (Divisor)(Quotient) + ___________ = ___________.

6. When a polynomial function f is divided by $x - c$, the remainder is ___________.

7. If a function f, whose domain is all real numbers, is even and if 4 is a zero of f, then ___________ is also a zero.

8. ***True or False*** Every polynomial function of degree 3 with real coefficients has exactly three real zeros.

9. If f is a polynomial function and $x - 4$ is a factor of f, then $f(4) =$ ___________.

10. ***True or False*** If f is a polynomial function of degree 4 and if $f(2) = 5$, then

$$\frac{f(x)}{x-2} = p(x) + \frac{5}{x-2}$$

where $p(x)$ is a polynomial of degree 3.

Skill Building

In Problems 11–20, use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - c$. Then use the Factor Theorem to determine whether $x - c$ is a factor of $f(x)$.

11. $f(x) = 4x^3 - 3x^2 - 8x + 4;\ x - 2$
12. $f(x) = -4x^3 + 5x^2 + 8;\ \ x + 3$
13. $f(x) = 3x^4 - 6x^3 - 5x + 10;\ x - 2$
14. $f(x) = 4x^4 - 15x^2 - 4;\ x - 2$
15. $f(x) = 3x^6 + 82x^3 + 27;\ x + 3$
16. $f(x) = 2x^6 - 18x^4 + x^2 - 9;\ x + 3$
17. $f(x) = 4x^6 - 64x^4 + x^2 - 15;\ x + 4$
18. $f(x) = x^6 - 16x^4 + x^2 - 16;\ x + 4$
19. $f(x) = 2x^4 - x^3 + 2x - 1;\ x - \frac{1}{2}$
20. $f(x) = 3x^4 + x^3 - 3x + 1;\ x + \frac{1}{3}$

In Problems 21–32, tell the maximum number of real zeros that each polynomial function may have. Do not attempt to find the zeros.

21. $f(x) = -4x^7 + x^3 - x^2 + 2$
22. $f(x) = 5x^4 + 2x^2 - 6x - 5$
23. $f(x) = 2x^6 - 3x^2 - x + 1$
24. $f(x) = -3x^5 + 4x^4 + 2$
25. $f(x) = 3x^3 - 2x^2 + x + 2$
26. $f(x) = -x^3 - x^2 + x + 1$
27. $f(x) = -x^4 + x^2 - 1$
28. $f(x) = x^4 + 5x^3 - 2$
29. $f(x) = x^5 + x^4 + x^2 + x + 1$
30. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$
31. $f(x) = x^6 - 1$
32. $f(x) = x^6 + 1$

In Problems 33–44, list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

33. $f(x) = 3x^4 - 3x^3 + x^2 - x + 1$
34. $f(x) = x^5 - x^4 + 2x^2 + 3$
35. $f(x) = x^5 - 6x^2 + 9x - 3$
36. $f(x) = 2x^5 - x^4 - x^2 + 1$
37. $f(x) = -4x^3 - x^2 + x + 2$
38. $f(x) = 6x^4 - x^2 + 2$
39. $f(x) = 6x^4 - x^2 + 9$
40. $f(x) = -4x^3 + x^2 + x + 6$
41. $f(x) = 2x^5 - x^3 + 2x^2 + 12$
42. $f(x) = 3x^5 - x^2 + 2x + 18$
43. $f(x) = 6x^4 + 2x^3 - x^2 + 20$
44. $f(x) = -6x^3 - x^2 + x + 10$

In Problems 45–56, use the Rational Zeros Theorem to find all the real zeros of each polynomial function. Use the zeros to factor f over the real numbers.

45. $f(x) = x^3 + 2x^2 - 5x - 6$
46. $f(x) = x^3 + 8x^2 + 11x - 20$
47. $f(x) = 2x^3 - x^2 + 2x - 1$
48. $f(x) = 2x^3 + x^2 + 2x + 1$
49. $f(x) = 2x^3 - 4x^2 - 10x + 20$
50. $f(x) = 3x^3 + 6x^2 - 15x - 30$
51. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
52. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$
53. $f(x) = x^4 + x^3 - 3x^2 - x + 2$
54. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$
55. $f(x) = 4x^4 + 5x^3 + 9x^2 + 10x + 2$
56. $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$

In Problems 57–68, solve each equation in the real number system.

57. $x^4 - x^3 + 2x^2 - 4x - 8 = 0$
58. $2x^3 + 3x^2 + 2x + 3 = 0$
59. $3x^3 + 4x^2 - 7x + 2 = 0$
60. $2x^3 - 3x^2 - 3x - 5 = 0$

61. $3x^3 - x^2 - 15x + 5 = 0$

62. $2x^3 - 11x^2 + 10x + 8 = 0$

63. $x^4 + 4x^3 + 2x^2 - x + 6 = 0$

64. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

65. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

66. $x^3 + \frac{3}{2}x^2 + 3x - 2 = 0$

67. $2x^4 - 19x^3 + 57x^2 - 64x + 20 = 0$

68. $2x^4 + x^3 - 24x^2 + 20x + 16 = 0$

In Problems 69–76, find bounds on the real zeros of each polynomial function.

69. $f(x) = x^4 - 3x^2 - 4$

70. $f(x) = x^4 - 5x^2 - 36$

71. $f(x) = x^4 + x^3 - x - 1$

72. $f(x) = x^4 - x^3 + x - 1$

73. $f(x) = 3x^4 + 3x^3 - x^2 - 12x - 12$

74. $f(x) = 3x^4 - 3x^3 - 5x^2 + 27x - 36$

75. $f(x) = 4x^5 - x^4 + 2x^3 - 2x^2 + x - 1$

76. $f(x) = 4x^5 + x^4 + x^3 + x^2 - 2x - 2$

In Problems 77–82, use the Intermediate Value Theorem to show that each polynomial function has a zero in the given interval.

77. $f(x) = 8x^4 - 2x^2 + 5x - 1; [0, 1]$

78. $f(x) = x^4 + 8x^3 - x^2 + 2; [-1, 0]$

79. $f(x) = 2x^3 + 6x^2 - 8x + 2; [-5, -4]$

80. $f(x) = 3x^3 - 10x + 9; [-3, -2]$

81. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18; [1.4, 1.5]$

82. $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2; [1.7, 1.8]$

In Problems 83–86, each equation has a solution r in the interval indicated. Use the method of Example 9 to approximate this solution correct to two decimal places.

83. $8x^4 - 2x^2 + 5x - 1 = 0; 0 \le r \le 1$

84. $x^4 + 8x^3 - x^2 + 2 = 0; -1 \le r \le 0$

85. $2x^3 + 6x^2 - 8x + 2 = 0; -5 \le r \le -4$

86. $3x^3 - 10x + 9 = 0; -3 \le r \le -2$

In Problems 87–90, each polynomial function has exactly one positive zero. Use the method of Example 9 to approximate the zero correct to two decimal places.

87. $f(x) = x^3 + x^2 + x - 4$

88. $f(x) = 2x^4 + x^2 - 1$

89. $f(x) = 2x^4 - 3x^3 - 4x^2 - 8$

90. $f(x) = 3x^3 - 2x^2 - 20$

Mixed Practice

In Problems 91–102, graph each polynomial function.

91. $f(x) = x^3 + 2x^2 - 5x - 6$

92. $f(x) = x^3 + 8x^2 + 11x - 20$

93. $f(x) = 2x^3 - x^2 + 2x - 1$

94. $f(x) = 2x^3 + x^2 + 2x + 1$

95. $f(x) = x^4 + x^2 - 2$

96. $f(x) = x^4 - 3x^2 - 4$

97. $f(x) = 4x^4 + 7x^2 - 2$

98. $f(x) = 4x^4 + 15x^2 - 4$

99. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

100. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

101. $f(x) = 4x^5 - 8x^4 - x + 2$

102. $f(x) = 4x^5 + 12x^4 - x - 3$

Applications and Extensions

103. Find k such that $f(x) = x^3 - kx^2 + kx + 2$ has the factor $x - 2$.

104. Find k such that $f(x) = x^4 - kx^3 + kx^2 + 1$ has the factor $x + 2$.

105. What is the remainder when $f(x) = 2x^{20} - 8x^{10} + x - 2$ is divided by $x - 1$?

106. What is the remainder when $f(x) = -3x^{17} + x^9 - x^5 + 2x$ is divided by $x + 1$?

107. Use the Factor Theorem to prove that $x - c$ is a factor of $x^n - c^n$ for any positive integer n.

108. Use the Factor Theorem to prove that $x + c$ is a factor of $x^n + c^n$ if $n \ge 1$ is an odd integer.

109. One solution of the equation $x^3 - 8x^2 + 16x - 3 = 0$ is 3. Find the sum of the remaining solutions.

110. One solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$ is -2. Find the sum of the remaining solutions.