Trigonometric Functions

6

Outline

Length of Day Revisited

The length of a day depends upon the day of the year as well as the latitude of the location. Latitude gives the location of a point on Earth north or south of the equator. In Chapter 4, we found a model that describes the relation between the length of day and latitude for a specific day of the year. In the Internet Project at the end of this chapter, we will find a model that describes the relation between the length of day and day of the year for a specific latitude.

—See the Internet-based Chapter Project I—

◁ A Look Back In Chapter 2, we began our discussion of functions. We defined domain and range and independent and dependent variables; we found the value of a function and graphed functions. We continued our study of functions by listing properties that a function might have, like being even or odd, and we created a library of functions, naming key functions and listing their properties, including the graph.

A Look Ahead ▷ In this chapter we define the trigonometric functions, six functions that have wide application. We shall talk about their domain and range, see how to find values, graph them, and develop a list of their properties.

There are two widely accepted approaches to the development of the trigonometric functions: one uses right triangles; the other uses circles, especially the unit circle. In this book, we develop the trigonometric functions using the unit circle. In Chapter 8, we present right triangle trigonometry.

6.1 Angles and Their Measure

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Circumference and Area of a Circle (Appendix A, Section A.2, p. A16)
- Uniform Motion (Appendix A, Section A.8, pp. A65–A67)

Now Work the **'Are You Prepared?'** problems on page 359.

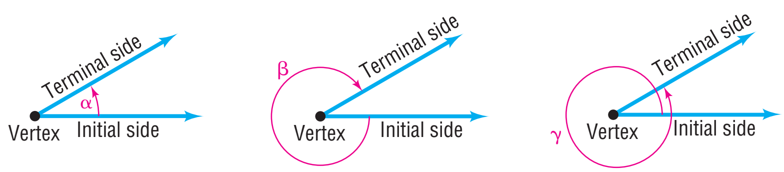
OBJECTIVES
1. Convert between Decimals and Degrees, Minutes, Seconds Measures for Angles (p. 352)
2. Find the Length of an Arc of a Circle (p. 354)
3. Convert from Degrees to Radians and from Radians to Degrees (p. 354)
4. Find the Area of a Sector of a Circle (p. 357)
5. Find the Linear Speed of an Object Traveling in Circular Motion (p. 358)

Figure 1

A **ray,** or **half-line,** is that portion of a line that starts at a point V on the line and extends indefinitely in one direction. The starting point V of a ray is called its **vertex.** See Figure 1.

If two rays are drawn with a common vertex, they form an **angle.** We call one ray of an angle the **initial side** and the other the **terminal side.** The angle formed is identified by showing the direction and amount of rotation from the initial side to the terminal side. If the rotation is in the counterclockwise direction, the angle is **positive;** if the rotation is clockwise, the angle is **negative.** See Figure 2.

Figure 2



Counterclockwise rotation
Positive angle
(a)

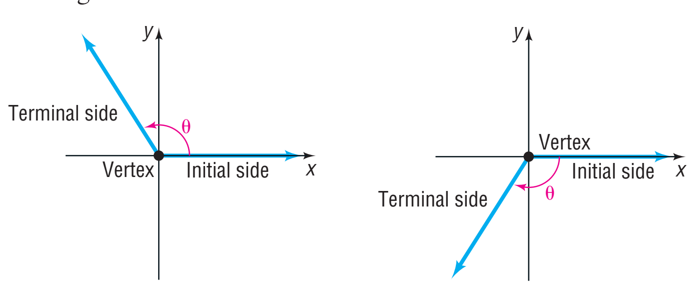
Clockwise rotation
Negative angle
(b)

Counterclockwise rotation
Positive angle
(c)

Lowercase Greek letters, such as α (alpha), β (beta), γ (gamma), and θ (theta), will often be used to denote angles. Notice in Figure 2(a) that the angle α is positive because the direction of the rotation from the initial side to the terminal side is counterclockwise. The angle β in Figure 2(b) is negative because the rotation is clockwise. The angle γ in Figure 2(c) is positive. Notice that the angle α in Figure 2(a) and the angle γ in Figure 2(c) have the same initial side and the same terminal side. However, α and γ are unequal, because the amount of rotation required to go from the initial side to the terminal side is greater for angle γ than for angle α.

An angle θ is said to be in **standard position** if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive x-axis. See Figure 3.

Figure 3

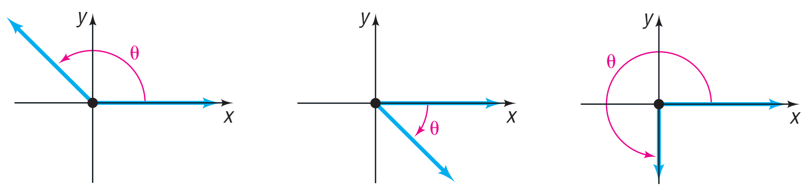


(a) θ is in standard position; θ is positive

(b) θ is in standard position; θ is negative

When an angle θ is in standard position, the terminal side will lie either in a quadrant, in which case we say that θ **lies in that quadrant,** or the terminal side will lie on the x-axis or the y-axis, in which case we say that θ is a **quadrantal angle.** For example, the angle θ in Figure 4(a) lies in quadrant II, the angle θ in Figure 4(b) lies in quadrant IV, and the angle θ in Figure 4(c) is a quadrantal angle.

Figure 4



(a) θ lies in quadrant II **(b)** θ lies in quadrant IV **(c)** θ is a quadrantal angle

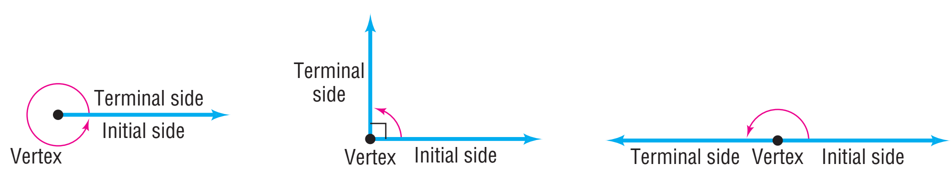
We measure angles by determining the amount of rotation needed for the initial side to become coincident with the terminal side. The two commonly used measures for angles are *degrees* and *radians*.

Degrees

HISTORICAL NOTE One counterclockwise rotation is 360° due to the Babylonian year, which had 360 days. ■

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (1 revolution) is said to measure 360 degrees, abbreviated 360°. **One degree, 1°**, is $\frac{1}{360}$ revolution. A **right angle** is an angle that measures 90°, or $\frac{1}{4}$ revolution; a **straight angle** is an angle that measures 180°, or $\frac{1}{2}$ revolution. See Figure 5. As Figure 5(b) shows, it is customary to indicate a right angle by using the symbol ⊾.

Figure 5



(a) 1 revolution counterclockwise, 360° **(b)** right angle, $\frac{1}{4}$ revolution counterclockwise, 90° **(c)** straight angle, $\frac{1}{2}$ revolution counterclockwise, 180°

It is also customary to refer to an angle that measures θ degrees as an angle *of* θ degrees.

EXAMPLE 1 Drawing an Angle

Draw each angle.

(a) 45° (b) −90° (c) 225° (d) 405°

Solution (a) An angle of 45° is $\frac{1}{2}$ of a right angle. See Figure 6.

(b) An angle of −90° is $\frac{1}{4}$ revolution in the clockwise direction. See Figure 7.

Figure 6

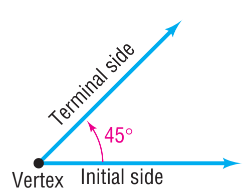
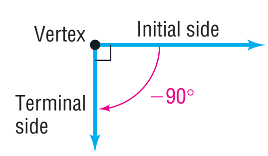


Figure 7



(c) An angle of 225° consists of a rotation through 180° followed by a rotation through 45°. See Figure 8.

(d) An angle of 405° consists of 1 revolution (360°) followed by a rotation through 45°. See Figure 9.

Figure 8

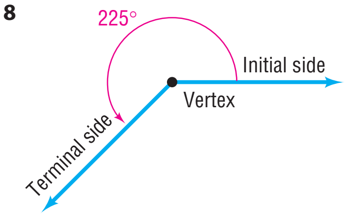
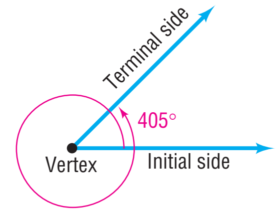


Figure 9



Now Work PROBLEM 11

1 Convert between Decimals and Degrees, Minutes, Seconds Measures for Angles

Although subdivisions of a degree may be obtained by using decimals, we also may use the notion of *minutes* and *seconds*. **One minute,** denoted by **1′**, is defined as $\frac{1}{60}$ degree. **One second,** denoted by **1″**, is defined as $\frac{1}{60}$ minute, or equivalently, $\frac{1}{3600}$ degree. An angle of, say, 30 degrees, 40 minutes, 10 seconds is written compactly as $30°40'10''$. To summarize:

$$\begin{gathered}1 \text{ counterclockwise revolution} = 360° \\ 1° = 60' \qquad 1' = 60''\end{gathered} \tag{1}$$

It is sometimes necessary to convert from the degree, minute, second notation $(D°M'S'')$ to a decimal form, and vice versa. Check your calculator; it should be capable of doing the conversion for you.

Before using your calculator, though, you must set the mode to degrees because there are two common ways to measure angles: degree mode and radian mode. (We will define radians shortly.) Usually, a menu is used to change from one mode to another. Check your owner's manual to find out how your particular calculator works.

To convert from the degree, minute, second notation $(D°M'S'')$ to a decimal form, and vice versa, follow these examples:

$$15°30' = 15.5° \quad \text{because} \quad 30' = 30 \cdot 1' \underset{\substack{\uparrow \\ 1' = \left(\frac{1}{60}\right)^{\circ}}}{=} 30 \cdot \left(\frac{1}{60}\right)^{\circ} = 0.5°$$

$$32.25° = 32°15' \quad \text{because} \quad 0.25° = \left(\frac{1}{4}\right)^{\circ} = \frac{1}{4} \cdot 1° \underset{\substack{\uparrow \\ 1° = 60'}}{=} \frac{1}{4}(60') = 15'$$

EXAMPLE 2 **Converting between Degrees, Minutes, Seconds, and Decimal Forms**

(a) Convert $50°6'21''$ to a decimal in degrees. Round the answer to four decimal places.

(b) Convert 21.256° to the $D°M'S''$ form. Round the answer to the nearest second.

Solution (a) Because $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{60}\cdot\frac{1}{60}\right)^\circ$, we convert as follows:

$$\begin{aligned} 50°6'21'' &= 50° + 6' + 21'' \\ &= 50° + 6\cdot\left(\frac{1}{60}\right)^\circ + 21\cdot\left(\frac{1}{60}\cdot\frac{1}{60}\right)^\circ \\ &\approx 50° + 0.1° + 0.0058° \\ &= 50.1058° \end{aligned}$$

(b) We proceed as follows:

$$\begin{aligned} 21.256° &= 21° + 0.256° \\ &= 21° + (0.256)(60') && \text{Convert fraction of degree to minutes; } 1° = 60'. \\ &= 21° + 15.36' \\ &= 21° + 15' + 0.36' \\ &= 21° + 15' + (0.36)(60'') && \text{Convert fraction of minute to seconds; } 1' = 60''. \\ &= 21° + 15' + 21.6'' \\ &\approx 21°15'22'' && \text{Round to the nearest second.} \end{aligned}$$

Now Work PROBLEMS 23 AND 29

In many applications, such as describing the exact location of a star or the precise position of a ship at sea, angles measured in degrees, minutes, and even seconds are used. For calculation purposes, these are transformed to decimal form. In other applications, especially those in calculus, angles are measured using *radians*.

Radians

A **central angle** is a positive angle whose vertex is at the center of a circle. The rays of a central angle subtend (intersect) an arc on the circle. If the radius of the circle is r and the length of the arc subtended by the central angle is also r, then the measure of the angle is **1 radian.** See Figure 10(a).

For a circle of radius 1, the rays of a central angle with measure 1 radian subtend an arc of length 1. For a circle of radius 3, the rays of a central angle with measure 1 radian subtend an arc of length 3. See Figure 10(b).

Figure 10

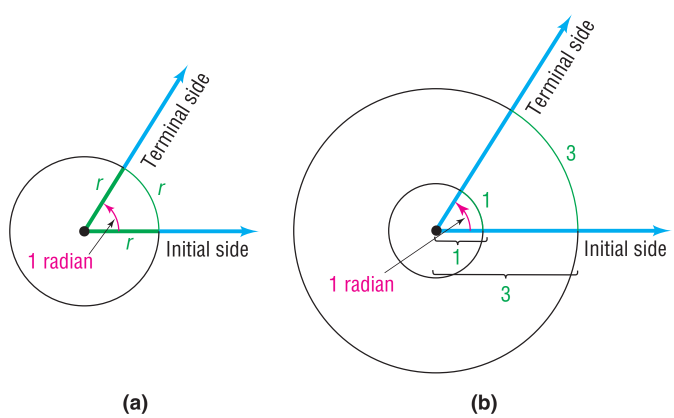
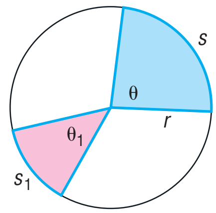


Figure 11
$\frac{\theta}{\theta_1} = \frac{s}{s_1}$



2 Find the Length of an Arc of a Circle

Now consider a circle of radius r and two central angles, θ and θ_1, measured in radians. Suppose that these central angles subtend arcs of lengths s and s_1, respectively, as shown in Figure 11. From geometry, we know that the ratio of the measures of the angles equals the ratio of the corresponding lengths of the arcs subtended by these angles; that is,

$$\frac{\theta}{\theta_1} = \frac{s}{s_1} \tag{2}$$

Suppose that $\theta_1 = 1$ radian. Refer again to Figure 10(a). The length s_1 of the arc subtended by the central angle $\theta_1 = 1$ radian equals the radius r of the circle. Then $s_1 = r$, so equation (2) reduces to

$$\frac{\theta}{1} = \frac{s}{r} \quad \text{or} \quad s = r\theta \tag{3}$$

THEOREM

Arc Length

For a circle of radius r, a central angle of θ radians subtends an arc whose length s is

$$\boxed{s = r\theta} \tag{4}$$

NOTE Formulas must be consistent with regard to the units used. In equation (4), we write

$$s = r\theta$$

To see the units, however, we must go back to equation (3) and write

$$\frac{\theta \text{ radians}}{1 \text{ radian}} = \frac{s \text{ length units}}{r \text{ length units}}$$

$$s \text{ length units} = r \text{ length units} \frac{\theta \text{ radians}}{1 \text{ radian}}$$

Since the radians divide out, we are left with

$$s \text{ length units} = (r \text{ length units})\theta \qquad s = r\theta$$

where θ appears to be "dimensionless" but, in fact, is measured in radians. So, in using the formula $s = r\theta$, the dimension for θ is radians, and any convenient unit of length (such as inches or meters) may be used for s and r. ■

EXAMPLE 3 Finding the Length of an Arc of a Circle

Find the length of the arc of a circle of radius 2 meters subtended by a central angle of 0.25 radian.

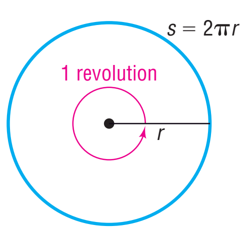
Solution Use equation (4) with $r = 2$ meters and $\theta = 0.25$. The length s of the arc is

$$s = r\theta = 2(0.25) = 0.5 \text{ meter}$$

Now Work PROBLEM 71

3 Convert from Degrees to Radians and from Radians to Degrees

Figure 12
1 revolution $= 2\pi$ radians



With two ways to measure angles, it is important to be able to convert from one to the other. Consider a circle of radius r. A central angle of 1 revolution will subtend an arc equal to the circumference of the circle (Figure 12). Because the circumference of a circle of radius r equals $2\pi r$, we substitute $2\pi r$ for s in equation (4) to find that, for an angle θ of 1 revolution,

$$\begin{aligned} s &= r\theta \\ 2\pi r &= r\theta && \theta = 1 \text{ revolution; } s = 2\pi r \\ \theta &= 2\pi \text{ radians} && \text{Solve for } \theta. \end{aligned}$$

From this, we have

$$1 \text{ revolution} = 2\pi \text{ radians} \qquad \textbf{(5)}$$

Since 1 revolution $= 360°$, we have

$$360° = 2\pi \text{ radians}$$

Dividing both sides by 2 yields

$$180° = \pi \text{ radians} \qquad \textbf{(6)}$$

Divide both sides of equation (6) by 180. Then

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian}$$

Divide both sides of (6) by π. Then

$$\frac{180}{\pi} \text{ degrees} = 1 \text{ radian}$$

We have the following two conversion formulas:*

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian} \qquad 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \qquad \textbf{(7)}$$

EXAMPLE 4 Converting from Degrees to Radians

Convert each angle in degrees to radians.

(a) $60°$ (b) $150°$ (c) $-45°$ (d) $90°$ (e) $107°$

Solution

(a) $60° = 60 \cdot 1 \text{ degree} = 60 \cdot \frac{\pi}{180} \text{ radian} = \frac{\pi}{3} \text{ radians}$

(b) $150° = 150 \cdot 1° = 150 \cdot \frac{\pi}{180} \text{ radian} = \frac{5\pi}{6} \text{ radians}$

(c) $-45° = -45 \cdot \frac{\pi}{180} \text{ radian} = -\frac{\pi}{4} \text{ radian}$

(d) $90° = 90 \cdot \frac{\pi}{180} \text{ radian} = \frac{\pi}{2} \text{ radians}$

(e) $107° = 107 \cdot \frac{\pi}{180} \text{ radian} \approx 1.868 \text{ radians}$

Example 4, parts (a)–(d), illustrates that angles that are "nice" fractions of a revolution are expressed in radian measure as fractional multiples of π, rather than as decimals. For example, a right angle, as in Example 4(d), is left in the form $\frac{\pi}{2}$ radians, which is exact, rather than using the approximation $\frac{\pi}{2} \approx \frac{3.1416}{2} = 1.5708$ radians. When the fractions are not "nice," we use the decimal approximation of the angle, as in Example 4(e).

Now Work PROBLEMS 35 AND 61

* Some students prefer instead to use the proportion $\frac{\text{Degree}}{180°} = \frac{\text{Radian}}{\pi}$. Then substitute for what is given and solve for the measurement sought.

EXAMPLE 5 Converting Radians to Degrees

Convert each angle in radians to degrees.

(a) $\frac{\pi}{6}$ radian (b) $\frac{3\pi}{2}$ radians (c) $-\frac{3\pi}{4}$ radians

(d) $\frac{7\pi}{3}$ radians (e) 3 radians

Solution

(a) $\frac{\pi}{6}\text{ radian} = \frac{\pi}{6}\cdot 1\text{ radian} = \frac{\pi}{6}\cdot\frac{180}{\pi}\text{ degrees} = 30°$

(b) $\frac{3\pi}{2}\text{ radians} = \frac{3\pi}{2}\cdot\frac{180}{\pi}\text{ degrees} = 270°$

(c) $-\frac{3\pi}{4}\text{ radians} = -\frac{3\pi}{4}\cdot\frac{180}{\pi}\text{ degrees} = -135°$

(d) $\frac{7\pi}{3}\text{ radians} = \frac{7\pi}{3}\cdot\frac{180}{\pi}\text{ degrees} = 420°$

(e) $3\text{ radians} = 3\cdot\frac{180}{\pi}\text{ degrees} \approx 171.89°$

Now Work PROBLEM 47

Table 1 lists the degree and radian measures of some commonly encountered angles. You should learn to feel equally comfortable using degree or radian measure for these angles.

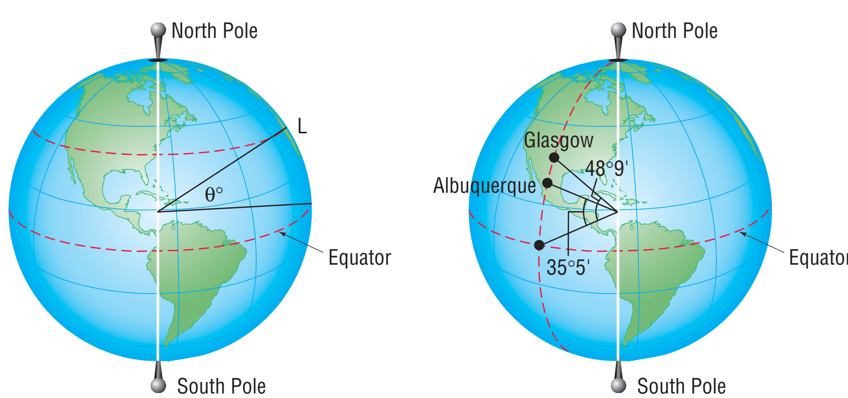
Table 1

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees		210°	225°	240°	270°	300°	315°	330°	360°
Radians		$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

EXAMPLE 6 Finding the Distance between Two Cities

The latitude of a location L is the measure of the angle formed by a ray drawn from the center of Earth to the Equator and a ray drawn from the center of Earth to L. See Figure 13(a). Glasgow, Montana, is due north of Albuquerque, New Mexico. Find the distance between Glasgow (48°9′ north latitude) and Albuquerque (35°5′ north latitude). See Figure 13(b). Assume that the radius of Earth is 3960 miles.

Figure 13



(a) **(b)**

Solution The measure of the central angle between the two cities is $48°9' - 35°5' = 13°4'$. Use equation (4), $s = r\theta$. But remember we must first convert the angle of $13°4'$ to radians.

$$\theta = 13°4' \approx \underset{\uparrow \atop 4' = 4\left(\frac{1}{60}\right)^{\circ}}{13.0667°} = 13.0667 \cdot \frac{\pi}{180} \text{ radian} \approx 0.228 \text{ radian}$$

Use $\theta = 0.228$ radian and $r = 3960$ miles in equation (4). The distance between the two cities is

$$s = r\theta = 3960 \cdot 0.228 \approx 903 \text{ miles}$$

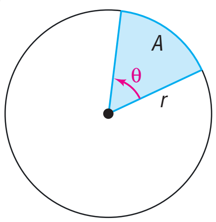
COMMENT If the measure of an angle is given as 5, it is understood to mean 5 radians; if the measure of an angle is given as $5°$, it means 5 degrees.

When an angle is measured in degrees, the degree symbol will always be shown. However, when an angle is measured in radians, we will follow the usual practice and omit the word *radians*. So, if the measure of an angle is given as $\frac{\pi}{6}$, it is understood to mean $\frac{\pi}{6}$ radian.

Now Work PROBLEM 101

4 Find the Area of a Sector of a Circle

Figure 14

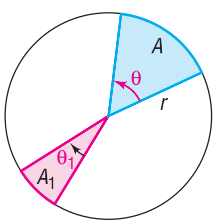


Consider a circle of radius r. Suppose that θ, measured in radians, is a central angle of this circle. See Figure 14. We seek a formula for the area A of the sector (shown in blue) formed by the angle θ.

Now consider a circle of radius r and two central angles θ and θ_1, both measured in radians. See Figure 15. From geometry, we know that the ratio of the measures of the angles equals the ratio of the corresponding areas of the sectors formed by these angles. That is,

Figure 15

$\frac{\theta}{\theta_1} = \frac{A}{A_1}$



$$\frac{\theta}{\theta_1} = \frac{A}{A_1}$$

Suppose that $\theta_1 = 2\pi$ radians. Then $A_1 =$ area of the circle $= \pi r^2$. Solving for A, we find

$$A = A_1 \frac{\theta}{\theta_1} \underset{\uparrow \atop A_1 = \pi r^2 \atop \theta_1 = 2\pi}{=} \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

THEOREM

Area of a Sector

The area A of the sector of a circle of radius r formed by a central angle of θ radians is

$$\boxed{A = \frac{1}{2} r^2 \theta \qquad \textbf{(8)}}$$

EXAMPLE 7 **Finding the Area of a Sector of a Circle**

Find the area of the sector of a circle of radius 2 feet formed by an angle of $30°$. Round the answer to two decimal places.

Solution Use equation (8) with $r = 2$ feet and $\theta = 30° = \frac{\pi}{6}$ radian. [Remember, in equation (8), θ must be in radians.]

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} (2)^2 \frac{\pi}{6} = \frac{\pi}{3} \approx 1.05$$

The area A of the sector is 1.05 square feet, rounded to two decimal places.

Now Work PROBLEM 79

5 Find the Linear Speed of an Object Traveling in Circular Motion

Earlier we defined the average speed of an object as the distance traveled divided by the elapsed time. For motion along a circle, we distinguish between **linear speed** and **angular speed.**

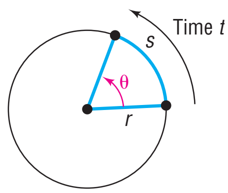
DEFINITION

Suppose that an object moves around a circle of radius r at a constant speed. If s is the distance traveled in time t around this circle, then the **linear speed** v of the object is defined as

$$v = \frac{s}{t} \tag{9}$$

Figure 16

$v = \frac{s}{t}$



As this object travels around the circle, suppose that θ (measured in radians) is the central angle swept out in time t. See Figure 16.

DEFINITION

The **angular speed** ω (the Greek letter omega) of this object is the angle θ (measured in radians) swept out, divided by the elapsed time t; that is,

$$\omega = \frac{\theta}{t} \tag{10}$$

Angular speed is the way the turning rate of an engine is described. For example, an engine idling at 900 rpm (revolutions per minute) is one that rotates at an angular speed of

$$900\,\frac{\text{revolutions}}{\text{minute}} = 900\,\frac{\cancel{\text{revolutions}}}{\text{minute}} \cdot 2\pi\,\frac{\text{radians}}{\cancel{\text{revolution}}} = 1800\pi\,\frac{\text{radians}}{\text{minute}}$$

There is an important relationship between linear speed and angular speed:

$$\text{linear speed} = v \underset{\substack{\uparrow \\ (9)}}{=} \frac{s}{t} \underset{\substack{\uparrow \\ s = r\theta}}{=} \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) \underset{\substack{\uparrow \\ (10)}}{=} r \cdot \omega$$

$$v = r\omega \tag{11}$$

where ω is measured in radians per unit time.

When using equation (11), remember that $v = \frac{s}{t}$ (the linear speed) has the dimensions of length per unit of time (such as feet per second or miles per hour), r (the radius of the circular motion) has the same length dimension as s, and ω (the angular speed) has the dimensions of radians per unit of time. If the angular speed is given in terms of *revolutions* per unit of time (as is often the case), be sure to convert it to *radians* per unit of time using the fact that 1 revolution $= 2\pi$ radians before attempting to use equation (11).

EXAMPLE 8 **Finding Linear Speed**

A child is spinning a rock at the end of a 2-foot rope at the rate of 180 revolutions per minute (rpm). Find the linear speed of the rock when it is released.

Solution Look at Figure 17. The rock is moving around a circle of radius $r = 2$ feet. The angular speed ω of the rock is

$$\omega = 180\,\frac{\text{revolutions}}{\text{minute}} = 180\,\frac{\cancel{\text{revolutions}}}{\text{minute}} \cdot 2\pi\,\frac{\text{radians}}{\cancel{\text{revolution}}} = 360\pi\,\frac{\text{radians}}{\text{minute}}$$

Figure 17

From equation (11), the linear speed v of the rock is

$$v = r\omega = 2 \text{ feet} \cdot 360\pi \frac{\text{radians}}{\text{minute}} = 720\pi \frac{\text{feet}}{\text{minute}} \approx 2262 \frac{\text{feet}}{\text{minute}}$$

The linear speed of the rock when it is released is 2262 ft/min $\approx$ 25.7 mi/hr.

Now Work PROBLEM 97

Historical Feature

Trigonometry was developed by Greek astronomers, who regarded the sky as the inside of a sphere, so it was natural that triangles on a sphere were investigated early (by Menelaus of Alexandria about AD 100) and that triangles in the plane were studied much later. The first book containing a systematic treatment of plane and spherical trigonometry was written by the Persian astronomer Nasir Eddin (about AD 1250).

Regiomontanus (1436–1476) is the person most responsible for moving trigonometry from astronomy into mathematics. His work was improved by Copernicus (1473–1543) and Copernicus's student Rhaeticus (1514–1576). Rhaeticus's book was the first to define the six trigonometric functions as ratios of sides of triangles, although he did not give the functions their present names. Credit for this is due to Thomas Finck (1583), but Finck's notation was by no means universally accepted at the time. The notation was finally stabilized by the textbooks of Leonhard Euler (1707–1783).

Trigonometry has since evolved from its use by surveyors, navigators, and engineers to present applications involving ocean tides, the rise and fall of food supplies in certain ecologies, brain wave patterns, and many other phenomena.

6.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. What is the formula for the circumference C of a circle of radius r? What is the formula for the area A of a circle of radius r? (p. A16)

2. If a particle has a speed of r feet per second and travels a distance d (in feet) in time t (in seconds), then $d =$ _____. (pp. A65–A67)

Concepts and Vocabulary

3. An angle θ is in ___________ __________ if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive x-axis.

4. A _________ _________ is a positive angle whose vertex is at the center of a circle.

5. If the radius of a circle is r and the length of the arc subtended by a central angle is also r, then the measure of the angle is 1 ________.

6. On a circle of radius r, a central angle of θ radians subtends an arc of length $s =$ ____; the area of the sector formed by this angle θ is $A =$ _______.

7. $180° =$ ____ radians

8. An object travels around a circle of radius r with constant speed. If s is the distance traveled in time t around the circle and θ is the central angle (in radians) swept out in time t, then the linear speed of the object is $v =$ ___ and the angular speed of the object is $\omega =$ ___.

9. ***True or False*** The angular speed ω of an object traveling around a circle of radius r is the angle θ (measured in radians) swept out, divided by the elapsed time t.

10. ***True or False*** For circular motion on a circle of radius r, linear speed equals angular speed divided by r.

Skill Building

In Problems 11–22, draw each angle.

11. 30°	**12.** 60°	**13.** 135°	**14.** −120°	**15.** 450°	**16.** 540°
17. $\frac{3\pi}{4}$	**18.** $\frac{4\pi}{3}$	**19.** $-\frac{\pi}{6}$	**20.** $-\frac{2\pi}{3}$	**21.** $\frac{16\pi}{3}$	**22.** $\frac{21\pi}{4}$

In Problems 23–28, convert each angle to a decimal in degrees. Round your answer to two decimal places.

23. 40°10′25″	**24.** 61°42′21″	**25.** 1°2′3″	**26.** 73°40′40″	**27.** 9°9′9″	**28.** 98°22′45″

In Problems 29–34, convert each angle to D°M′S″ form. Round your answer to the nearest second.

29. 40.32° **30.** 61.24° **31.** 18.255° **32.** 29.411° **33.** 19.99° **34.** 44.01°

In Problems 35–46, convert each angle in degrees to radians. Express your answer as a multiple of π.

35. 30° **36.** 120° **37.** 240° **38.** 330° **39.** −60° **40.** −30°

41. 180° **42.** 270° **43.** −135° **44.** −225° **45.** −90° **46.** −180°

In Problems 47–58, convert each angle in radians to degrees.

47. $\frac{\pi}{3}$ **48.** $\frac{5\pi}{6}$ **49.** $-\frac{5\pi}{4}$ **50.** $-\frac{2\pi}{3}$ **51.** $\frac{\pi}{2}$ **52.** 4π

53. $\frac{\pi}{12}$ **54.** $\frac{5\pi}{12}$ **55.** $-\frac{\pi}{2}$ **56.** $-\pi$ **57.** $-\frac{\pi}{6}$ **58.** $-\frac{3\pi}{4}$

In Problems 59–64, convert each angle in degrees to radians. Express your answer in decimal form, rounded to two decimal places.

59. 17° **60.** 73° **61.** −40° **62.** −51° **63.** 125° **64.** 350°

In Problems 65–70, convert each angle in radians to degrees. Express your answer in decimal form, rounded to two decimal places.

65. 3.14 **66.** 0.75 **67.** 2 **68.** 3 **69.** 6.32 **70.** $\sqrt{2}$

In Problems 71–78, s denotes the length of the arc of a circle of radius r subtended by the central angle θ. Find the missing quantity. Round answers to three decimal places.

71. $r = 10$ meters, $\theta = \frac{1}{2}$ radian, $s = ?$

72. $r = 6$ feet, $\theta = 2$ radians, $s = ?$

73. $\theta = \frac{1}{3}$ radian, $s = 2$ feet, $r = ?$

74. $\theta = \frac{1}{4}$ radian, $s = 6$ centimeters, $r = ?$

75. $r = 5$ miles, $s = 3$ miles, $\theta = ?$

76. $r = 6$ meters, $s = 8$ meters, $\theta = ?$

77. $r = 2$ inches, $\theta = 30°$, $s = ?$

78. $r = 3$ meters, $\theta = 120°$, $s = ?$

In Problems 79–86, A denotes the area of the sector of a circle of radius r formed by the central angle θ. Find the missing quantity. Round answers to three decimal places.

79. $r = 10$ meters, $\theta = \frac{1}{2}$ radian, $A = ?$

80. $r = 6$ feet, $\theta = 2$ radians, $A = ?$

81. $\theta = \frac{1}{3}$ radian, $A = 2$ square feet, $r = ?$

82. $\theta = \frac{1}{4}$ radian, $A = 6$ square centimeters, $r = ?$

83. $r = 5$ miles, $A = 3$ square miles, $\theta = ?$

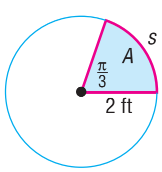
84. $r = 6$ meters, $A = 8$ square meters, $\theta = ?$

85. $r = 2$ inches, $\theta = 30°$, $A = ?$

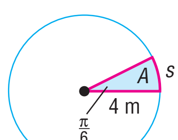
86. $r = 3$ meters, $\theta = 120°$, $A = ?$

In Problems 87–90, find the length s and area A. Round answers to three decimal places.

87.



88.



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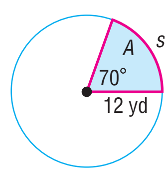
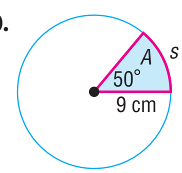
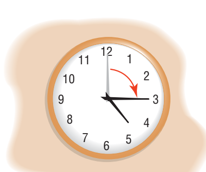


90.



Applications and Extensions

91. Movement of a Minute Hand The minute hand of a clock is 6 inches long. How far does the tip of the minute hand move in 15 minutes? How far does it move in 25 minutes? Round answers to two decimal places.

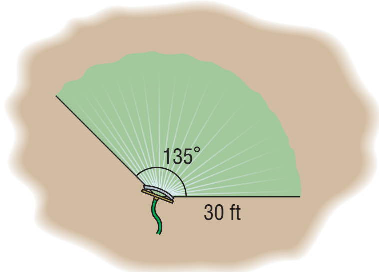


92. Movement of a Pendulum A pendulum swings through an angle of 20° each second. If the pendulum is 40 inches long, how far does its tip move each second? Round answers to two decimal places.

93. Area of a Sector Find the area of the sector of a circle of radius 4 meters formed by an angle of 45°. Round the answer to two decimal places.

94. Area of a Sector Find the area of the sector of a circle of radius 3 centimeters formed by an angle of 60°. Round the answer to two decimal places.

95. Watering a Lawn A water sprinkler sprays water over a distance of 30 feet while rotating through an angle of 135°. What area of lawn receives water?



96. Designing a Water Sprinkler An engineer is asked to design a water sprinkler that will cover a field of 100 square yards that is in the shape of a sector of a circle of radius 15 yards. Through what angle should the sprinkler rotate?

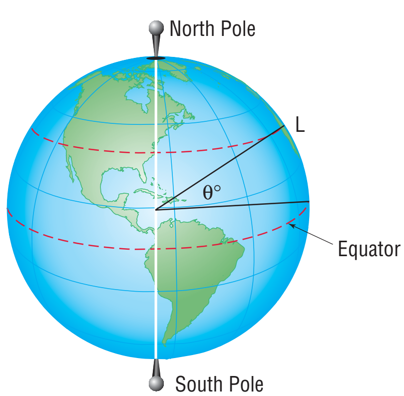
97. Motion on a Circle An object is traveling around a circle with a radius of 5 centimeters. If in 20 seconds a central angle of $\frac{1}{3}$ radian is swept out, what is the angular speed of the object? What is its linear speed?

98. Motion on a Circle An object is traveling around a circle with a radius of 2 meters. If in 20 seconds the object travels 5 meters, what is its angular speed? What is its linear speed?

99. Bicycle Wheels The diameter of each wheel of a bicycle is 26 inches. If you are traveling at a speed of 35 miles per hour on this bicycle, through how many revolutions per minute are the wheels turning?

100. Car Wheels The radius of each wheel of a car is 15 inches. If the wheels are turning at the rate of 3 revolutions per second, how fast is the car moving? Express your answer in inches per second and in miles per hour.

In Problems 101–104, the latitude of a location L is the angle formed by a ray drawn from the center of Earth to the Equator and a ray drawn from the center of Earth to L. See the figure.



101. Distance between Cities Memphis, Tennessee, is due north of New Orleans, Louisiana. Find the distance between Memphis (35°9′ north latitude) and New Orleans (29°57′ north latitude). Assume that the radius of Earth is 3960 miles.

102. Distance between Cities Charleston, West Virginia, is due north of Jacksonville, Florida. Find the distance between Charleston (38°21′ north latitude) and Jacksonville (30°20′ north latitude). Assume that the radius of Earth is 3960 miles.

103. Linear Speed on Earth Earth rotates on an axis through its poles. The distance from the axis to a location on Earth 30° north latitude is about 3429.5 miles. Therefore, a location on Earth at 30° north latitude is spinning on a circle of radius 3429.5 miles. Compute the linear speed on the surface of Earth at 30° north latitude.

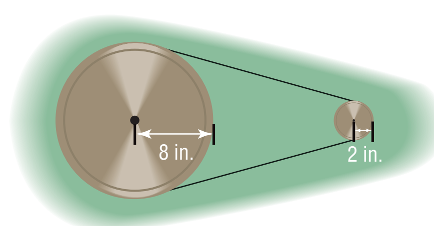
104. Linear Speed on Earth Earth rotates on an axis through its poles. The distance from the axis to a location on Earth 40° north latitude is about 3033.5 miles. Therefore, a location on Earth at 40° north latitude is spinning on a circle of radius 3033.5 miles. Compute the linear speed on the surface of Earth at 40° north latitude.

105. Speed of the Moon The mean distance of the moon from Earth is 2.39×10^5 miles. Assuming that the orbit of the moon around Earth is circular and that 1 revolution takes 27.3 days, find the linear speed of the moon. Express your answer in miles per hour.

106. Speed of Earth The mean distance of Earth from the Sun is 9.29×10^7 miles. Assuming that the orbit of Earth around the Sun is circular and that 1 revolution takes 365 days, find the linear speed of Earth. Express your answer in miles per hour.

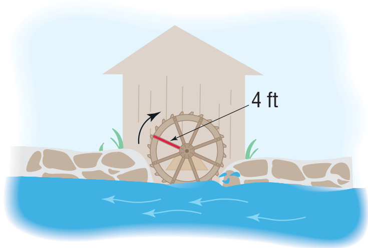
107. Pulleys Two pulleys, one with radius 2 inches and the other with radius 8 inches, are connected by a belt. (See the figure.) If the 2-inch pulley is caused to rotate at 3 revolutions per minute, determine the revolutions per minute of the 8-inch pulley.

[**Hint:** The linear speeds of the pulleys are the same; both equal the speed of the belt.]



108. Ferris Wheels A neighborhood carnival has a Ferris wheel whose radius is 30 feet. You measure the time it takes for one revolution to be 70 seconds. What is the linear speed (in feet per second) of this Ferris wheel? What is the angular speed in radians per second?

109. Computing the Speed of a River Current To approximate the speed of the current of a river, a circular paddle wheel with radius 4 feet is lowered into the water. If the current causes the wheel to rotate at a speed of 10 revolutions per minute, what is the speed of the current? Express your answer in miles per hour.

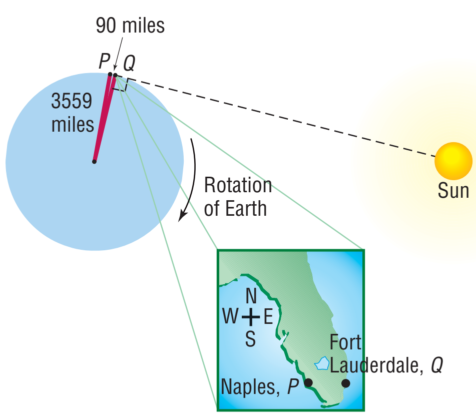


110. Spin Balancing Tires A spin balancer rotates the wheel of a car at 480 revolutions per minute. If the diameter of the wheel is 26 inches, what road speed is being tested? Express your answer in miles per hour. At how many revolutions per minute should the balancer be set to test a road speed of 80 miles per hour?

111. The Cable Cars of San Francisco At the Cable Car Museum you can see the four cable lines that are used to pull cable cars up and down the hills of San Francisco. Each cable travels at a speed of 9.55 miles per hour, caused by a rotating wheel whose diameter is 8.5 feet. How fast is the wheel rotating? Express your answer in revolutions per minute.

112. Difference in Time of Sunrise Naples, Florida, is approximately 90 miles due west of Ft. Lauderdale. How much sooner would a person in Ft. Lauderdale first see the rising Sun than a person in Naples? See the hint.

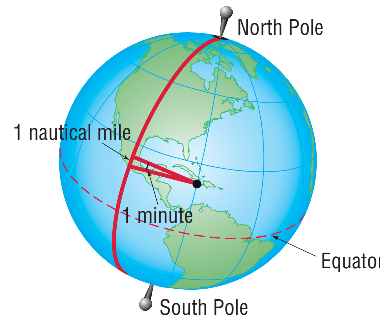
[**Hint:** Consult the figure. When a person at Q sees the first rays of the Sun, a person at P is still in the dark. The person at P sees the first rays after Earth has rotated so that P is at the location Q. Now use the fact that at the latitude of Ft. Lauderdale in 24 hours an arc of length $2\pi(3559)$ miles is subtended.]



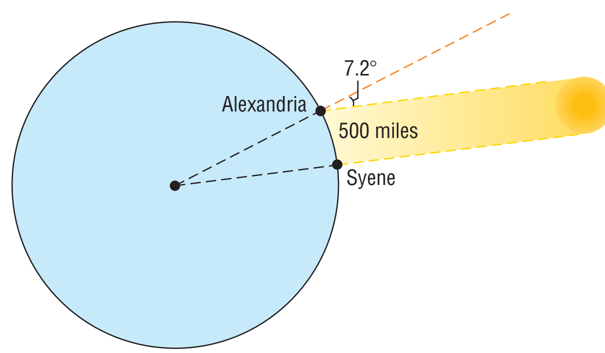
113. Keeping Up with the Sun How fast would you have to travel on the surface of Earth at the equator to keep up with the Sun (that is, so that the Sun would appear to remain in the same position in the sky)?

114. Nautical Miles A **nautical mile** equals the length of arc subtended by a central angle of 1 minute on a great circle† on the surface of Earth. See the figure. If the radius of Earth is taken as 3960 miles, express 1 nautical mile in terms of ordinary, or **statute**, miles.

† Any circle drawn on the surface of Earth that divides Earth into two equal hemispheres.



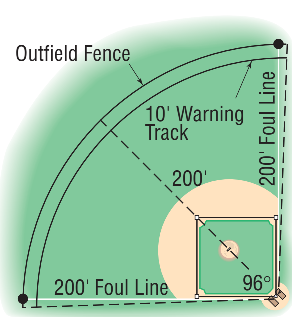
115. Approximating the Circumference of Earth Eratosthenes of Cyrene (276–195 BC) was a Greek scholar who lived and worked in Cyrene and Alexandria. One day while visiting in Syene he noticed that the Sun's rays shone directly down a well. On this date 1 year later, in Alexandria, which is 500 miles due north of Syene he measured the angle of the Sun to be about 7.2 degrees. See the figure. Use this information to approximate the radius and circumference of Earth.



116. Designing a Little League Field For a 60-foot Little League Baseball field, the distance from home base to the nearest fence (or other obstruction) in fair territory should be a minimum of 200 feet. The commissioner of parks and recreation is making plans for a new 60-foot field. Because of limited ground availability, he will use the minimum required distance to the outfield fence. To increase safety, however, he plans to include a 10-foot-wide warning track on the inside of the fence. To further increase safety, the fence and warning track will extend both directions into foul territory. In total, the arc formed by the outfield fence (including the extensions into the foul territories) will be subtended by a central angle at home plate measuring 96°, as illustrated.

(a) Determine the length of the outfield fence.

(b) Determine the area of the warning track.



Source: www.littleleague.org

[**Note:** There is a 90° angle between the two foul lines. Then there are two 3° angles between the foul lines and the dotted lines shown. The angle between the two dotted lines outside the 200-foot foul lines is 96°.]

117. Pulleys Two pulleys, one with radius r_1 and the other with radius r_2, are connected by a belt. The pulley with radius r_1 rotates at ω_1 revolutions per minute, whereas the pulley with radius r_2 rotates at ω_2 revolutions per minute. Show that $\frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}$.

Explaining Concepts: Discussion and Writing

118. Do you prefer to measure angles using degrees or radians? Provide justification and a rationale for your choice.

119. What is 1 radian? What is 1 degree?

120. Which angle has the larger measure: 1 degree or 1 radian? Or are they equal?

121. Explain the difference between linear speed and angular speed.

122. For a circle of radius r, a central angle of θ degrees subtends an arc whose length s is $s = \frac{\pi}{180}r\theta$. Discuss whether this is a true or false statement. Give reasons to defend your position.

123. Discuss why ships and airplanes use nautical miles to measure distance. Explain the difference between a nautical mile and a statute mile.

124. Investigate the way that speed bicycles work. In particular, explain the differences and similarities between 5-speed and 9-speed derailleurs. Be sure to include a discussion of linear speed and angular speed.

125. In Example 6, we found that the distance between Albuquerque, New Mexico, and Glasgow, Montana, is approximately 903 miles. According to *mapquest.com*, the distance is approximately 1300 miles. What might account for the difference?

'Are You Prepared?' Answers

1. $C = 2\pi r$; $A = \pi r^2$ **2.** $r \cdot t$

6.2 Trigonometric Functions: Unit Circle Approach

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Geometry Essentials (Appendix A, Section A.2, pp. A14–A19)
- Unit Circle (Section 1.4, p. 35)
- Symmetry (Section 1.2, pp. 12–14)
- Functions (Section 2.1, pp. 46–56)

Now Work the 'Are You Prepared?' problems on page 375.

OBJECTIVES

1. Find the Exact Values of the Trigonometric Functions Using a Point on the Unit Circle (p. 365)
2. Find the Exact Values of the Trigonometric Functions of Quadrantal Angles (p. 366)
3. Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$ (p. 368)
4. Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$ (p. 369)
5. Find the Exact Values of the Trigonometric Functions for Integer Multiples of $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$ (p. 372)
6. Use a Calculator to Approximate the Value of a Trigonometric Function (p. 373)
7. Use a Circle of Radius r to Evaluate the Trigonometric Functions (p. 374)

We now introduce the trigonometric functions using the unit circle.

The Unit Circle

Recall that the unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system. Also recall that any circle of radius r has circumference of length $2\pi r$. Therefore, the unit circle (radius = 1) has a circumference of length 2π. In other words, for 1 revolution around the unit circle the length of the arc is 2π units.

The following discussion sets the stage for defining the trigonometric functions using the unit circle.

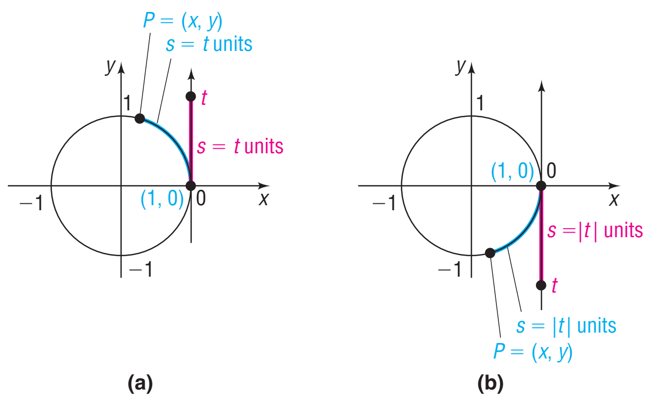
Let t be any real number. We position the t-axis so that it is vertical with the positive direction up. We place this t-axis in the xy-plane so that $t = 0$ is located at the point $(1, 0)$ in the xy-plane.

If $t \geq 0$, let s be the distance from the origin to t on the t-axis. See the red portion of Figure 18(a).

Now look at the unit circle in Figure 18(a). Beginning at the point $(1, 0)$ on the unit circle, travel $s = t$ units in the counterclockwise direction along the circle, to arrive at the point $P = (x, y)$. In this sense, the length $s = t$ units is being **wrapped** around the unit circle.

If $t < 0$, we begin at the point $(1, 0)$ on the unit circle and travel $s = |t|$ units in the clockwise direction to arrive at the point $P = (x, y)$. See Figure 18(b).

Figure 18



If $t > 2\pi$ or if $t < -2\pi$, it will be necessary to travel around the unit circle more than once before arriving at the point P. Do you see why?

Let's describe this process another way. Picture a string of length $s = |t|$ units being wrapped around a circle of radius 1 unit. We start wrapping the string around the circle at the point $(1, 0)$. If $t \geq 0$, we wrap the string in the counterclockwise direction; if $t < 0$, we wrap the string in the clockwise direction. The point $P = (x, y)$ is the point where the string ends.

This discussion tells us that, for any real number t, we can locate a unique point $P = (x, y)$ on the unit circle. We call P **the point on the unit circle that corresponds to *t*.** This is the important idea here. No matter what real number t is chosen, there is a unique point P on the unit circle corresponding to it. We use the coordinates of the point $P = (x, y)$ on the unit circle corresponding to the real number t to define the **six trigonometric functions of *t*.**

DEFINITION

Let t be a real number and let $P = (x, y)$ be the point on the unit circle that corresponds to t.

The **sine function** associates with t the y-coordinate of P and is denoted by

$$\sin t = y$$

The **cosine function** associates with t the x-coordinate of P and is denoted by

$$\cos t = x$$

In Words
The sine function takes as input a real number t that corresponds to a point $P = (x, y)$ on the unit circle and outputs the y-coordinate. The cosine function takes as input a real number t that corresponds to a point $P = (x, y)$ on the unit circle and outputs the x-coordinate.

If $x \neq 0$, the **tangent function** associates with t the ratio of the y-coordinate to the x-coordinate of P and is denoted by

$$\tan t = \frac{y}{x}$$

If $y \neq 0$, the **cosecant function** is defined as

$$\csc t = \frac{1}{y}$$

If $x \neq 0$, the **secant function** is defined as

$$\sec t = \frac{1}{x}$$

If $y \neq 0$, the **cotangent function** is defined as

$$\cot t = \frac{x}{y}$$

Notice in these definitions that if $x = 0$, that is, if the point P is on the y-axis, then the tangent function and the secant function are undefined. Also, if $y = 0$, that is, if the point P is on the x-axis, then the cosecant function and the cotangent function are undefined.

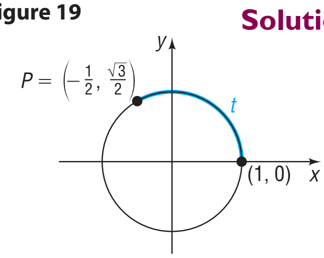
Because we use the unit circle in these definitions of the trigonometric functions, they are sometimes referred to as **circular functions.**

1 Find the Exact Values of the Trigonometric Functions Using a Point on the Unit Circle

EXAMPLE 1 Finding the Values of the Six Trigonometric Functions Using a Point on the Unit Circle

Let t be a real number and let $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ be the point on the unit circle that corresponds to t. Find the values of $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, and $\cot t$.

Figure 19



Solution See Figure 19. We follow the definition of the six trigonometric functions, using $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (x, y)$. Then, with $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$, we have

$$\sin t = y = \frac{\sqrt{3}}{2} \qquad \cos t = x = -\frac{1}{2} \qquad \tan t = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3} \qquad \sec t = \frac{1}{x} = \frac{1}{-\frac{1}{2}} = -2 \qquad \cot t = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3}$$

WARNING When writing the values of the trigonometric functions, do not forget the argument of the function.

$\sin t = \frac{\sqrt{3}}{2}$ correct

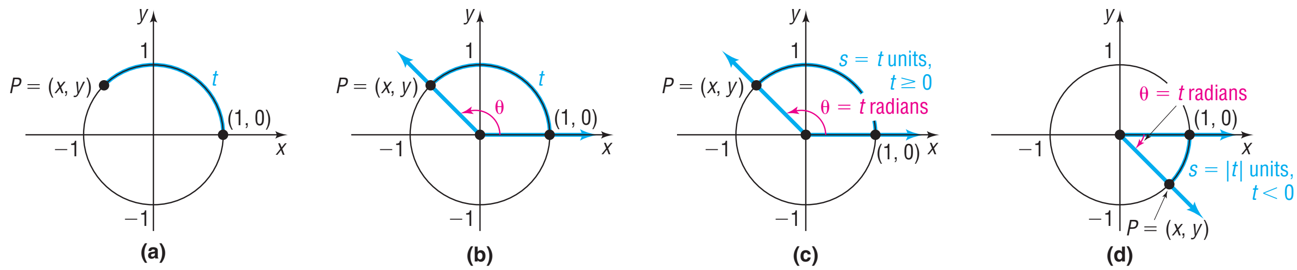
$\sin = \frac{\sqrt{3}}{2}$ incorrect ■

Now Work PROBLEM 13

Trigonometric Functions of Angles

Let $P = (x, y)$ be the point on the unit circle corresponding to the real number t. See Figure 20(a). Let θ be the angle in standard position, measured in radians, whose terminal side is the ray from the origin through P and whose arc length is $|t|$. See Figure 20(b). Since the unit circle has radius 1 unit, if $s = |t|$ units, then from the arc length formula $s = r\theta$, we have $\theta = t$ radians. See Figures 20(c) and (d).

Figure 20



The point $P = (x, y)$ on the unit circle that corresponds to the real number t is also the point P on the terminal side of the angle $\theta = t$ radians. As a result, we can say that

$$\underset{\text{Real number}}{\sin t} = \underset{\theta = t \text{ radians}}{\sin \theta}$$

and so on. We can now define the trigonometric functions of the angle θ.

DEFINITION

If $\theta = t$ radians, the **six trigonometric functions of the angle θ** are defined as

$$\sin\theta = \sin t \qquad \cos\theta = \cos t \qquad \tan\theta = \tan t$$
$$\csc\theta = \csc t \qquad \sec\theta = \sec t \qquad \cot\theta = \cot t$$

Even though the trigonometric functions can be viewed both as functions of real numbers and as functions of angles, it is customary to refer to trigonometric functions of real numbers and trigonometric functions of angles collectively as the *trigonometric functions*. We shall follow this practice from now on.

If an angle θ is measured in degrees, we shall use the degree symbol when writing a trigonometric function of θ, as, for example, in $\sin 30°$ and $\tan 45°$. If an angle θ is measured in radians, then no symbol is used when writing a trigonometric function of θ, as, for example, in $\cos\pi$ and $\sec\frac{\pi}{3}$.

Finally, since the values of the trigonometric functions of an angle θ are determined by the coordinates of the point $P = (x, y)$ on the unit circle corresponding to θ, the units used to measure the angle θ are irrelevant. For example, it does not matter whether we write $\theta = \frac{\pi}{2}$ radians or $\theta = 90°$. The point on the unit circle corresponding to this angle is $P = (0, 1)$. As a result,

$$\sin\frac{\pi}{2} = \sin 90° = 1 \quad \text{and} \quad \cos\frac{\pi}{2} = \cos 90° = 0$$

2 Find the Exact Values of the Trigonometric Functions of Quadrantal Angles

To find the exact value of a trigonometric function of an angle θ or a real number t requires that we locate the point $P = (x, y)$ on the unit circle that corresponds to t. This is not always easy to do. In the examples that follow, we will evaluate the trigonometric functions of certain angles or real numbers for which this process is relatively easy. A calculator will be used to evaluate the trigonometric functions of most other angles.

EXAMPLE 2 Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles

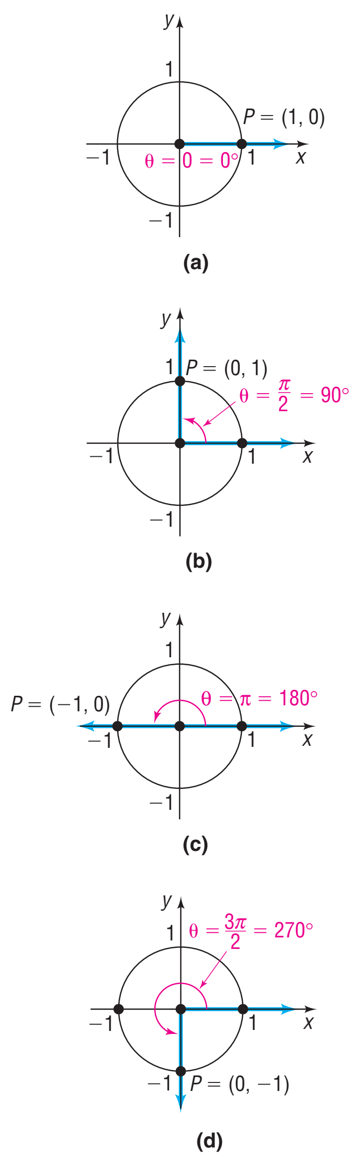
Find the exact values of the six trigonometric functions of:

(a) $\theta = 0 = 0°$ (b) $\theta = \dfrac{\pi}{2} = 90°$

(c) $\theta = \pi = 180°$ (d) $\theta = \dfrac{3\pi}{2} = 270°$

Solution

Figure 21



(a) The point on the unit circle that corresponds to $\theta = 0 = 0°$ is $P = (1, 0)$. See Figure 21(a). Then

$$\sin 0 = \sin 0° = y = 0 \qquad \cos 0 = \cos 0° = x = 1$$

$$\tan 0 = \tan 0° = \frac{y}{x} = 0 \qquad \sec 0 = \sec 0° = \frac{1}{x} = 1$$

Since the y-coordinate of P is 0, $\csc 0$ and $\cot 0$ are not defined.

(b) The point on the unit circle that corresponds to $\theta = \dfrac{\pi}{2} = 90°$ is $P = (0, 1)$. See Figure 21(b). Then

$$\sin\frac{\pi}{2} = \sin 90° = y = 1 \qquad \cos\frac{\pi}{2} = \cos 90° = x = 0$$

$$\csc\frac{\pi}{2} = \csc 90° = \frac{1}{y} = 1 \qquad \cot\frac{\pi}{2} = \cot 90° = \frac{x}{y} = 0$$

Since the x-coordinate of P is 0, $\tan\dfrac{\pi}{2}$ and $\sec\dfrac{\pi}{2}$ are not defined.

(c) The point on the unit circle that corresponds to $\theta = \pi = 180°$ is $P = (-1, 0)$. See Figure 21(c). Then

$$\sin\pi = \sin 180° = y = 0 \qquad \cos\pi = \cos 180° = x = -1$$

$$\tan\pi = \tan 180° = \frac{y}{x} = 0 \qquad \sec\pi = \sec 180° = \frac{1}{x} = -1$$

Since the y-coordinate of P is 0, $\csc\pi$ and $\cot\pi$ are not defined.

(d) The point on the unit circle that corresponds to $\theta = \dfrac{3\pi}{2} = 270°$ is $P = (0, -1)$. See Figure 21(d). Then

$$\sin\frac{3\pi}{2} = \sin 270° = y = -1 \qquad \cos\frac{3\pi}{2} = \cos 270° = x = 0$$

$$\csc\frac{3\pi}{2} = \csc 270° = \frac{1}{y} = -1 \qquad \cot\frac{3\pi}{2} = \cot 270° = \frac{x}{y} = 0$$

Since the x-coordinate of P is 0, $\tan\dfrac{3\pi}{2}$ and $\sec\dfrac{3\pi}{2}$ are not defined.

Table 2 on the next page summarizes the values of the trigonometric functions found in Example 2.

Table 2

Quadrantal Angles							
θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0°	0	1	0	Not defined	1	Not defined
$\frac{\pi}{2}$	90°	1	0	Not defined	1	Not defined	0
π	180°	0	-1	0	Not defined	-1	Not defined
$\frac{3\pi}{2}$	270°	-1	0	Not defined	-1	Not defined	0

There is no need to memorize Table 2. To find the value of a trigonometric function of a quadrantal angle, draw the angle and apply the definition, as we did in Example 2.

EXAMPLE 3 Finding Exact Values of the Trigonometric Functions of Angles That Are Integer Multiples of Quadrantal Angles

Find the exact value of:

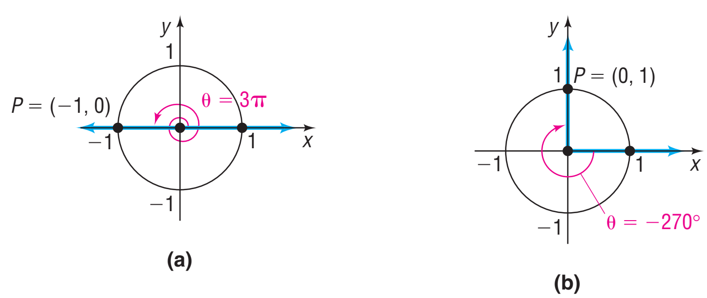
(a) $\sin(3\pi)$ (b) $\cos(-270°)$

Solution

(a) See Figure 22(a). The point P on the unit circle that corresponds to $\theta = 3\pi$ is $P = (-1, 0)$, so $\sin(3\pi) = y = 0$.

(b) See Figure 22(b). The point P on the unit circle that corresponds to $\theta = -270°$ is $P = (0, 1)$, so $\cos(-270°) = x = 0$.

Figure 22



Now Work PROBLEMS 21 AND 61

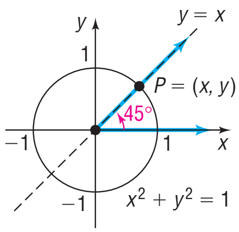
3 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

EXAMPLE 4 Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

Find the exact values of the six trigonometric functions of $\frac{\pi}{4} = 45°$.

Solution We seek the coordinates of the point $P = (x, y)$ on the unit circle that corresponds to $\theta = \frac{\pi}{4} = 45°$. See Figure 23. First, observe that P lies on the line $y = x$. (Do you

Figure 23



see why? Since $\theta = 45° = \frac{1}{2} \cdot 90°$, P must lie on the line that bisects quadrant I.) Since $P = (x, y)$ also lies on the unit circle, $x^2 + y^2 = 1$, it follows that

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + x^2 &= 1 \quad y = x, x > 0, y > 0 \\ 2x^2 &= 1 \\ x = \frac{1}{\sqrt{2}} &= \frac{\sqrt{2}}{2} \qquad y = \frac{\sqrt{2}}{2} \end{aligned}$$

Then

$$\sin\frac{\pi}{4} = \sin 45° = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \cos 45° = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \tan 45° = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\csc\frac{\pi}{4} = \csc 45° = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \qquad \sec\frac{\pi}{4} = \sec 45° = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \qquad \cot\frac{\pi}{4} = \cot 45° = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

EXAMPLE 5 Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) $\sin 45° \cos 180°$

(b) $\tan\frac{\pi}{4} - \sin\frac{3\pi}{2}$

(c) $\left(\sec\frac{\pi}{4}\right)^2 + \csc\frac{\pi}{2}$

Solution

(a) $\sin 45° \cos 180° = \underbrace{\frac{\sqrt{2}}{2}}_{\text{From Example 4}} \cdot \underbrace{(-1)}_{\text{From Table 2}} = -\frac{\sqrt{2}}{2}$

(b) $\tan\frac{\pi}{4} - \sin\frac{3\pi}{2} = \underbrace{1}_{\text{From Example 4}} - \underbrace{(-1)}_{\text{From Table 2}} = 2$

(c) $\left(\sec\frac{\pi}{4}\right)^2 + \csc\frac{\pi}{2} = \left(\sqrt{2}\right)^2 + 1 = 2 + 1 = 3$

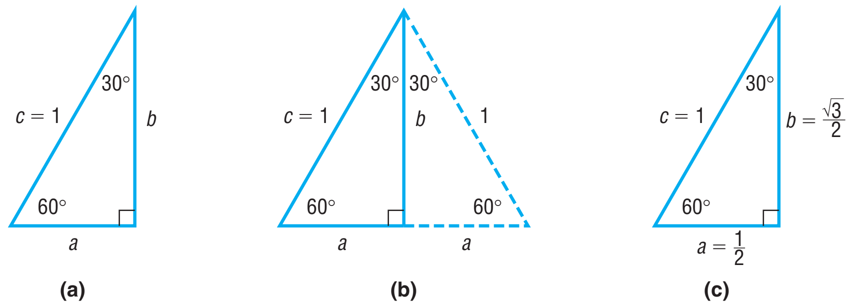
Now Work PROBLEM 35

4 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$

Consider a right triangle in which one of the angles is $\frac{\pi}{6} = 30°$. It then follows that the third angle is $\frac{\pi}{3} = 60°$. Figure 24(a) illustrates such a triangle with hypotenuse of length 1. Our problem is to determine a and b.

We begin by placing next to this triangle another triangle congruent to the first, as shown in Figure 24(b). Notice that we now have a triangle whose three angles each equal 60°. This triangle is therefore equilateral, so each side is of length 1.

Figure 24



This means the base is $2a = 1$, and so $a = \frac{1}{2}$. By the Pythagorean Theorem, b satisfies the equation $a^2 + b^2 = c^2$, so we have

$$a^2 + b^2 = c^2$$

$$\frac{1}{4} + b^2 = 1 \qquad a = \frac{1}{2}, c = 1$$

$$b^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$b = \frac{\sqrt{3}}{2}$$

$b > 0$ because b is the length of the side of a triangle.

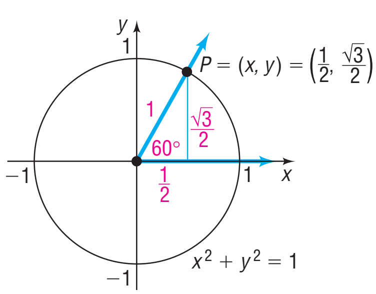
This results in Figure 24(c).

EXAMPLE 6 Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{3} = 60°$

Find the exact values of the six trigonometric functions of $\frac{\pi}{3} = 60°$.

Solution Position the triangle in Figure 24(c) so that the 60° angle is in standard position. See Figure 25. The point on the unit circle that corresponds to $\theta = \frac{\pi}{3} = 60°$ is $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then

Figure 25



$$\sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \cos 60° = \frac{1}{2}$$

$$\csc\frac{\pi}{3} = \csc 60° = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec\frac{\pi}{3} = \sec 60° = \frac{1}{\frac{1}{2}} = 2$$

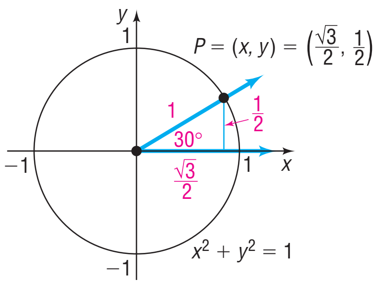
$$\tan\frac{\pi}{3} = \tan 60° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \cot\frac{\pi}{3} = \cot 60° = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

EXAMPLE 7 Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$

Find the exact values of the trigonometric functions of $\frac{\pi}{6} = 30°$.

Solution Position the triangle in Figure 24(c) so that the 30° angle is in standard position. See Figure 26. The point on the unit circle that corresponds to $\theta = \frac{\pi}{6} = 30°$ is $P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Then

Figure 26



$$\sin\frac{\pi}{6} = \sin 30° = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\csc\frac{\pi}{6} = \csc 30° = \frac{1}{\frac{1}{2}} = 2 \qquad \sec\frac{\pi}{6} = \sec 30° = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{6} = \tan 30° = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Table 3 summarizes the information just derived for $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$. Until you memorize the entries in Table 3, you should draw an appropriate diagram to determine the values given in the table.

Table 3

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

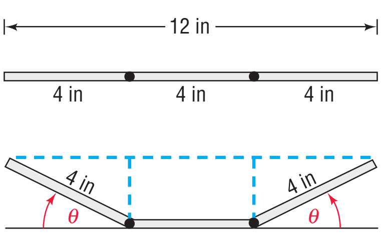
Now Work PROBLEM 41

EXAMPLE 8 Constructing a Rain Gutter

A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. See Figure 27. The area A of the opening may be expressed as a function of θ as

$$A(\theta) = 16 \sin\theta(\cos\theta + 1)$$

Figure 27



Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$.

Solution For $\theta = 30°$: $A(30°) = 16 \sin 30°(\cos 30° + 1)$

$$= 16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + 1\right) = 4\sqrt{3} + 8 \approx 14.9$$

The area of the opening for $\theta = 30°$ is about 14.9 square inches.

$$\text{For } \theta = 45°: \quad A(45°) = 16 \sin 45°(\cos 45° + 1)$$
$$= 16\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + 1\right) = 8 + 8\sqrt{2} \approx 19.3$$

The area of the opening for $\theta = 45°$ is about 19.3 square inches.

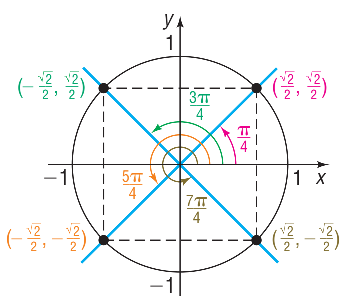
$$\text{For } \theta = 60°: \quad A(60°) = 16 \sin 60°(\cos 60° + 1)$$
$$= 16\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} + 1\right) = 12\sqrt{3} \approx 20.8$$

The area of the opening for $\theta = 60°$ is about 20.8 square inches.

5 Find the Exact Values of the Trigonometric Functions for Integer Multiples of $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$

We know the exact values of the trigonometric functions of $\frac{\pi}{4} = 45°$. Using symmetry, we can find the exact values of the trigonometric functions of $\frac{3\pi}{4} = 135°$, $\frac{5\pi}{4} = 225°$, and $\frac{7\pi}{4} = 315°$.

Figure 28



See Figure 28. Using symmetry with respect to the y-axis, the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is the point on the unit circle that corresponds to the angle $\frac{3\pi}{4} = 135°$. Similarly, using symmetry with respect to the origin, the point $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is the point on the unit circle that corresponds to the angle $\frac{5\pi}{4} = 225°$. Finally, using symmetry with respect to the x-axis, the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is the point on the unit circle that corresponds to the angle $\frac{7\pi}{4} = 315°$.

EXAMPLE 9 **Finding Exact Values for Multiples of $\frac{\pi}{4} = 45°$**

Find the exact value of each expression.

(a) $\cos\frac{5\pi}{4}$ (b) $\sin 135°$ (c) $\tan 315°$ (d) $\sin\left(-\frac{\pi}{4}\right)$ (e) $\cos\frac{11\pi}{4}$

Solution

(a) From Figure 28, we see the point $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ corresponds to $\frac{5\pi}{4}$, so $\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$.

(b) Since $135° = \frac{3\pi}{4}$, the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ corresponds to $135°$, so $\sin 135° = \frac{\sqrt{2}}{2}$.

(c) Since $315° = \frac{7\pi}{4}$, the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ corresponds to $315°$, so

$$\tan 315° = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1.$$

(d) The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ corresponds to $-\frac{\pi}{4}$, so $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

(e) The point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ corresponds to $\frac{11\pi}{4}$, so $\cos\frac{11\pi}{4} = -\frac{\sqrt{2}}{2}$.

Now Work PROBLEMS 51 AND 55

The use of symmetry also provides information about certain integer multiples of the angles $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$. See Figures 29 and 30.

Figure 29

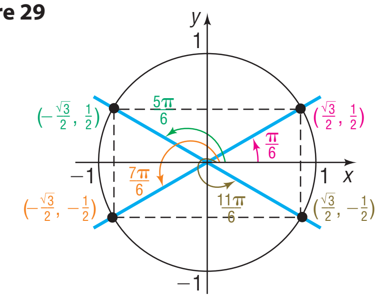
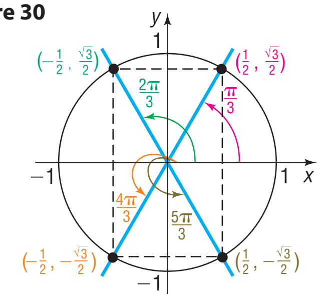


Figure 30



EXAMPLE 10

Finding Exact Values for Multiples of $\frac{\pi}{6} = 30°$ or $\frac{\pi}{3} = 60°$

Based on Figures 29 and 30, we see that

(a) $\cos 210° = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

(b) $\sin(-60°) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

(c) $\tan\frac{5\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

(d) $\cos\underbrace{\frac{8\pi}{3}}_{\frac{8\pi}{3} = 2\pi + \frac{2\pi}{3}} = \cos\frac{2\pi}{3} = -\frac{1}{2}$

Now Work PROBLEM 47

6 Use a Calculator to Approximate the Value of a Trigonometric Function

Before getting started, you must first decide whether to enter the angle in the calculator using radians or degrees and then set the calculator to the correct MODE. Check your instruction manual to find out how your calculator handles degrees and radians. Your calculator has keys marked $\boxed{\sin}$, $\boxed{\cos}$, and $\boxed{\tan}$. To find the values of the remaining three trigonometric functions, secant, cosecant, and cotangent, we use the fact that, if $P = (x, y)$ is a point on the unit circle on the terminal side of θ, then

WARNING On your calculator the second functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ do not represent the reciprocal of sin, cos, and tan.

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{y} = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan\theta}$$

EXAMPLE 11

Using a Calculator to Approximate the Value of a Trigonometric Function

Use a calculator to find the approximate value of:

(a) $\cos 48°$ (b) $\csc 21°$ (c) $\tan\frac{\pi}{12}$

Express your answer rounded to two decimal places.

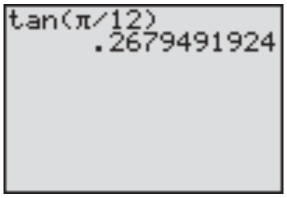
Solution (a) First, we set the MODE to receive degrees. Rounded to two decimal places,

$$\cos 48° = 0.6691306 \approx 0.67$$

(b) Most calculators do not have a csc key. The manufacturers assume that the user knows some trigonometry. To find the value of csc 21°, use the fact that $\csc 21° = \dfrac{1}{\sin 21°}$. Rounded to two decimal places,

$$\csc 21° \approx 2.79$$

Figure 31



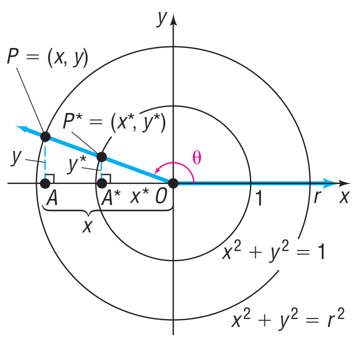
(c) Set the MODE to receive radians. Figure 31 shows the solution using a TI-84 Plus graphing calculator. Rounded to two decimal places,

$$\tan \frac{\pi}{12} \approx 0.27$$

Now Work PROBLEM 65

7 Use a Circle of Radius *r* to Evaluate the Trigonometric Functions

Figure 32



Until now, finding the exact value of a trigonometric function of an angle θ required that we locate the corresponding point $P = (x, y)$ on the unit circle. In fact, though, any circle whose center is at the origin can be used.

Let θ be any nonquadrantal angle placed in standard position. Let $P = (x, y)$ be the point on the circle $x^2 + y^2 = r^2$ that corresponds to θ, and let $P^* = (x^*, y^*)$ be the point on the unit circle that corresponds to θ. See Figure 32, where θ is shown in quadrant II.

Notice that the triangles OA^*P^* and OAP are similar; as a result, the ratios of corresponding sides are equal.

$$\frac{y^*}{1} = \frac{y}{r} \qquad \frac{x^*}{1} = \frac{x}{r} \qquad \frac{y^*}{x^*} = \frac{y}{x}$$

$$\frac{1}{y^*} = \frac{r}{y} \qquad \frac{1}{x^*} = \frac{r}{x} \qquad \frac{x^*}{y^*} = \frac{x}{y}$$

These results lead us to formulate the following theorem:

THEOREM

For an angle θ in standard position, let $P = (x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then

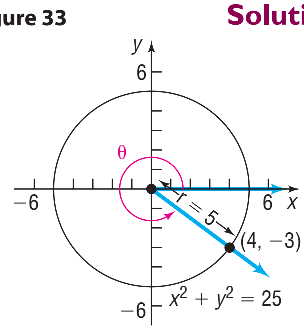
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad x \neq 0$$

$$\csc\theta = \frac{r}{y} \quad y \neq 0 \qquad \sec\theta = \frac{r}{x} \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad y \neq 0$$

EXAMPLE 12 **Finding the Exact Values of the Six Trigonometric Functions**

Find the exact values of each of the six trigonometric functions of an angle θ if $(4, -3)$ is a point on its terminal side in standard position.

Figure 33



Solution See Figure 33. The point $(4, -3)$ is on a circle of radius $r = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ with the center at the origin.

For the point $(x, y) = (4, -3)$, we have $x = 4$ and $y = -3$. Since $r = 5$, we find

$$\sin\theta = \frac{y}{r} = -\frac{3}{5} \qquad \cos\theta = \frac{x}{r} = \frac{4}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{3}{4}$$

$$\csc\theta = \frac{r}{y} = -\frac{5}{3} \qquad \sec\theta = \frac{r}{x} = \frac{5}{4} \qquad \cot\theta = \frac{x}{y} = -\frac{4}{3}$$

Now Work PROBLEM 77

Historical Feature

The name *sine* for the sine function is due to a medieval confusion. The name comes from the Sanskrit word *jīva* (meaning chord), first used in India by Araybhata the Elder (AD 510). He really meant half-chord, but abbreviated it. This was brought into Arabic as *jība*, which was meaningless. Because the proper Arabic word *jaib* would be written the same way (short vowels are not written out in Arabic), *jība* was pronounced as *jaib*, which meant bosom or hollow, and *jība* remains as the Arabic word for sine to this day. Scholars translating the Arabic works into Latin found that the word *sinus* also meant bosom or hollow, and from *sinus* we get the word *sine*.

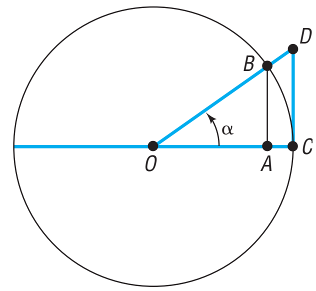
The name *tangent*, due to Thomas Finck (1583), can be understood by looking at Figure 34. The line segment $\overline{DC}$ is tangent to the circle at C. If $d(O, B) = d(O, C) = 1$, then the length of the line segment $\overline{DC}$ is

$$d(D, C) = \frac{d(D, C)}{1} = \frac{d(D, C)}{d(O, C)} = \tan \alpha$$

The old name for the tangent is *umbra versa* (meaning turned shadow), referring to the use of the tangent in solving height problems with shadows.

The names of the remaining functions came about as follows. If α and β are complementary angles, then $\cos \alpha = \sin \beta$. Because β is the complement of α, it was natural to write the cosine of α as *sin co* α. Probably for reasons involving ease of pronunciation, the *co* migrated to the front, and then cosine received a three-letter abbreviation to match sin, sec, and tan. The two other cofunctions were similarly treated, except that the long forms *cotan* and *cosec* survive to this day in some countries.

Figure 34



6.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. In a right triangle, with legs a and b and hypotenuse c, the Pythagorean Theorem states that ____________. (p. A14)

2. The value of the function $f(x) = 3x - 7$ at 5 is ________. (pp. 46–56)

3. ***True or False*** For a function $y = f(x)$, for each x in the domain, there is exactly one element y in the range. (pp. 46–56)

4. If two triangles are similar, then corresponding angles are _______ and the lengths of corresponding sides are _____________. (pp. A14–A19)

5. What point is symmetric with respect to the y-axis to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$? (pp. 12–14)

6. If (x, y) is a point on the unit circle in quadrant IV and if $x = \frac{\sqrt{3}}{2}$, what is y? (p. 35)

Concepts and Vocabulary

7. The _______ function takes as input a real number t that corresponds to a point $P = (x, y)$ on the unit circle and outputs the x-coordinate.

8. The point on the unit circle that corresponds to $\theta = \frac{\pi}{2}$ is $P =$ ______.

9. The point on the unit circle that corresponds to $\theta = \frac{\pi}{4}$ is $P =$ _____________.

10. The point on the unit circle that corresponds to $\theta = \frac{\pi}{3}$ is $P =$ ____________.

11. For any angle θ in standard position, let $P = (x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then, $\sin\theta =$ ____ and $\cos\theta =$ ____.

12. ***True or False*** Exact values can be found for the sine of any angle.

Skill Building

In Problems 13–20, $P = (x, y)$ is the point on the unit circle that corresponds to a real number t. Find the exact values of the six trigonometric functions of t.

13. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **14.** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **15.** $\left(-\frac{2}{5}, \frac{\sqrt{21}}{5}\right)$ **16.** $\left(-\frac{1}{5}, \frac{2\sqrt{6}}{5}\right)$

17. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **18.** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **19.** $\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ **20.** $\left(-\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$

In Problems 21–30, find the exact value. Do not use a calculator.

21. $\sin \frac{11\pi}{2}$ **22.** $\cos(7\pi)$ **23.** $\tan(6\pi)$ **24.** $\cot \frac{7\pi}{2}$ **25.** $\csc \frac{11\pi}{2}$

26. $\sec(8\pi)$ **27.** $\cos\left(-\frac{3\pi}{2}\right)$ **28.** $\sin(-3\pi)$ **29.** $\sec(-\pi)$ **30.** $\tan(-3\pi)$

In Problems 31–46, find the exact value of each expression. Do not use a calculator.

31. $\sin 45° + \cos 60°$ **32.** $\sin 30° - \cos 45°$ **33.** $\sin 90° + \tan 45°$ **34.** $\cos 180° - \sin 180°$

35. $\sin 45° \cos 45°$ **36.** $\tan 45° \cos 30°$ **37.** $\csc 45° \tan 60°$ **38.** $\sec 30° \cot 45°$

39. $4 \sin 90° - 3 \tan 180°$ **40.** $5 \cos 90° - 8 \sin 270°$ **41.** $2 \sin \frac{\pi}{3} - 3 \tan \frac{\pi}{6}$ **42.** $2 \sin \frac{\pi}{4} + 3 \tan \frac{\pi}{4}$

43. $2 \sec \frac{\pi}{4} + 4 \cot \frac{\pi}{3}$ **44.** $3 \csc \frac{\pi}{3} + \cot \frac{\pi}{4}$ **45.** $\csc \frac{\pi}{2} + \cot \frac{\pi}{2}$ **46.** $\sec \pi - \csc \frac{\pi}{2}$

In Problems 47–64, find the exact values of the six trigonometric functions of the given angle. If any are not defined, say "not defined." Do not use a calculator.

47. $\frac{2\pi}{3}$ **48.** $\frac{5\pi}{6}$ **49.** $210°$ **50.** $240°$ **51.** $\frac{3\pi}{4}$ **52.** $\frac{11\pi}{4}$

53. $\frac{8\pi}{3}$ **54.** $\frac{13\pi}{6}$ **55.** $405°$ **56.** $390°$ **57.** $-\frac{\pi}{6}$ **58.** $-\frac{\pi}{3}$

59. $-135°$ **60.** $-240°$ **61.** $\frac{5\pi}{2}$ **62.** 5π **63.** $-\frac{14\pi}{3}$ **64.** $-\frac{13\pi}{6}$

In Problems 65–76, use a calculator to find the approximate value of each expression rounded to two decimal places.

65. $\sin 28°$ **66.** $\cos 14°$ **67.** $\sec 21°$ **68.** $\cot 70°$

69. $\tan \frac{\pi}{10}$ **70.** $\sin \frac{\pi}{8}$ **71.** $\cot \frac{\pi}{12}$ **72.** $\csc \frac{5\pi}{13}$

73. $\sin 1$ **74.** $\tan 1$ **75.** $\sin 1°$ **76.** $\tan 1°$

In Problems 77–84, a point on the terminal side of an angle θ in standard position is given. Find the exact value of each of the six trigonometric functions of θ.

77. $(-3, 4)$ **78.** $(5, -12)$ **79.** $(2, -3)$ **80.** $(-1, -2)$

81. $(-2, -2)$ **82.** $(-1, 1)$ **83.** $\left(\frac{1}{3}, \frac{1}{4}\right)$ **84.** $(0.3, 0.4)$

---

85. Find the exact value of:
$$\sin 45° + \sin 135° + \sin 225° + \sin 315°$$

86. Find the exact value of:
$$\tan 60° + \tan 150°$$

87. Find the exact value of:
$$\sin 40° + \sin 130° + \sin 220° + \sin 310°$$

88. Find the exact value of:
$$\tan 40° + \tan 140°$$

89. If $f(\theta) = \sin \theta = 0.1$, find $f(\theta + \pi)$.

90. If $f(\theta) = \cos \theta = 0.3$, find $f(\theta + \pi)$.

91. If $f(\theta) = \tan \theta = 3$, find $f(\theta + \pi)$.

92. If $f(\theta) = \cot \theta = -2$, find $f(\theta + \pi)$.

93. If $\sin \theta = \frac{1}{5}$, find $\csc \theta$.

94. If $\cos \theta = \frac{2}{3}$, find $\sec \theta$.

In Problems 95–106, $f(\theta) = \sin \theta$ and $g(\theta) = \cos \theta$. Find the exact value of each function below if $\theta = 60°$. Do not use a calculator.

95. $f(\theta)$ **96.** $g(\theta)$ **97.** $f\left(\frac{\theta}{2}\right)$

98. $g\left(\frac{\theta}{2}\right)$ **99.** $[f(\theta)]^2$ **100.** $[g(\theta)]^2$

101. $f(2\theta)$ **102.** $g(2\theta)$ **103.** $2f(\theta)$

104. $2g(\theta)$ **105.** $f(-\theta)$ **106.** $g(-\theta)$

Mixed Practice

In Problems 107–114, $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = 2x$, and $p(x) = \frac{x}{2}$. *Find the value of each of the following:*

107. $(f + g)(30°)$

108. $(f - g)(60°)$

109. $(f \cdot g)\left(\frac{3\pi}{4}\right)$

110. $(f \cdot g)\left(\frac{4\pi}{3}\right)$

111. $(f \circ h)\left(\frac{\pi}{6}\right)$

112. $(g \circ p)(60°)$

113. $(p \circ g)(315°)$

114. $(h \circ f)\left(\frac{5\pi}{6}\right)$

115. (a) Find $f\left(\frac{\pi}{4}\right)$. What point is on the graph of f?

(b) Assuming f is one-to-one*, use the result of part (a) to find a point on the graph of f^{-1}.

(c) What point is on the graph of $y = f\left(x + \frac{\pi}{4}\right) - 3$ if $x = \frac{\pi}{4}$?

116. (a) Find $g\left(\frac{\pi}{6}\right)$. What point is on the graph of g?

(b) Assuming g is one-to-one*, use the result of part (a) to find a point on the graph of g^{-1}.

(c) What point is on the graph of $y = 2g\left(x - \frac{\pi}{6}\right)$ if $x = \frac{\pi}{6}$?

Applications and Extensions

117. Find two negative and three positive angles, expressed in radians, for which the point on the unit circle that corresponds to each angle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

118. Find two negative and three positive angles, expressed in radians, for which the point on the unit circle that corresponds to each angle is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

119. Use a calculator in radian mode to complete the following table.

What can you conclude about the value of $f(\theta) = \frac{\sin\theta}{\theta}$ as θ approaches 0?

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\sin\theta$								
$f(\theta) = \frac{\sin\theta}{\theta}$								

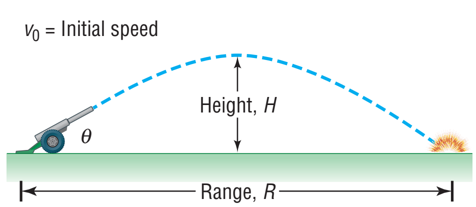
120. Use a calculator in radian mode to complete the following table.

What can you conclude about the value of $g(\theta) = \frac{\cos\theta - 1}{\theta}$ as θ approaches 0?

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\cos\theta - 1$								
$g(\theta) = \frac{\cos\theta - 1}{\theta}$								

For Problems 121–124, use the following discussion.

Projectile Motion *The path of a projectile fired at an inclination θ to the horizontal with initial speed v_0 is a parabola (see the figure).*



The range R of the projectile, that is, the horizontal distance that the projectile travels, is found by using the function

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

where $g \approx 32.2$ feet per second per second ≈ 9.8 meters per second per second is the acceleration due to gravity. The maximum height H of the projectile is given by the function

$$H(\theta) = \frac{v_0^2 (\sin\theta)^2}{2g}$$

*In Section 7.1, we discuss the necessary domain restriction so that the function is one-to-one.

In Problems 121–124, find the range R and maximum height H.

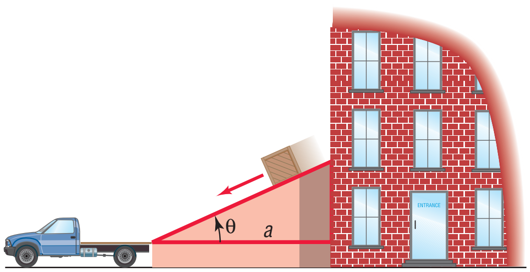
121. The projectile is fired at an angle of $45°$ to the horizontal with an initial speed of 100 feet per second.

122. The projectile is fired at an angle of $30°$ to the horizontal with an initial speed of 150 meters per second.

123. The projectile is fired at an angle of $25°$ to the horizontal with an initial speed of 500 meters per second.

124. The projectile is fired at an angle of $50°$ to the horizontal with an initial speed of 200 feet per second.

125. Inclined Plane See the figure.



If friction is ignored, the time t (in seconds) required for a block to slide down an inclined plane is given by the function

$$t(\theta) = \sqrt{\frac{2a}{g \sin\theta \cos\theta}}$$

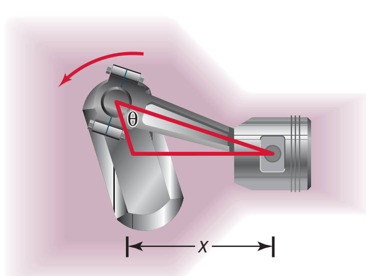
where a is the length (in feet) of the base and $g \approx 32$ feet per second per second is the acceleration due to gravity. How long does it take a block to slide down an inclined plane with base $a = 10$ feet when:

(a) $\theta = 30°$?
(b) $\theta = 45°$?
(c) $\theta = 60°$?

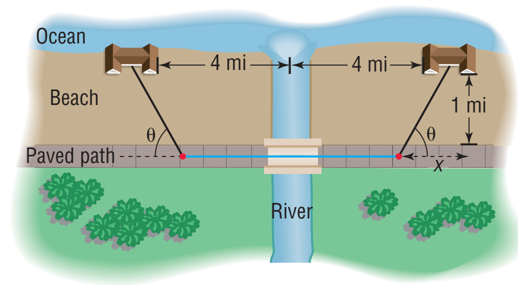
126. Piston Engines In a certain piston engine, the distance x (in centimeters) from the center of the drive shaft to the head of the piston is given by the function

$$x(\theta) = \cos\theta + \sqrt{16 + 0.5\cos(2\theta)}$$

where θ is the angle between the crank and the path of the piston head. See the figure. Find x when $\theta = 30°$ and when $\theta = 45°$.



127. Calculating the Time of a Trip Two oceanfront homes are located 8 miles apart on a straight stretch of beach, each a distance of 1 mile from a paved road that parallels the ocean. See the figure.



Sally can jog 8 miles per hour along the paved road, but only 3 miles per hour in the sand on the beach. Because of a river directly between the two houses, it is necessary to jog in the sand to the road, continue on the road, and then jog directly back in the sand to get from one house to the other. The time T to get from one house to the other as a function of the angle θ shown in the illustration is

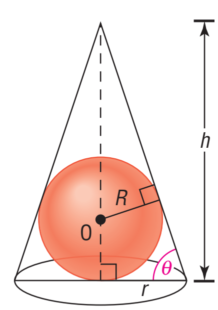
$$T(\theta) = 1 + \frac{2}{3\sin\theta} - \frac{1}{4\tan\theta}, \qquad 0° < \theta < 90°$$

(a) Calculate the time T for $\theta = 30°$. How long is Sally on the paved road?
(b) Calculate the time T for $\theta = 45°$. How long is Sally on the paved road?
(c) Calculate the time T for $\theta = 60°$. How long is Sally on the paved road?
(d) Calculate the time T for $\theta = 90°$. Describe the path taken. Why can't the formula for T be used?

128. Designing Fine Decorative Pieces A designer of decorative art plans to market solid gold spheres encased in clear crystal cones. Each sphere is of fixed radius R and will be enclosed in a cone of height h and radius r. See the illustration. Many cones can be used to enclose the sphere, each having a different slant angle θ. The volume V of the cone can be expressed as a function of the slant angle θ of the cone as

$$V(\theta) = \frac{1}{3}\pi R^3 \frac{(1 + \sec\theta)^3}{(\tan\theta)^2}, \qquad 0° < \theta < 90°$$

What volume V is required to enclose a sphere of radius 2 centimeters in a cone whose slant angle θ is $30°$? $45°$? $60°$?



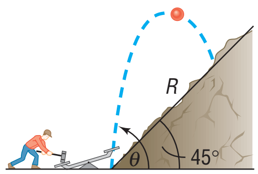
129. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of an inclined plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane as a function of θ is given by

$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

(a) Find the distance R that the object travels along the inclined plane if the initial velocity is 32 feet per second and $\theta = 60°$.

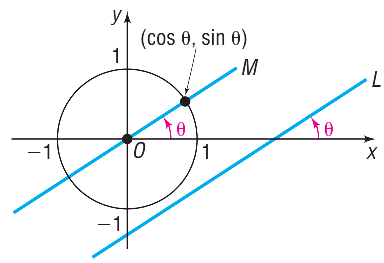
(b) Graph $R = R(\theta)$ if the initial velocity is 32 feet per second.

(c) What value of θ makes R largest?

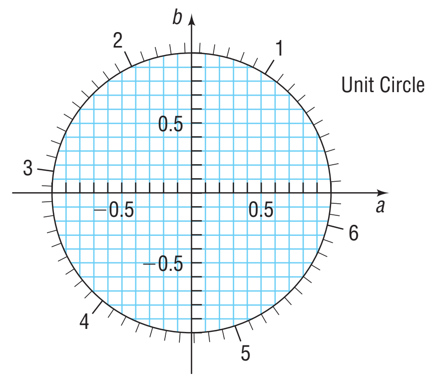


130. If θ, $0 < \theta < \pi$, is the angle between the positive x-axis and a nonhorizontal, nonvertical line L, show that the slope m of L equals $\tan\theta$. The angle θ is called the **inclination** of L.

[**Hint:** See the illustration, where we have drawn the line M parallel to L and passing through the origin. Use the fact that M intersects the unit circle at the point $(\cos\theta, \sin\theta)$.]



In Problems 131 and 132, use the figure to approximate the value of the six trigonometric functions at t to the nearest tenth. Then use a calculator to approximate each of the six trigonometric functions at t.



131. (a) $t = 1$ (b) $t = 5.1$

132. (a) $t = 2$ (b) $t = 4$

Explaining Concepts: Discussion and Writing

133. Write a brief paragraph that explains how to quickly compute the trigonometric functions of 30°, 45°, and 60°.

134. Write a brief paragraph that explains how to quickly compute the trigonometric functions of 0°, 90°, 180°, and 270°.

135. How would you explain the meaning of the sine function to a fellow student who has just completed college algebra?

'Are You Prepared?' Answers

1. $c^2 = a^2 + b^2$ **2.** 8 **3.** True **4.** equal; proportional **5.** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **6.** $-\frac{1}{2}$

6.3 Properties of the Trigonometric Functions

Preparing for this section *Before getting started, review the following:*

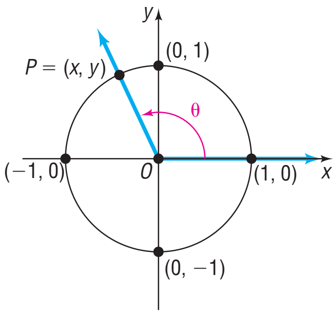
- Functions (Section 2.1, pp. 46–56)
- Identity (Appendix A, Section A.6, p. A44)
- Even and Odd Functions (Section 2.3, pp. 69–70)

Now Work the **'Are You Prepared?'** problems on page 390.

OBJECTIVES
1 Determine the Domain and the Range of the Trigonometric Functions (p. 380)
2 Determine the Period of the Trigonometric Functions (p. 381)
3 Determine the Signs of the Trigonometric Functions in a Given Quadrant (p. 383)
4 Find the Values of the Trigonometric Functions Using Fundamental Identities (p. 384)
5 Find the Exact Values of the Trigonometric Functions of an Angle Given One of the Functions and the Quadrant of the Angle (p. 386)
6 Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions (p. 389)

1 Determine the Domain and the Range of the Trigonometric Functions

Figure 35



Let θ be an angle in standard position, and let $P = (x, y)$ be the point on the unit circle that corresponds to θ. See Figure 35. Then, by the definition given earlier,

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x} \quad x \neq 0$$

$$\csc\theta = \frac{1}{y} \quad y \neq 0 \qquad \sec\theta = \frac{1}{x} \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad y \neq 0$$

For $\sin\theta$ and $\cos\theta$, there is no concern about dividing by 0, so θ can be any angle. It follows that the domain of the sine function and cosine function is the set of all real numbers.

> The domain of the sine function is the set of all real numbers.
> The domain of the cosine function is the set of all real numbers.

For the tangent function and secant function, the x-coordinate of $P = (x, y)$ cannot be 0 since this results in division by 0. See Figure 35. On the unit circle, there are two such points, $(0, 1)$ and $(0, -1)$. These two points correspond to the angles $\frac{\pi}{2}(90^\circ)$ and $\frac{3\pi}{2}(270^\circ)$ or, more generally, to any angle that is an odd integer multiple of $\frac{\pi}{2}(90^\circ)$, such as $\pm\frac{\pi}{2}(\pm 90^\circ)$, $\pm\frac{3\pi}{2}(\pm 270^\circ)$, $\pm\frac{5\pi}{2}(\pm 450^\circ)$, and so on. Such angles must therefore be excluded from the domain of the tangent function and secant function.

> The domain of the tangent function is the set of all real numbers, except odd integer multiples of $\frac{\pi}{2}(90^\circ)$.
> The domain of the secant function is the set of all real numbers, except odd integer multiples of $\frac{\pi}{2}(90^\circ)$.

For the cotangent function and cosecant function, the y-coordinate of $P = (x, y)$ cannot be 0 since this results in division by 0. See Figure 35. On the unit circle, there are two such points, $(1, 0)$ and $(-1, 0)$. These two points correspond to the angles $0(0^\circ)$ and $\pi(180^\circ)$ or, more generally, to any angle that is an integer multiple of $\pi(180^\circ)$, such as $0(0^\circ)$, $\pm\pi(\pm 180^\circ)$, $\pm 2\pi(\pm 360^\circ)$, $\pm 3\pi(\pm 540^\circ)$, and so on. Such angles must therefore be excluded from the domain of the cotangent function and cosecant function.

> The domain of the cotangent function is the set of all real numbers, except integer multiples of $\pi(180^\circ)$.
> The domain of the cosecant function is the set of all real numbers, except integer multiples of $\pi(180^\circ)$.

Next we determine the range of each of the six trigonometric functions. Refer again to Figure 35. Let $P = (x, y)$ be the point on the unit circle that corresponds to the angle θ. It follows that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Since $\sin\theta = y$ and $\cos\theta = x$, we have

$$-1 \leq \sin\theta \leq 1 \qquad -1 \leq \cos\theta \leq 1$$

The range of both the sine function and the cosine function consists of all real numbers between -1 and 1, inclusive. Using absolute value notation, we have $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

If θ is not an integer multiple of $\pi(180°)$, then $\csc\theta = \frac{1}{y}$. Since $y = \sin\theta$ and $|y| = |\sin\theta| \le 1$, it follows that $|\csc\theta| = \frac{1}{|\sin\theta|} = \frac{1}{|y|} \ge 1$ $\left(\frac{1}{y} \le -1 \text{ or } \frac{1}{y} \ge 1\right)$. Since $\csc\theta = \frac{1}{y}$, the range of the cosecant function consists of all real numbers less than or equal to -1 or greater than or equal to 1. That is,

$$\csc\theta \le -1 \quad \text{or} \quad \csc\theta \ge 1$$

If θ is not an odd integer multiple of $\frac{\pi}{2}(90°)$, then $\sec\theta = \frac{1}{x}$. Since $x = \cos\theta$ and $|x| = |\cos\theta| \le 1$, it follows that $|\sec\theta| = \frac{1}{|\cos\theta|} = \frac{1}{|x|} \ge 1$ $\left(\frac{1}{x} \le -1 \text{ or } \frac{1}{x} \ge 1\right)$. Since $\sec\theta = \frac{1}{x}$, the range of the secant function consists of all real numbers less than or equal to -1 or greater than or equal to 1. That is,

$$\sec\theta \le -1 \quad \text{or} \quad \sec\theta \ge 1$$

The range of both the tangent function and the cotangent function is the set of all real numbers.

$$-\infty < \tan\theta < \infty \qquad -\infty < \cot\theta < \infty$$

You are asked to prove this in Problems 121 and 122.

Table 4 summarizes these results.

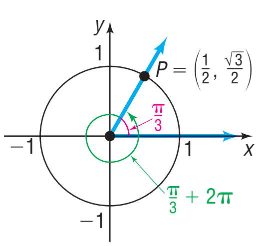
Table 4

Function	Symbol	Domain	Range
sine	$f(\theta) = \sin\theta$	All real numbers	All real numbers from -1 to 1, inclusive
cosine	$f(\theta) = \cos\theta$	All real numbers	All real numbers from -1 to 1, inclusive
tangent	$f(\theta) = \tan\theta$	All real numbers, except odd integer multiples of $\frac{\pi}{2}(90°)$	All real numbers
cosecant	$f(\theta) = \csc\theta$	All real numbers, except integer multiples of $\pi(180°)$	All real numbers greater than or equal to 1 or less than or equal to -1
secant	$f(\theta) = \sec\theta$	All real numbers, except odd integer multiples of $\frac{\pi}{2}(90°)$	All real numbers greater than or equal to 1 or less than or equal to -1
cotangent	$f(\theta) = \cot\theta$	All real numbers, except integer multiples of $\pi(180°)$	All real numbers

Now Work PROBLEM 97

2 Determine the Period of the Trigonometric Functions

Figure 36

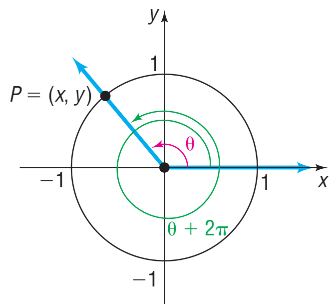


Look at Figure 36. This figure shows that for an angle of $\frac{\pi}{3}$ radians the corresponding point P on the unit circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Notice that, for an angle of $\frac{\pi}{3} + 2\pi$ radians, the corresponding point P on the unit circle is also $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3} + 2\pi\right) = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \cos\left(\frac{\pi}{3} + 2\pi\right) = \frac{1}{2}$$

This example illustrates a more general situation. For a given angle θ, measured in radians, suppose that we know the corresponding point $P = (x, y)$ on the unit circle. Now add 2π to θ. The point on the unit circle corresponding to $\theta + 2\pi$ is

Figure 37



identical to the point P on the unit circle corresponding to θ. See Figure 37. The values of the trigonometric functions of $\theta + 2\pi$ are equal to the values of the corresponding trigonometric functions of θ.

If we add (or subtract) integer multiples of 2π to θ, the values of the sine and cosine function remain unchanged. That is, for all θ

$$\sin(\theta + 2\pi k) = \sin\theta \qquad \cos(\theta + 2\pi k) = \cos\theta \tag{1}$$
where k is any integer.

Functions that exhibit this kind of behavior are called *periodic functions*.

DEFINITION

A function f is called **periodic** if there is a positive number p such that, whenever θ is in the domain of f, so is $\theta + p$, and

$$f(\theta + p) = f(\theta)$$

If there is a smallest such number p, this smallest value is called the **(fundamental) period** of f.

Based on equation (1), the sine and cosine functions are periodic. In fact, the sine and cosine functions have period 2π. You are asked to prove this fact in Problems 123 and 124. The secant and cosecant functions are also periodic with period 2π, and the tangent and cotangent functions are periodic with period π. You are asked to prove these statements in Problems 125 through 128.

These facts are summarized as follows:

In Words

Tangent and cotangent have period π; the others have period 2π.

Periodic Properties

$$\sin(\theta + 2\pi) = \sin\theta \quad \cos(\theta + 2\pi) = \cos\theta \quad \tan(\theta + \pi) = \tan\theta$$
$$\csc(\theta + 2\pi) = \csc\theta \quad \sec(\theta + 2\pi) = \sec\theta \quad \cot(\theta + \pi) = \cot\theta$$

Because the sine, cosine, secant, and cosecant functions have period 2π, once we know their values over an interval of length 2π, we know all their values; similarly, since the tangent and cotangent functions have period π, once we know their values over an interval of length π, we know all their values.

EXAMPLE 1 Finding Exact Values Using Periodic Properties

Find the exact value of:

(a) $\sin\dfrac{17\pi}{4}$ (b) $\cos(5\pi)$ (c) $\tan\dfrac{5\pi}{4}$

Solution

(a) It is best to sketch the angle first, as shown in Figure 38(a). Since the period of the sine function is 2π, each full revolution can be ignored. This leaves the angle $\dfrac{\pi}{4}$. Then

$$\sin\frac{17\pi}{4} = \sin\left(\frac{\pi}{4} + 4\pi\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

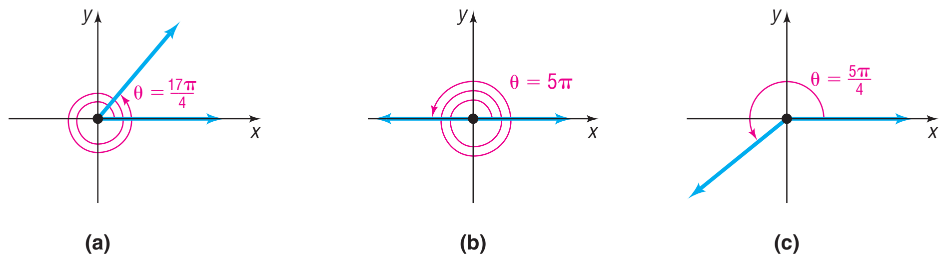
(b) See Figure 38(b). Since the period of the cosine function is 2π, each full revolution can be ignored. This leaves the angle π. Then

$$\cos(5\pi) = \cos(\pi + 4\pi) = \cos\pi = -1$$

(c) See Figure 38(c). Since the period of the tangent function is π, each half-revolution can be ignored. This leaves the angle $\frac{\pi}{4}$. Then

$$\tan\frac{5\pi}{4} = \tan\left(\frac{\pi}{4} + \pi\right) = \tan\frac{\pi}{4} = 1$$

Figure 38

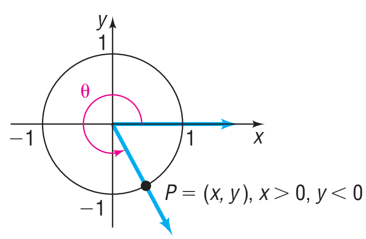


The periodic properties of the trigonometric functions will be very helpful to us when we study their graphs later in the chapter.

Now Work PROBLEM 11

3 Determine the Signs of the Trigonometric Functions in a Given Quadrant

Figure 39



Let $P = (x, y)$ be the point on the unit circle that corresponds to the angle θ. If we know in which quadrant the point P lies, then we can determine the signs of the trigonometric functions of θ. For example, if $P = (x, y)$ lies in quadrant IV, as shown in Figure 39, then we know that $x > 0$ and $y < 0$. Consequently,

$$\sin\theta = y < 0 \qquad \cos\theta = x > 0 \qquad \tan\theta = \frac{y}{x} < 0$$

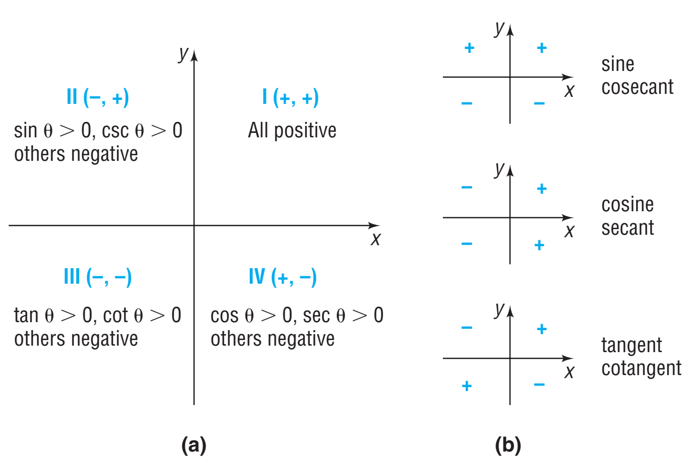
$$\csc\theta = \frac{1}{y} < 0 \qquad \sec\theta = \frac{1}{x} > 0 \qquad \cot\theta = \frac{x}{y} < 0$$

Table 5 lists the signs of the six trigonometric functions for each quadrant. See also Figure 40.

Table 5

Quadrant of P	$\sin\theta$, $\csc\theta$	$\cos\theta$, $\sec\theta$	$\tan\theta$, $\cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative

Figure 40



EXAMPLE 2 **Finding the Quadrant in Which an Angle θ Lies**

If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which the angle θ lies.

Solution Let $P = (x, y)$ be the point on the unit circle corresponding to θ. Then $\sin\theta = y < 0$ and $\cos\theta = x < 0$. Because points in quadrant III have $x < 0$ and $y < 0$, θ lies in quadrant III.

Now Work PROBLEM 27

4 Find the Values of the Trigonometric Functions Using Fundamental Identities

If $P = (x, y)$ is the point on the unit circle corresponding to θ, then

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x} \quad \text{if } x \neq 0$$

$$\csc\theta = \frac{1}{y} \quad \text{if } y \neq 0 \qquad \sec\theta = \frac{1}{x} \quad \text{if } x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad \text{if } y \neq 0$$

Based on these definitions, we have the **reciprocal identities:**

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \tag{2}$$

Two other fundamental identities are the **quotient identities.**

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \tag{3}$$

The proofs of identities (2) and (3) follow from the definitions of the trigonometric functions. (See Problems 129 and 130.)

If $\sin\theta$ and $\cos\theta$ are known, identities (2) and (3) make it easy to find the values of the remaining trigonometric functions.

EXAMPLE 3 **Finding Exact Values Using Identities When Sine and Cosine Are Given**

Given $\sin\theta = \dfrac{\sqrt{5}}{5}$ and $\cos\theta = \dfrac{2\sqrt{5}}{5}$, find the exact values of the four remaining trigonometric functions of θ using identities.

Solution Based on a quotient identity from (3), we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\sqrt{5}}{5}}{\dfrac{2\sqrt{5}}{5}} = \frac{1}{2}$$

Then we use the reciprocal identities from (2) to get

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{5}}{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2\sqrt{5}}{5}} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{1}{2}} = 2$$

Now Work PROBLEM 35

The equation of the unit circle is $x^2 + y^2 = 1$ or, equivalently,

$$y^2 + x^2 = 1$$

If $P = (x, y)$ is the point on the unit circle that corresponds to the angle θ, then $y = \sin\theta$ and $x = \cos\theta$, so we have

$$(\sin\theta)^2 + (\cos\theta)^2 = 1 \tag{4}$$

It is customary to write $\sin^2\theta$ instead of $(\sin\theta)^2$, $\cos^2\theta$ instead of $(\cos\theta)^2$, and so on. With this notation, we can rewrite equation (4) as

$$\sin^2\theta + \cos^2\theta = 1 \tag{5}$$

If $\cos\theta \neq 0$, we can divide each side of equation (5) by $\cos^2\theta$.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

Now use identities (2) and (3) to get

$$\tan^2\theta + 1 = \sec^2\theta \tag{6}$$

Similarly, if $\sin\theta \neq 0$, we can divide equation (5) by $\sin^2\theta$ and use identities (2) and (3) to get $1 + \cot^2\theta = \csc^2\theta$, which we write as

$$\cot^2\theta + 1 = \csc^2\theta \tag{7}$$

Collectively, the identities in (5), (6), and (7) are referred to as the **Pythagorean identities.**

Let's pause here to summarize the fundamental identities.

Fundamental Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

EXAMPLE 4 Finding the Exact Value of a Trigonometric Expression Using Identities

Find the exact value of each expression. Do not use a calculator.

(a) $\tan 20° - \dfrac{\sin 20°}{\cos 20°}$ (b) $\sin^2\dfrac{\pi}{12} + \dfrac{1}{\sec^2\dfrac{\pi}{12}}$

Solution (a) $\tan 20° - \dfrac{\sin 20°}{\cos 20°} = \tan 20° - \tan 20° = 0$

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

(b) $\sin^2\dfrac{\pi}{12} + \dfrac{1}{\sec^2\dfrac{\pi}{12}} = \sin^2\dfrac{\pi}{12} + \cos^2\dfrac{\pi}{12} = 1$

$\cos\theta = \dfrac{1}{\sec\theta}$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

Now Work PROBLEM 79

5 Find the Exact Values of the Trigonometric Functions of an Angle Given One of the Functions and the Quadrant of the Angle

Many problems require finding the exact values of the remaining trigonometric functions when the value of one of them is known and the quadrant in which θ lies can be found. There are two approaches to solving such problems. One approach uses a circle of radius r; the other uses identities.

When using identities, sometimes a rearrangement is required. For example, the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

can be solved for $\sin\theta$ in terms of $\cos\theta$ (or vice versa) as follows:

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

where the $+$ sign is used if $\sin\theta > 0$ and the $-$ sign is used if $\sin\theta < 0$. Similarly, in $\tan^2\theta + 1 = \sec^2\theta$, we can solve for $\tan\theta$ (or $\sec\theta$), and in $\cot^2\theta + 1 = \csc^2\theta$, we can solve for $\cot\theta$ (or $\csc\theta$).

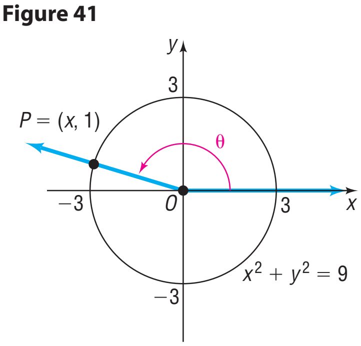
EXAMPLE 5 Finding Exact Values Given One Value and the Sign of Another

Given that $\sin\theta = \dfrac{1}{3}$ and $\cos\theta < 0$, find the exact value of each of the remaining five trigonometric functions.

Solution 1 Using a Circle

Suppose that $P = (x, y)$ is the point on a circle that corresponds to θ. Since $\sin\theta = \dfrac{1}{3} > 0$ and $\cos\theta < 0$, the point $P = (x, y)$ is in quadrant II. Because $\sin\theta = \dfrac{1}{3} = \dfrac{y}{r}$, we let $y = 1$ and $r = 3$. The point $P = (x, y) = (x, 1)$ that corresponds to θ lies on the circle of radius 3 centered at the origin: $x^2 + y^2 = 9$. See Figure 41.

Figure 41



To find x, we use the fact that $x^2 + y^2 = 9$, $y = 1$, and P is in quadrant II (so $x < 0$).

$$x^2 + y^2 = 9$$
$$x^2 + 1^2 = 9 \qquad y = 1$$
$$x^2 = 8$$
$$x = -2\sqrt{2} \qquad x < 0$$

Since $x = -2\sqrt{2}$, $y = 1$, and $r = 3$, we find that

$$\cos\theta = \frac{x}{r} = -\frac{2\sqrt{2}}{3} \qquad \tan\theta = \frac{y}{x} = \frac{1}{-2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

$$\csc\theta = \frac{r}{y} = \frac{3}{1} = 3 \qquad \sec\theta = \frac{r}{x} = \frac{3}{-2\sqrt{2}} = -\frac{3\sqrt{2}}{4} \qquad \cot\theta = \frac{x}{y} = \frac{-2\sqrt{2}}{1} = -2\sqrt{2}$$

Solution 2 Using Identities

First, solve the identity $\sin^2\theta + \cos^2\theta = 1$ for $\cos\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1 - \sin^2\theta} \end{aligned}$$

Because $\cos\theta < 0$, choose the minus sign and use the fact that $\sin\theta = \frac{1}{3}$.

$$\cos\theta = -\sqrt{1 - \sin^2\theta} \underset{\sin\theta = \frac{1}{3}}{\uparrow}= -\sqrt{1 - \frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

Now we know the values of $\sin\theta$ and $\cos\theta$, so we can use quotient and reciprocal identities to get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{3}}{\frac{-2\sqrt{2}}{3}} = \frac{1}{-2\sqrt{2}} = -\frac{\sqrt{2}}{4} \qquad \cot\theta = \frac{1}{\tan\theta} = -2\sqrt{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{-2\sqrt{2}}{3}} = \frac{-3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{3}} = 3$$

Finding the Values of the Trigonometric Functions of θ When the Value of One Function Is Known and the Quadrant of θ Is Known

Given the value of one trigonometric function and the quadrant in which θ lies, the exact value of each of the remaining five trigonometric functions can be found in either of two ways.

Method 1 Using a Circle of Radius *r*

STEP 1: Draw a circle centered at the origin showing the location of the angle θ and the point $P = (x, y)$ that corresponds to θ. The radius of the circle that contains $P = (x, y)$ is $r = \sqrt{x^2 + y^2}$.

STEP 2: Assign a value to two of the three variables x, y, r based on the value of the given trigonometric function and the location of P.

STEP 3: Use the fact that P lies on the circle $x^2 + y^2 = r^2$ to find the value of the missing variable.

STEP 4: Apply the theorem on page 374 to find the values of the remaining trigonometric functions.

Method 2 Using Identities

Use appropriately selected identities to find the value of each remaining trigonometric function.

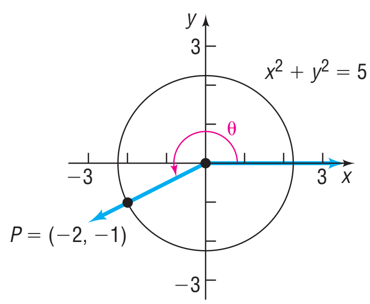
EXAMPLE 6 Given the Value of One Trigonometric Function and the Sign of Another, Find the Values of the Remaining Ones

Given that $\tan\theta = \frac{1}{2}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trigonometric functions of θ.

Solution 1 Using a Circle

Step 1: Since $\tan\theta = \frac{1}{2} > 0$ and $\sin\theta < 0$, the point $P = (x, y)$ that corresponds to θ lies in quadrant III. See Figure 42.

Figure 42



Step 2: Since $\tan\theta = \frac{1}{2} = \frac{y}{x}$ and θ lies in quadrant III, let $x = -2$ and $y = -1$.

Step 3: With $x = -2$ and $y = -1$, then $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$, P lies on the circle $x^2 + y^2 = 5$.

Step 4: So apply the theorem on page 374 using $x = -2$, $y = -1$, and $r = \sqrt{5}$.

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5} \qquad \cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-1} = -\sqrt{5} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2} \qquad \cot\theta = \frac{x}{y} = \frac{-2}{-1} = 2$$

Solution 2 Using Identities

Because we know the value of $\tan\theta$, we use the Pythagorean identity that involves $\tan\theta$, that is, $\tan^2\theta + 1 = \sec^2\theta$. Since $\tan\theta = \frac{1}{2} > 0$ and $\sin\theta < 0$, then θ lies in quadrant III, where $\sec\theta < 0$.

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{Pythagorean identity}$$

$$\left(\frac{1}{2}\right)^2 + 1 = \sec^2\theta \qquad \tan\theta = \frac{1}{2}$$

$$\sec^2\theta = \frac{1}{4} + 1 = \frac{5}{4} \qquad \text{Proceed to solve for } \sec\theta.$$

$$\sec\theta = -\frac{\sqrt{5}}{2} \qquad \sec\theta < 0$$

Now we know $\tan\theta = \frac{1}{2}$ and $\sec\theta = -\frac{\sqrt{5}}{2}$. Using reciprocal identities, we find

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\frac{\sqrt{5}}{2}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{1}{2}} = 2$$

To find $\sin\theta$, use the following reasoning:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{so} \quad \sin\theta = \tan\theta\cdot\cos\theta = \left(\frac{1}{2}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{\sqrt{5}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{\sqrt{5}}{5}} = -\frac{5}{\sqrt{5}} = -\sqrt{5}$$

Now Work PROBLEM 43

6 Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions

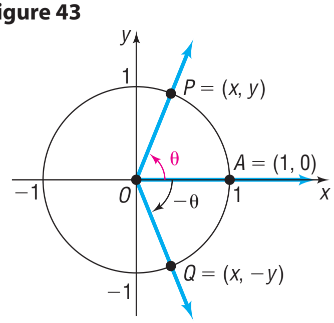
Recall that a function f is even if $f(-\theta) = f(\theta)$ for all θ in the domain of f; a function f is odd if $f(-\theta) = -f(\theta)$ for all θ in the domain of f. We will now show that the trigonometric functions sine, tangent, cotangent, and cosecant are odd functions and the functions cosine and secant are even functions.

In Words
Cosine and secant are even functions; the others are odd functions.

Even–Odd Properties

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Figure 43



Proof Let $P = (x, y)$ be the point on the unit circle that corresponds to the angle θ. See Figure 43. Using symmetry, the point Q on the unit circle that corresponds to the angle $-\theta$ will have coordinates $(x, -y)$. Using the definition of the trigonometric functions, we have

$$\sin\theta = y \qquad \sin(-\theta) = -y \qquad \cos\theta = x \qquad \cos(-\theta) = x$$

so

$$\sin(-\theta) = -y = -\sin\theta \qquad \cos(-\theta) = x = \cos\theta$$

Now, using these results and some of the fundamental identities, we have

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta \qquad \cot(-\theta) = \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta$$
$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta \qquad \csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta$$

■

EXAMPLE 7 **Finding Exact Values Using Even–Odd Properties**

Find the exact value of:

(a) $\sin(-45°)$ (b) $\cos(-\pi)$ (c) $\cot\left(-\frac{3\pi}{2}\right)$ (d) $\tan\left(-\frac{37\pi}{4}\right)$

Solution

(a) $\sin(-45°) = -\sin 45° = -\frac{\sqrt{2}}{2}$
Odd function

(b) $\cos(-\pi) = \cos\pi = -1$
Even function

(c) $\cot\left(-\frac{3\pi}{2}\right) = -\cot\frac{3\pi}{2} = 0$
Odd function

(d) $\tan\left(-\frac{37\pi}{4}\right) = -\tan\frac{37\pi}{4} = -\tan\left(\frac{\pi}{4} + 9\pi\right) = -\tan\frac{\pi}{4} = -1$
Odd function Period is π.

Now Work PROBLEM 59

6.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The domain of the function $f(x) = \dfrac{x+1}{2x+1}$ is ____________. (pp. 46–56)
2. A function for which $f(x) = f(-x)$ for all x in the domain of f is called a(n) _______ function. (pp. 69–70)
3. ***True or False*** The function $f(x) = \sqrt{x}$ is even. (pp. 69–70)
4. ***True or False*** The equation $x^2 + 2x = (x+1)^2 - 1$ is an identity. (p. A44)

Concepts and Vocabulary

5. The sine, cosine, cosecant, and secant functions have period _____; the tangent and cotangent functions have period ____.
6. The domain of the tangent function is _____.
7. The range of the sine function is ________.
8. ***True or False*** The only even trigonometric functions are the cosine and secant functions.
9. $\sin^2\theta + \cos^2\theta =$ _______.
10. ***True or False*** $\sec\theta = \dfrac{1}{\sin\theta}$

Skill Building

In Problems 11–26, use the fact that the trigonometric functions are periodic to find the exact value of each expression. Do not use a calculator.

11. $\sin 405°$
12. $\cos 420°$
13. $\tan 405°$
14. $\sin 390°$
15. $\csc 450°$
16. $\sec 540°$
17. $\cot 390°$
18. $\sec 420°$
19. $\cos \dfrac{33\pi}{4}$
20. $\sin \dfrac{9\pi}{4}$
21. $\tan(21\pi)$
22. $\csc \dfrac{9\pi}{2}$
23. $\sec \dfrac{17\pi}{4}$
24. $\cot \dfrac{17\pi}{4}$
25. $\tan \dfrac{19\pi}{6}$
26. $\sec \dfrac{25\pi}{6}$

In Problems 27–34, name the quadrant in which the angle θ lies.

27. $\sin\theta > 0,\quad \cos\theta < 0$
28. $\sin\theta < 0,\quad \cos\theta > 0$
29. $\sin\theta < 0,\quad \tan\theta < 0$
30. $\cos\theta > 0,\quad \tan\theta > 0$
31. $\cos\theta > 0,\quad \tan\theta < 0$
32. $\cos\theta < 0,\quad \tan\theta > 0$
33. $\sec\theta < 0,\quad \sin\theta > 0$
34. $\csc\theta > 0,\quad \cos\theta < 0$

In Problems 35–42, $\sin\theta$ *and* $\cos\theta$ *are given. Find the exact value of each of the four remaining trigonometric functions.*

35. $\sin\theta = -\dfrac{3}{5},\quad \cos\theta = \dfrac{4}{5}$
36. $\sin\theta = \dfrac{4}{5},\quad \cos\theta = -\dfrac{3}{5}$
37. $\sin\theta = \dfrac{2\sqrt{5}}{5},\quad \cos\theta = \dfrac{\sqrt{5}}{5}$
38. $\sin\theta = -\dfrac{\sqrt{5}}{5},\quad \cos\theta = -\dfrac{2\sqrt{5}}{5}$
39. $\sin\theta = \dfrac{1}{2},\quad \cos\theta = \dfrac{\sqrt{3}}{2}$
40. $\sin\theta = \dfrac{\sqrt{3}}{2},\quad \cos\theta = \dfrac{1}{2}$
41. $\sin\theta = -\dfrac{1}{3},\quad \cos\theta = \dfrac{2\sqrt{2}}{3}$
42. $\sin\theta = \dfrac{2\sqrt{2}}{3},\quad \cos\theta = -\dfrac{1}{3}$

In Problems 43–58, find the exact value of each of the remaining trigonometric functions of θ.

43. $\sin\theta = \dfrac{12}{13},\quad \theta$ in quadrant II
44. $\cos\theta = \dfrac{3}{5},\quad \theta$ in quadrant IV
45. $\cos\theta = -\dfrac{4}{5},\quad \theta$ in quadrant III
46. $\sin\theta = -\dfrac{5}{13},\quad \theta$ in quadrant III
47. $\sin\theta = \dfrac{5}{13},\quad 90° < \theta < 180°$
48. $\cos\theta = \dfrac{4}{5},\quad 270° < \theta < 360°$
49. $\cos\theta = -\dfrac{1}{3},\quad \dfrac{\pi}{2} < \theta < \pi$
50. $\sin\theta = -\dfrac{2}{3},\quad \pi < \theta < \dfrac{3\pi}{2}$
51. $\sin\theta = \dfrac{2}{3},\quad \tan\theta < 0$
52. $\cos\theta = -\dfrac{1}{4},\quad \tan\theta > 0$
53. $\sec\theta = 2,\quad \sin\theta < 0$
54. $\csc\theta = 3,\quad \cot\theta < 0$
55. $\tan\theta = \dfrac{3}{4},\quad \sin\theta < 0$
56. $\cot\theta = \dfrac{4}{3},\quad \cos\theta < 0$
57. $\tan\theta = -\dfrac{1}{3},\quad \sin\theta > 0$
58. $\sec\theta = -2,\quad \tan\theta > 0$

In Problems 59–76, use the even–odd properties to find the exact value of each expression. Do not use a calculator.

59. $\sin(-60°)$
60. $\cos(-30°)$
61. $\tan(-30°)$
62. $\sin(-135°)$
63. $\sec(-60°)$
64. $\csc(-30°)$
65. $\sin(-90°)$
66. $\cos(-270°)$
67. $\tan\left(-\frac{\pi}{4}\right)$
68. $\sin(-\pi)$
69. $\cos\left(-\frac{\pi}{4}\right)$
70. $\sin\left(-\frac{\pi}{3}\right)$
71. $\tan(-\pi)$
72. $\sin\left(-\frac{3\pi}{2}\right)$
73. $\csc\left(-\frac{\pi}{4}\right)$
74. $\sec(-\pi)$
75. $\sec\left(-\frac{\pi}{6}\right)$
76. $\csc\left(-\frac{\pi}{3}\right)$

In Problems 77–88, use properties of the trigonometric functions to find the exact value of each expression. Do not use a calculator.

77. $\sin^2 40° + \cos^2 40°$
78. $\sec^2 18° - \tan^2 18°$
79. $\sin 80° \csc 80°$
80. $\tan 10° \cot 10°$
81. $\tan 40° - \frac{\sin 40°}{\cos 40°}$
82. $\cot 20° - \frac{\cos 20°}{\sin 20°}$
83. $\cos 400° \cdot \sec 40°$
84. $\tan 200° \cdot \cot 20°$
85. $\sin\left(-\frac{\pi}{12}\right) \csc \frac{25\pi}{12}$
86. $\sec\left(-\frac{\pi}{18}\right) \cdot \cos \frac{37\pi}{18}$
87. $\frac{\sin(-20°)}{\cos 380°} + \tan 200°$
88. $\frac{\sin 70°}{\cos(-430°)} + \tan(-70°)$

89. If $\sin\theta = 0.3$, find the value of:

$$\sin\theta + \sin(\theta + 2\pi) + \sin(\theta + 4\pi)$$

90. If $\cos\theta = 0.2$, find the value of:

$$\cos\theta + \cos(\theta + 2\pi) + \cos(\theta + 4\pi)$$

91. If $\tan\theta = 3$, find the value of:

$$\tan\theta + \tan(\theta + \pi) + \tan(\theta + 2\pi)$$

92. If $\cot\theta = -2$, find the value of:

$$\cot\theta + \cot(\theta - \pi) + \cot(\theta - 2\pi)$$

93. Find the exact value of:

$$\sin 1° + \sin 2° + \sin 3° + \cdots + \sin 358° + \sin 359°$$

94. Find the exact value of:

$$\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 358° + \cos 359°$$

95. What is the domain of the sine function?

96. What is the domain of the cosine function?

97. For what numbers θ is $f(\theta) = \tan\theta$ not defined?

98. For what numbers θ is $f(\theta) = \cot\theta$ not defined?

99. For what numbers θ is $f(\theta) = \sec\theta$ not defined?

100. For what numbers θ is $f(\theta) = \csc\theta$ not defined?

101. What is the range of the sine function?

102. What is the range of the cosine function?

103. What is the range of the tangent function?

104. What is the range of the cotangent function?

105. What is the range of the secant function?

106. What is the range of the cosecant function?

107. Is the sine function even, odd, or neither? Is its graph symmetric? With respect to what?

108. Is the cosine function even, odd, or neither? Is its graph symmetric? With respect to what?

109. Is the tangent function even, odd, or neither? Is its graph symmetric? With respect to what?

110. Is the cotangent function even, odd, or neither? Is its graph symmetric? With respect to what?

111. Is the secant function even, odd, or neither? Is its graph symmetric? With respect to what?

112. Is the cosecant function even, odd, or neither? Is its graph symmetric? With respect to what?

Applications and Extensions

In Problems 113–118, use the periodic and even–odd properties.

113. If $f(\theta) = \sin\theta$ and $f(a) = \frac{1}{3}$, find the exact value of:

(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

114. If $f(\theta) = \cos\theta$ and $f(a) = \frac{1}{4}$, find the exact value of:

(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a - 2\pi)$

115. If $f(\theta) = \tan\theta$ and $f(a) = 2$, find the exact value of:

(a) $f(-a)$ (b) $f(a) + f(a + \pi) + f(a + 2\pi)$

116. If $f(\theta) = \cot\theta$ and $f(a) = -3$, find the exact value of:

(a) $f(-a)$ (b) $f(a) + f(a + \pi) + f(a + 4\pi)$

117. If $f(\theta) = \sec\theta$ and $f(a) = -4$, find the exact value of:

(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

118. If $f(\theta) = \csc\theta$ and $f(a) = 2$, find the exact value of:
(a) $f(-a)$ (b) $f(a) + f(a + 2\pi) + f(a + 4\pi)$

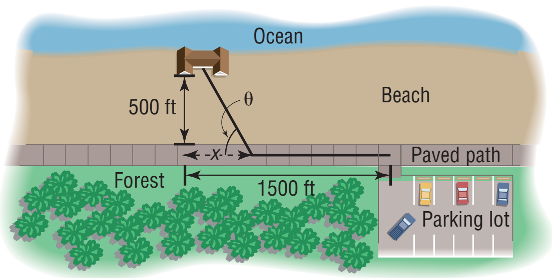
119. Calculating the Time of a Trip From a parking lot, you want to walk to a house on the beach. The house is located 1500 feet down a paved path that parallels the ocean, which is 500 feet away. See the illustration. Along the path you can walk 300 feet per minute, but in the sand on the beach you can only walk 100 feet per minute.

The time T to get from the parking lot to the beach house can be expressed as a function of the angle θ shown in the illustration and is

$$T(\theta) = 5 - \frac{5}{3\tan\theta} + \frac{5}{\sin\theta}, \qquad 0 < \theta < \frac{\pi}{2}$$

Calculate the time T if you walk directly from the parking lot to the house.

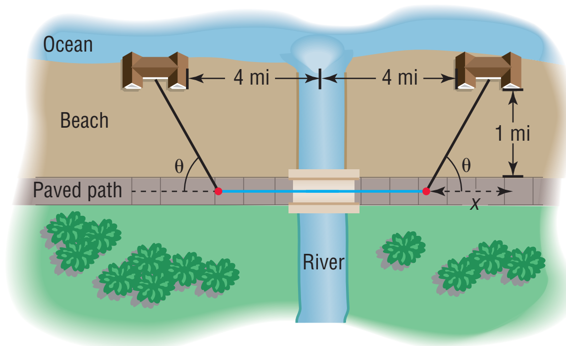
[**Hint:** $\tan\theta = \frac{500}{1500}$.]



120. Calculating the Time of a Trip Two oceanfront homes are located 8 miles apart on a straight stretch of beach, each a distance of 1 mile from a paved path that parallels the ocean. Sally can jog 8 miles per hour on the paved path, but only 3 miles per hour in the sand on the beach. Because a river flows directly between the two houses, it is necessary to jog in the sand to the road, continue on the path, and then jog directly back in the sand to get from one house to the other. See the illustration. The time T to get from one house to the other as a function of the angle θ shown in the illustration is

$$T(\theta) = 1 + \frac{2}{3\sin\theta} - \frac{1}{4\tan\theta} \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Calculate the time T for $\tan\theta = \frac{1}{4}$.
(b) Describe the path taken.
(c) Explain why θ must be larger than 14°.



121. Show that the range of the tangent function is the set of all real numbers.

122. Show that the range of the cotangent function is the set of all real numbers.

123. Show that the period of $f(\theta) = \sin\theta$ is 2π.
[**Hint:** Assume that $0 < p < 2\pi$ exists so that $\sin(\theta + p) = \sin\theta$ for all θ. Let $\theta = 0$ to find p. Then let $\theta = \frac{\pi}{2}$ to obtain a contradiction.]

124. Show that the period of $f(\theta) = \cos\theta$ is 2π.

125. Show that the period of $f(\theta) = \sec\theta$ is 2π.

126. Show that the period of $f(\theta) = \csc\theta$ is 2π.

127. Show that the period of $f(\theta) = \tan\theta$ is π.

128. Show that the period of $f(\theta) = \cot\theta$ is π.

129. Prove the reciprocal identities given in formula (2).

130. Prove the quotient identities given in formula (3).

131. Establish the identity:

$$(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = 1$$

Explaining Concepts: Discussion and Writing

132. Write down five properties of the tangent function. Explain the meaning of each.

133. Describe your understanding of the meaning of a periodic function.

134. Explain how to find the value of sin 390° using periodic properties.

135. Explain how to find the value of $\cos(-45°)$ using even–odd properties.

136. Explain how to find the value of sin 390° and $\cos(-45°)$ using the unit circle.

'Are You Prepared?' Answers

1. $\left\{x \middle| x \neq -\frac{1}{2}\right\}$ **2.** even **3.** False **4.** True

6.4 Graphs of the Sine and Cosine Functions*

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Graphing Techniques: Transformations (Section 2.5, pp. 90–99)

Now Work the **'Are You Prepared?'** problems on page 403.

OBJECTIVES
1. Graph Functions of the Form $y = A\sin(\omega x)$ Using Transformations (p. 394)
2. Graph Functions of the Form $y = A\cos(\omega x)$ Using Transformations (p. 396)
3. Determine the Amplitude and Period of Sinusoidal Functions (p. 397)
4. Graph Sinusoidal Functions Using Key Points (p. 398)
5. Find an Equation for a Sinusoidal Graph (p. 402)

Since we want to graph the trigonometric functions in the xy-plane, we shall use the traditional symbols x for the independent variable (or argument) and y for the dependent variable (or value at x) for each function. So we write the six trigonometric functions as

$$y = f(x) = \sin x \qquad y = f(x) = \cos x \qquad y = f(x) = \tan x$$
$$y = f(x) = \csc x \qquad y = f(x) = \sec x \qquad y = f(x) = \cot x$$

Here the independent variable x represents an angle, measured in radians. In calculus, x will usually be treated as a real number. As we said earlier, these are equivalent ways of viewing x.

The Graph of the Sine Function $y = \sin x$

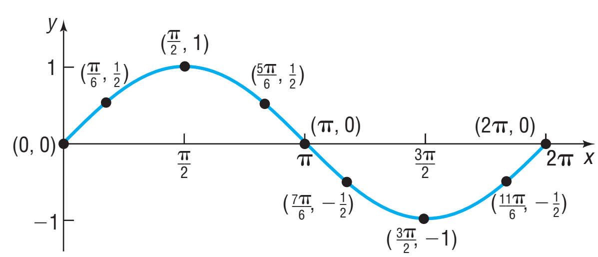
Since the sine function has period 2π, we only need to graph $y = \sin x$ on the interval $[0, 2\pi]$. The remainder of the graph will consist of repetitions of this portion of the graph.

We begin by constructing Table 6, which lists some points on the graph of $y = \sin x$, $0 \le x \le 2\pi$. As the table shows, the graph of $y = \sin x$, $0 \le x \le 2\pi$, begins at the origin. As x increases from 0 to $\frac{\pi}{2}$, the value of $y = \sin x$ increases from 0 to 1; as x increases from $\frac{\pi}{2}$ to π to $\frac{3\pi}{2}$, the value of y decreases from 1 to 0 to -1; as x increases from $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0. If we plot the points listed in Table 6 and connect them with a smooth curve, we obtain the graph shown in Figure 44.

Table 6

x	$y = \sin x$	(x, y)
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
π	0	$(\pi, 0)$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
2π	0	$(2\pi, 0)$

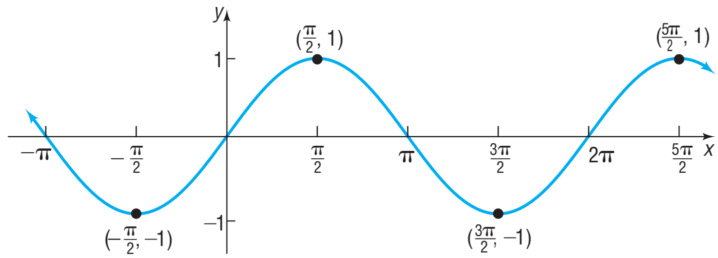
Figure 44
$y = \sin x, 0 \le x \le 2\pi$



The graph in Figure 44 is one period, or **cycle,** of the graph of $y = \sin x$. To obtain a more complete graph of $y = \sin x$, continue the graph in each direction, as shown in Figure 45.

* For those who wish to include phase shifts here, Section 6.6 can be covered immediately after Section 6.4 without loss of continuity.

Figure 45
$y = \sin x, -\infty < x < \infty$



The graph of $y = \sin x$ illustrates some of the facts that we already know about the sine function.

Properties of the Sine Function $y = \sin x$

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The sine function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The sine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
6. The absolute maximum is 1 and occurs at $x = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$; the absolute minimum is -1 and occurs at $x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$.

Now Work PROBLEM 9

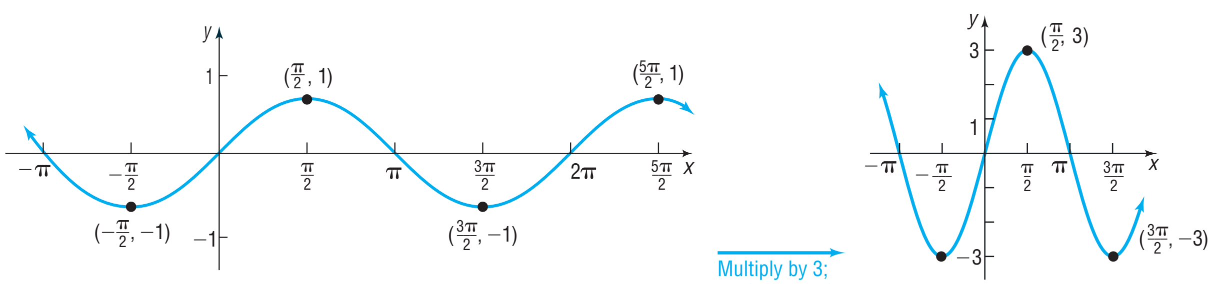
1 Graph Functions of the Form $y = A\sin(\omega x)$ Using Transformations

EXAMPLE 1 **Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations**

Graph $y = 3\sin x$ using transformations.

Solution Figure 46 illustrates the steps.

Figure 46



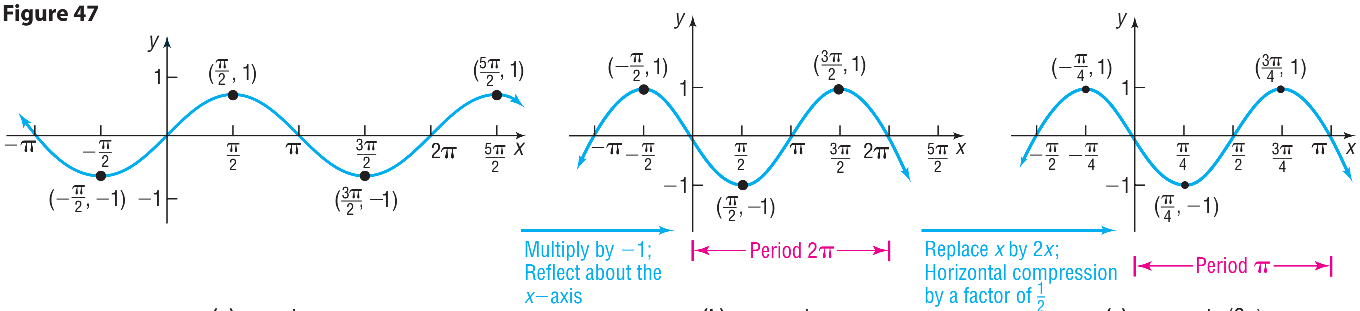
(a) $y = \sin x$ **(b)** $y = 3\sin x$

EXAMPLE 2 **Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations**

Graph $y = -\sin(2x)$ using transformations.

Solution Figure 47 illustrates the steps.

Figure 47



(a) $y = \sin x$ (b) $y = -\sin x$ (c) $y = -\sin(2x)$

Notice in Figure 47(c) that the period of the function $y = -\sin(2x)$ is π due to the horizontal compression of the original period 2π by a factor of $\frac{1}{2}$.

Now Work PROBLEM 37 USING TRANSFORMATIONS

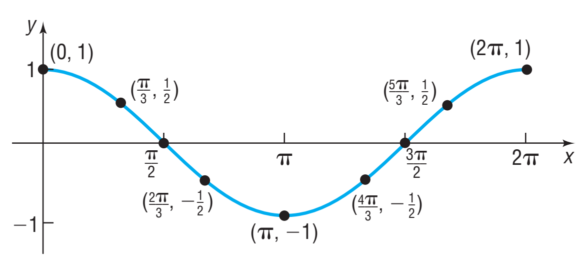
The Graph of the Cosine Function

The cosine function also has period 2π. We proceed as we did with the sine function by constructing Table 7, which lists some points on the graph of $y = \cos x$, $0 \le x \le 2\pi$. As the table shows, the graph of $y = \cos x$, $0 \le x \le 2\pi$, begins at the point $(0, 1)$. As x increases from 0 to $\frac{\pi}{2}$ to π, the value of y decreases from 1 to 0 to -1; as x increases from π to $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0 to 1. As before, plot the points in Table 7 to get one period or cycle of the graph. See Figure 48.

Table 7

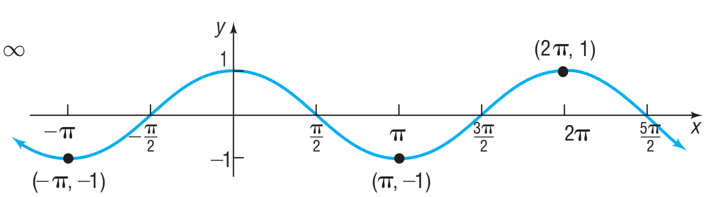
x	$y = \cos x$	(x, y)
0	1	$(0, 1)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3}, \frac{1}{2}\right)$
2π	1	$(2\pi, 1)$

Figure 48
$y = \cos x, 0 \le x \le 2\pi$



A more complete graph of $y = \cos x$ is obtained by continuing the graph in each direction, as shown in Figure 49.

Figure 49
$y = \cos x, -\infty < x < \infty$



The graph of $y = \cos x$ illustrates some of the facts that we already know about the cosine function.

Properties of the Cosine Function

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The cosine function is an even function, as the symmetry of the graph with respect to the y-axis indicates.
4. The cosine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$; the y-intercept is 1.
6. The absolute maximum is 1 and occurs at $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$; the absolute minimum is -1 and occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$.

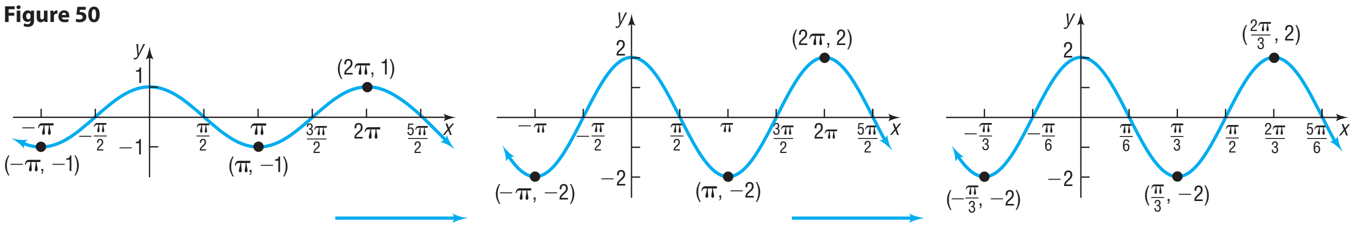
2 Graph Functions of the Form $y = A\cos(\omega x)$ Using Transformations

EXAMPLE 3 **Graphing Functions of the Form $y = A\cos(\omega x)$ Using Transformations**

Graph $y = 2\cos(3x)$ using transformations.

Solution Figure 50 shows the steps.

Figure 50



(a) $y = \cos x$ (b) $y = 2\cos x$ (c) $y = 2\cos(3x)$

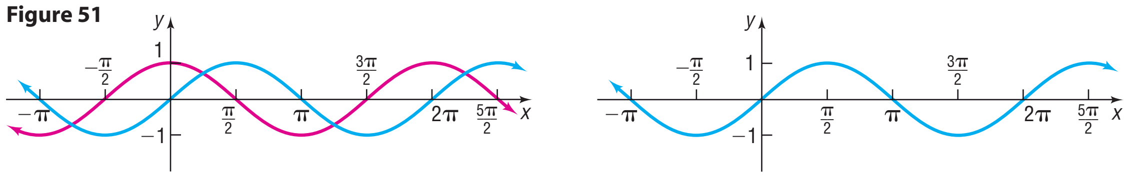
Notice in Figure 50(c) that the period of the function $y = 2\cos(3x)$ is $\frac{2\pi}{3}$ due to the compression of the original period 2π by a factor of $\frac{1}{3}$.

Now Work PROBLEM 45 USING TRANSFORMATIONS

Sinusoidal Graphs

Shift the graph of $y = \cos x$ to the right $\frac{\pi}{2}$ units to obtain the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$. See Figure 51(a). Now look at the graph of $y = \sin x$ in Figure 51(b). We see that the graph of $y = \sin x$ is the same as the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$.

Figure 51


(a) $y = \cos x$ $y = \cos\left(x - \frac{\pi}{2}\right)$ (b) $y = \sin x$

Based on Figure 51, we conjecture that

$$\sin x = \cos\left(x - \frac{\pi}{2}\right)$$

Seeing the Concept

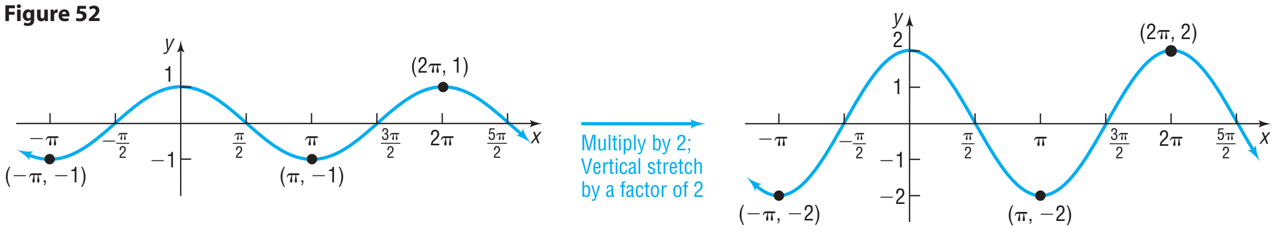
Graph $Y_1 = \sin x$ and $Y_2 = \cos\left(x - \frac{\pi}{2}\right)$. How many graphs do you see?

(We shall prove this fact in Chapter 7.) Because of this relationship, the graphs of functions of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ are referred to as **sinusoidal graphs.**

3 Determine the Amplitude and Period of Sinusoidal Functions

In Figure 52(b) we show the graph of $y = 2\cos x$. Notice that the values of $y = 2\cos x$ lie between -2 and 2, inclusive.

Figure 52



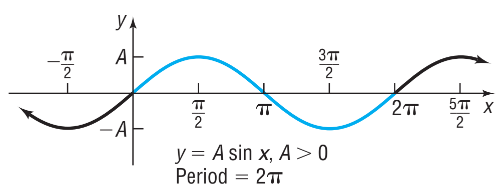
(a) $y = \cos x$ **(b)** $y = 2\cos x$

In general, the values of the functions $y = A\sin x$ and $y = A\cos x$, where $A \neq 0$, will always satisfy the inequalities

$$-|A| \leq A\sin x \leq |A| \quad \text{and} \quad -|A| \leq A\cos x \leq |A|$$

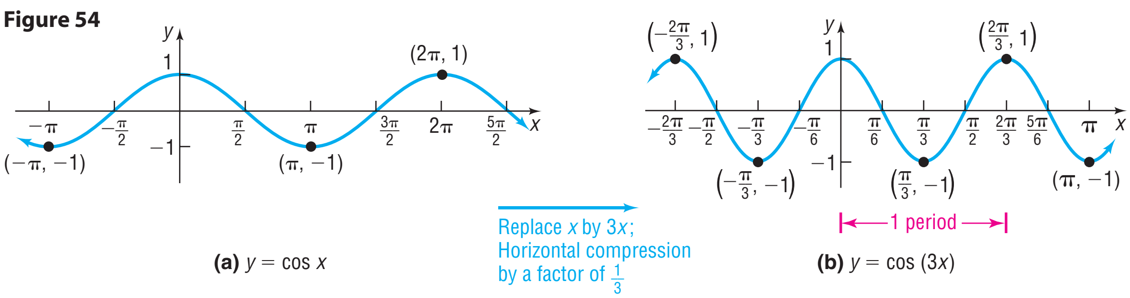
respectively. The number $|A|$ is called the **amplitude** of $y = A\sin x$ or $y = A\cos x$. See Figure 53.

Figure 53



In Figure 54(b), we show the graph of $y = \cos(3x)$. Notice that the period of this function is $\frac{2\pi}{3}$, due to the horizontal compression of the original period 2π by a factor of $\frac{1}{3}$.

Figure 54

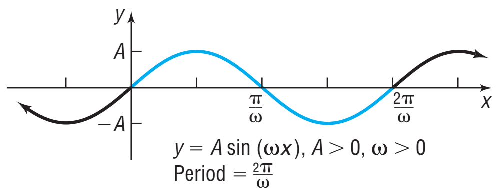


(a) $y = \cos x$ **(b)** $y = \cos(3x)$

In general, if $\omega > 0$, the functions $y = \sin(\omega x)$ and $y = \cos(\omega x)$ will have period $T = \frac{2\pi}{\omega}$. To see why, recall that the graph of $y = \sin(\omega x)$ is obtained from the graph of $y = \sin x$ by performing a horizontal compression or stretch by a factor $\frac{1}{\omega}$. This horizontal compression replaces the interval $[0, 2\pi]$, which contains one period of the graph of $y = \sin x$, by the interval $\left[0, \frac{2\pi}{\omega}\right]$, which contains one period of the graph of $y = \sin(\omega x)$. So, the function $y = \cos(3x)$, graphed in Figure 54(b), with $\omega = 3$, has period $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.

One period of the graph of $y = \sin(\omega x)$ or $y = \cos(\omega x)$ is called a **cycle.** Figure 55 illustrates the general situation. The blue portion of the graph is one cycle.

Figure 55



NOTE Recall that a function f is even if $f(-x) = f(x)$; a function f is odd if $f(-x) = -f(x)$. Since the sine function is odd, $\sin(-x) = -\sin x$; since the cosine function is even, $\cos(-x) = \cos x$.

When graphing $y = \sin(\omega x)$ or $y = \cos(\omega x)$, we want ω to be positive. To graph $y = \sin(-\omega x)$, $\omega > 0$ or $y = \cos(-\omega x)$, $\omega > 0$, we use the Even–Odd Properties of the sine and cosine functions as follows:

$$\sin(-\omega x) = -\sin(\omega x) \quad \text{and} \quad \cos(-\omega x) = \cos(\omega x)$$

This gives us an equivalent form in which the coefficient of x in the argument is positive. For example,

$$\sin(-2x) = -\sin(2x) \quad \text{and} \quad \cos(-\pi x) = \cos(\pi x)$$

Because of this, we can assume that $\omega > 0$.

THEOREM

If $\omega > 0$, the amplitude and period of $y = A\sin(\omega x)$ and $y = A\cos(\omega x)$ are given by

$$\text{Amplitude} = |A| \qquad \text{Period} = T = \frac{2\pi}{\omega} \tag{1}$$

EXAMPLE 4 **Finding the Amplitude and Period of a Sinusoidal Function**

Determine the amplitude and period of $y = 3\sin(4x)$.

Solution Comparing $y = 3\sin(4x)$ to $y = A\sin(\omega x)$, we find that $A = 3$ and $\omega = 4$. From equation (1),

$$\text{Amplitude} = |A| = 3 \qquad \text{Period} = T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Now Work PROBLEM 15

4 Graph Sinusoidal Functions Using Key Points

So far, we have graphed functions of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ using transformations. We now introduce another method that can be used to graph these functions.

Figure 56 shows one cycle of the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$. Notice that each graph consists of four parts corresponding to the four subintervals:

$$\left[0, \frac{\pi}{2}\right], \quad \left[\frac{\pi}{2}, \pi\right], \quad \left[\pi, \frac{3\pi}{2}\right], \quad \left[\frac{3\pi}{2}, 2\pi\right]$$

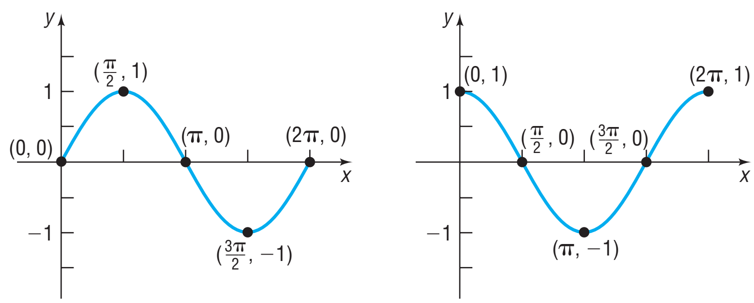
Each subinterval is of length $\frac{\pi}{2}$ (the period 2π divided by 4, the number of parts), and the endpoints of these intervals $x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi$ give rise to five key points on each graph:

For $y = \sin x$: $\quad (0, 0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right), (2\pi, 0)$

For $y = \cos x$: $\quad (0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$

Look again at Figure 56.

Figure 56



(a) $y = \sin x$ **(b)** $y = \cos x$

EXAMPLE 5 How to Graph a Sinusoidal Function Using Key Points

Graph $y = 3\sin(4x)$ using key points.

Step-by-Step Solution

Step 1: Determine the amplitude and period of the sinusoidal function.

Comparing $y = 3\sin(4x)$ to $y = A\sin(\omega x)$, we see that $A = 3$ and $\omega = 4$, so the amplitude is $|A| = 3$ and the period is $\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$. Because the amplitude is 3, the graph of $y = 3\sin(4x)$ will lie between -3 and 3 on the y-axis. Because the period is $\frac{\pi}{2}$, one cycle will begin at $x = 0$ and end at $x = \frac{\pi}{2}$.

Step 2: Divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

Divide the interval $\left[0, \frac{\pi}{2}\right]$ into four subintervals, each of length $\frac{\pi}{2} \div 4 = \frac{\pi}{8}$, as follows:

$$\left[0, \frac{\pi}{8}\right] \quad \left[\frac{\pi}{8}, \frac{\pi}{8} + \frac{\pi}{8}\right] = \left[\frac{\pi}{8}, \frac{\pi}{4}\right] \quad \left[\frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{8}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{8}\right] \quad \left[\frac{3\pi}{8}, \frac{3\pi}{8} + \frac{\pi}{8}\right] = \left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$$

The endpoints of the subintervals are $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$. These values represent the x-coordinates of the five key points on the graph.

Step 3: Use the endpoints of these subintervals to obtain five key points on the graph.

To obtain the y-coordinates of the five key points of $y = 3\sin(4x)$, multiply the y-coordinates of the five key points for $y = \sin x$ in Figure 56(a) by $A = 3$. The five key points are

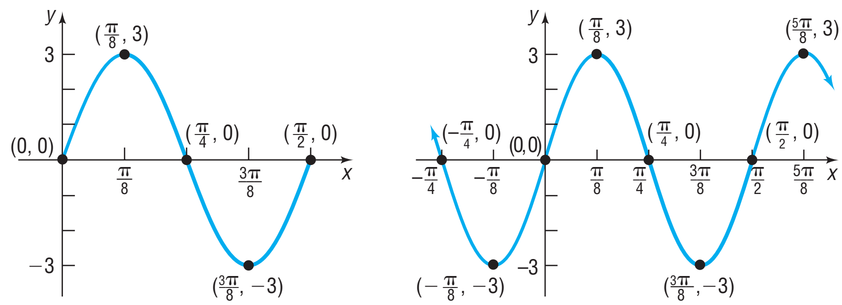
$$(0, 0) \quad \left(\frac{\pi}{8}, 3\right) \quad \left(\frac{\pi}{4}, 0\right) \quad \left(\frac{3\pi}{8}, -3\right) \quad \left(\frac{\pi}{2}, 0\right)$$

COMMENT We could also obtain the five key points by evaluating $y = 3\sin(4x)$ at each value of x. ■

Step 4: Plot the five key points and draw a sinusoidal graph to obtain the graph of one cycle. Extend the graph in each direction to make it complete.

Plot the five key points obtained in Step 3 and fill in the graph of the sine curve as shown in Figure 57(a). Extend the graph in each direction to obtain the complete graph shown in Figure 57(b). Notice that additional key points appear every $\frac{\pi}{8}$ radian.

Figure 57



(a) (b) $y = 3\sin(4x)$

✓**Check:** Graph $y = 3\sin(4x)$ using transformations. Which graphing method do you prefer?

Now Work PROBLEM 37 USING KEY POINTS

SUMMARY Steps for Graphing a Sinusoidal Function of the Form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ Using Key Points

STEP 1: Determine the amplitude and period of the sinusoidal function.

STEP 2: Divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

STEP 3: Use the endpoints of these subintervals to obtain five key points on the graph.

STEP 4: Plot the five key points and draw a sinusoidal graph to obtain the graph of one cycle. Extend the graph in each direction to make it complete.

EXAMPLE 6 Graphing a Sinusoidal Function Using Key Points

Graph $y = 2\sin\left(-\frac{\pi}{2}x\right)$ using key points.

Solution Since the sine function is odd, we can use the equivalent form:

$$y = -2\sin\left(\frac{\pi}{2}x\right)$$

STEP 1: Comparing $y = -2\sin\left(\frac{\pi}{2}x\right)$ to $y = A\sin(\omega x)$, we find that $A = -2$ and $\omega = \frac{\pi}{2}$. The amplitude is $|A| = |-2| = 2$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.

The graph of $y = -2\sin\left(\frac{\pi}{2}x\right)$ will lie between -2 and 2 on the y-axis. One cycle will begin at $x = 0$ and end at $x = 4$.

Step 2: Divide the interval $[0, 4]$ into four subintervals, each of length $4 \div 4 = 1$. The x-coordinates of the five key points are

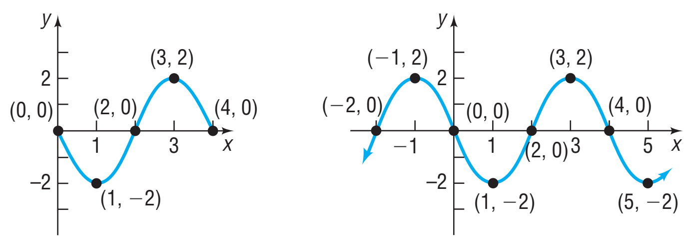
0	$0 + 1 = 1$	$1 + 1 = 2$	$2 + 1 = 3$	$3 + 1 = 4$
1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate

Step 3: Since $y = -2\sin\left(\frac{\pi}{2}x\right)$, multiply the y-coordinates of the five key points in Figure 56(a) by $A = -2$. The five key points on the graph are

$$(0, 0) \quad (1, -2) \quad (2, 0) \quad (3, 2) \quad (4, 0)$$

Step 4: Plot these five points and fill in the graph of the sine function as shown in Figure 58(a). Extend the graph in each direction to obtain Figure 58(b).

Figure 58



(a) **(b)** $y = 2\sin\left(-\frac{\pi}{2}x\right)$

COMMENT To graph a sinusoidal function of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ using a graphing utility, use the amplitude to set Y_{min} and Y_{max} and use the period to set X_{min} and X_{max}. ■

Check: Graph $y = 2\sin\left(-\frac{\pi}{2}x\right)$ using transformations. Which graphing method do you prefer?

Now Work PROBLEM 41 USING KEY POINTS

If the function to be graphed is of the form $y = A\sin(\omega x) + B$ [or $y = A\cos(\omega x) + B$], first graph $y = A\sin(\omega x)$ [or $y = A\cos(\omega x)$] and then use a vertical shift.

EXAMPLE 7 Graphing a Sinusoidal Function Using Key Points

Graph $y = -4\cos(\pi x) - 2$ using key points. Use the graph to determine the domain and the range of $y = -4\cos(\pi x) - 2$.

Solution Begin by graphing the function $y = -4\cos(\pi x)$. Comparing $y = -4\cos(\pi x)$ with $y = A\cos(\omega x)$, we find that $A = -4$ and $\omega = \pi$. The amplitude is $|A| = |-4| = 4$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.

The graph of $y = -4\cos(\pi x)$ will lie between -4 and 4 on the y-axis. One cycle will begin at $x = 0$ and end at $x = 2$.

Divide the interval $[0, 2]$ into four subintervals, each of length $2 \div 4 = \frac{1}{2}$. The x-coordinates of the five key points are

0	$0 + \frac{1}{2} = \frac{1}{2}$	$\frac{1}{2} + \frac{1}{2} = 1$	$1 + \frac{1}{2} = \frac{3}{2}$	$\frac{3}{2} + \frac{1}{2} = 2$
1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate

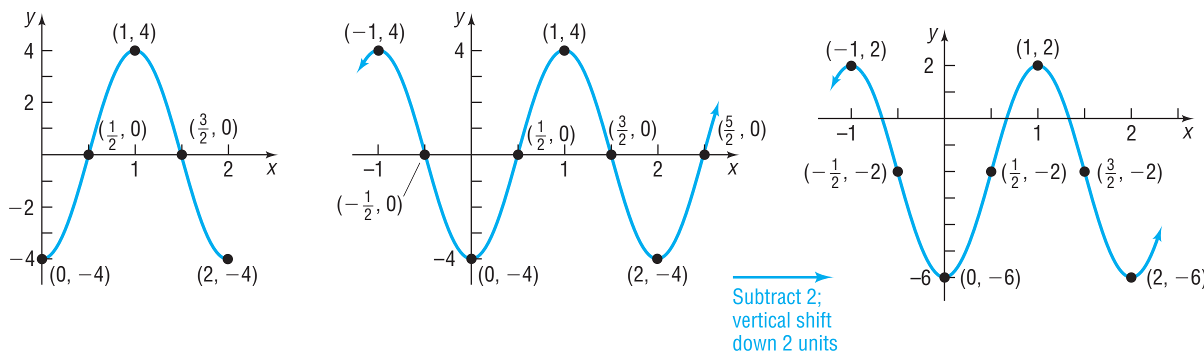
Since $y = -4\cos(\pi x)$, multiply the y-coordinates of the five key points of $y = \cos x$ shown in Figure 56(b) by $A = -4$ to obtain the five key points on the graph of $y = -4\cos(\pi x)$:

$$(0, -4) \quad \left(\frac{1}{2}, 0\right) \quad (1, 4) \quad \left(\frac{3}{2}, 0\right) \quad (2, -4)$$

Plot these five points and fill in the graph of the cosine function as shown in Figure 59(a). Extending the graph in each direction, we obtain Figure 59(b), the graph of $y = -4\cos(\pi x)$.

A vertical shift down 2 units gives the graph of $y = -4\cos(\pi x) - 2$, as shown in Figure 59(c).

Figure 59



(a) (b) $y = -4\cos(\pi x)$ (c) $y = -4\cos(\pi x) - 2$

The domain of $y = -4\cos(\pi x) - 2$ is the set of all real numbers or $(-\infty, \infty)$. The range of $y = -4\cos(\pi x) - 2$ is $\{y \mid -6 \le y \le 2\}$ or $[-6, 2]$.

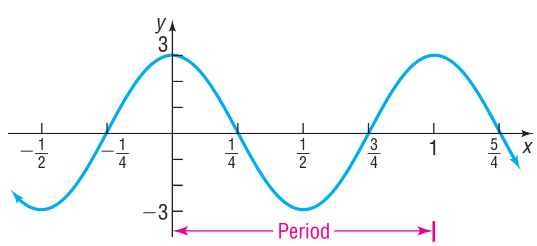
Now Work PROBLEM 51

5 Find an Equation for a Sinusoidal Graph

EXAMPLE 8 Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 60.

Figure 60



Solution The graph has the characteristics of a cosine function. Do you see why? The maximum value, 3, occurs at $x = 0$. So we view the equation as a cosine function $y = A\cos(\omega x)$ with $A = 3$ and period $T = 1$. Then $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. The cosine function whose graph is given in Figure 60 is

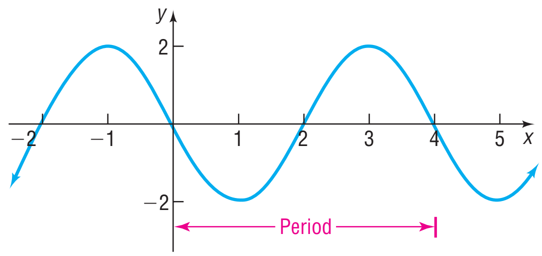
$$y = A\cos(\omega x) = 3\cos(2\pi x)$$

Check: Graph $Y_1 = 3\cos(2\pi x)$ and compare the result with Figure 60.

EXAMPLE 9 Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 61.

Figure 61



Solution The graph is sinusoidal, with amplitude $|A| = 2$. The period is 4, so $\frac{2\pi}{\omega} = 4$ or $\omega = \frac{\pi}{2}$. Since the graph passes through the origin, it is easier to view the equation as a sine function,[†] but notice that the graph is actually the reflection of a sine function about the x-axis (since the graph is decreasing near the origin). This requires that $A = -2$. The sine function whose graph is given in Figure 61 is

$$y = A\sin(\omega x) = -2\sin\left(\frac{\pi}{2}x\right)$$

Check: Graph $Y_1 = -2\sin\left(\frac{\pi}{2}x\right)$ and compare the result with Figure 61.

Now Work PROBLEMS 59 AND 63

6.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Use transformations to graph $y = 3x^2$. (pp. 90–99)

2. Use transformations to graph $y = \sqrt{2x}$. (pp. 90–99)

Concepts and Vocabulary

3. The maximum value of $y = \sin x, 0 \le x \le 2\pi$, is ________ and occurs at $x =$ ________.

4. The function $y = A\sin(\omega x)$, $A > 0$, has amplitude 3 and period 2; then $A =$ ________ and ω ________.

5. The function $y = 3\cos(6x)$ has amplitude ________ and period ________.

6. ***True or False*** The graphs of $y = \sin x$ and $y = \cos x$ are identical except for a horizontal shift.

7. ***True or False*** For $y = 2\sin(\pi x)$, the amplitude is 2 and the period is $\frac{\pi}{2}$.

8. ***True or False*** The graph of the sine function has infinitely many x-intercepts.

Skill Building

9. $f(x) = \sin x$

(a) What is the y-intercept of the graph of f?

(b) For what numbers x, $-\pi \le x \le \pi$, is the graph of f increasing?

(c) What is the absolute maximum of f?

(d) For what numbers x, $0 \le x \le 2\pi$, does $f(x) = 0$?

(e) For what numbers x, $-2\pi \le x \le 2\pi$, does $f(x) = 1$? Where does $f(x) = -1$?

(f) For what numbers x, $-2\pi \le x \le 2\pi$, does $f(x) = -\frac{1}{2}$?

(g) What are the x-intercepts of f?

[†] The equation could also be viewed as a cosine function with a horizontal shift, but viewing it as a sine function is easier.

10. $g(x) = \cos x$

(a) What is the y-intercept of the graph of g?

(b) For what numbers x, $-\pi \le x \le \pi$, is the graph of g decreasing?

(c) What is the absolute minimum of g?

(d) For what numbers x, $0 \le x \le 2\pi$, does $g(x) = 0$?

(e) For what numbers x, $-2\pi \le x \le 2\pi$, does $g(x) = 1$? Where does $g(x) = -1$?

(f) For what numbers x, $-2\pi \le x \le 2\pi$, does $g(x) = \dfrac{\sqrt{3}}{2}$?

(g) What are the x-intercepts of g?

In Problems 11–20, determine the amplitude and period of each function without graphing.

11. $y = 2\sin x$

12. $y = 3\cos x$

13. $y = -4\cos(2x)$

14. $y = -\sin\left(\dfrac{1}{2}x\right)$

15. $y = 6\sin(\pi x)$

16. $y = -3\cos(3x)$

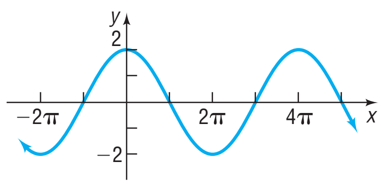
17. $y = -\dfrac{1}{2}\cos\left(\dfrac{3}{2}x\right)$

18. $y = \dfrac{4}{3}\sin\left(\dfrac{2}{3}x\right)$

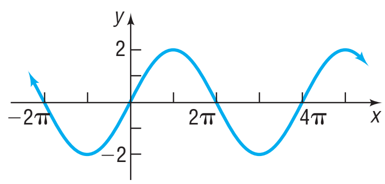
19. $y = \dfrac{5}{3}\sin\left(-\dfrac{2\pi}{3}x\right)$

20. $y = \dfrac{9}{5}\cos\left(-\dfrac{3\pi}{2}x\right)$

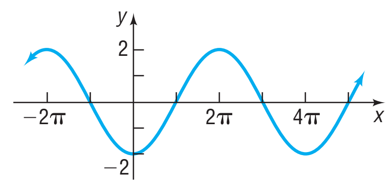
In Problems 21–30, match the given function to one of the graphs (A)–(J).



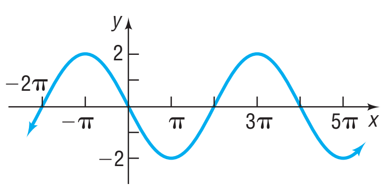
(A)



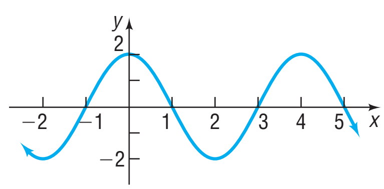
(B)



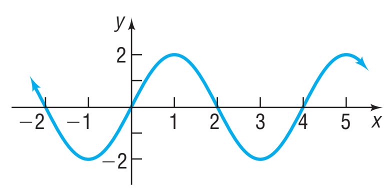
(C)



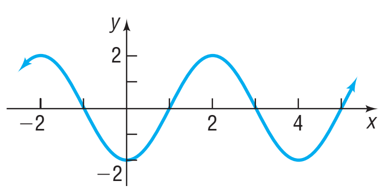
(D)



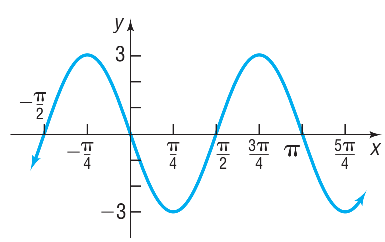
(E)



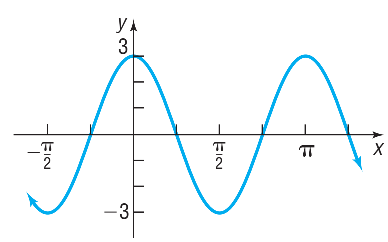
(F)



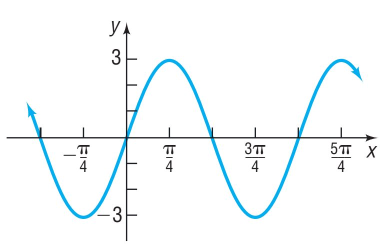
(G)



(H)



(I)



(J)

21. $y = 2\sin\left(\dfrac{\pi}{2}x\right)$

22. $y = 2\cos\left(\dfrac{\pi}{2}x\right)$

23. $y = 2\cos\left(\dfrac{1}{2}x\right)$

24. $y = 3\cos(2x)$

25. $y = -3\sin(2x)$

26. $y = 2\sin\left(\dfrac{1}{2}x\right)$

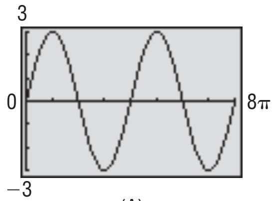
27. $y = -2\cos\left(\dfrac{1}{2}x\right)$

28. $y = -2\cos\left(\dfrac{\pi}{2}x\right)$

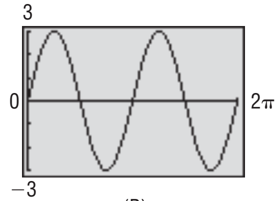
29. $y = 3\sin(2x)$

30. $y = -2\sin\left(\dfrac{1}{2}x\right)$

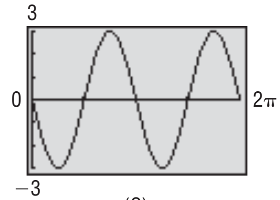
In Problems 31–34, match the given function to one of the graphs (A)–(D).



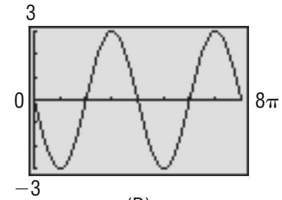
(A)



(B)



(C)



(D)

31. $y = 3\sin\left(\frac{1}{2}x\right)$ **32.** $y = -3\sin(2x)$ **33.** $y = 3\sin(2x)$ **34.** $y = -3\sin\left(\frac{1}{2}x\right)$

In Problems 35–58, graph each function. Be sure to label key points and show at least two cycles. Use the graph to determine the domain and the range of each function.

35. $y = 4\cos x$

36. $y = 3\sin x$

37. $y = -4\sin x$

38. $y = -3\cos x$

39. $y = \cos(4x)$

40. $y = \sin(3x)$

41. $y = \sin(-2x)$

42. $y = \cos(-2x)$

43. $y = 2\sin\left(\frac{1}{2}x\right)$

44. $y = 2\cos\left(\frac{1}{4}x\right)$

45. $y = -\frac{1}{2}\cos(2x)$

46. $y = -4\sin\left(\frac{1}{8}x\right)$

47. $y = 2\sin x + 3$

48. $y = 3\cos x + 2$

49. $y = 5\cos(\pi x) - 3$

50. $y = 4\sin\left(\frac{\pi}{2}x\right) - 2$

51. $y = -6\sin\left(\frac{\pi}{3}x\right) + 4$

52. $y = -3\cos\left(\frac{\pi}{4}x\right) + 2$

53. $y = 5 - 3\sin(2x)$

54. $y = 2 - 4\cos(3x)$

55. $y = \frac{5}{3}\sin\left(-\frac{2\pi}{3}x\right)$

56. $y = \frac{9}{5}\cos\left(-\frac{3\pi}{2}x\right)$

57. $y = -\frac{3}{2}\cos\left(\frac{\pi}{4}x\right) + \frac{1}{2}$

58. $y = -\frac{1}{2}\sin\left(\frac{\pi}{8}x\right) + \frac{3}{2}$

In Problems 59–62, write the equation of a sine function that has the given characteristics.

59. Amplitude: 3
Period: π

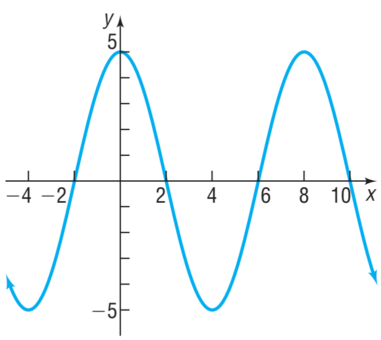
60. Amplitude: 2
Period: 4π

61. Amplitude: 3
Period: 2

62. Amplitude: 4
Period: 1

In Problems 63–76, find an equation for each graph.

63.



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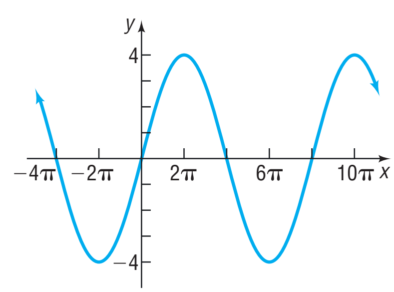

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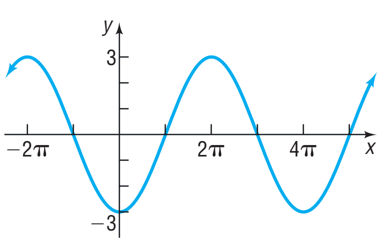

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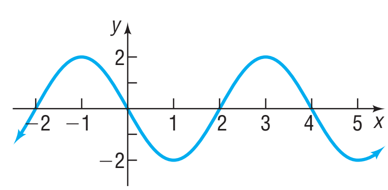

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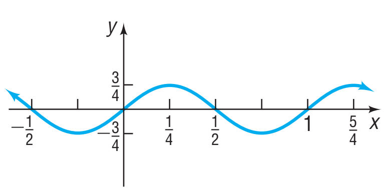

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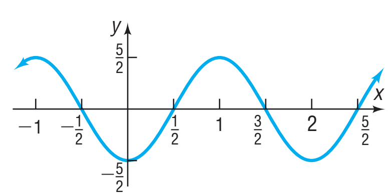

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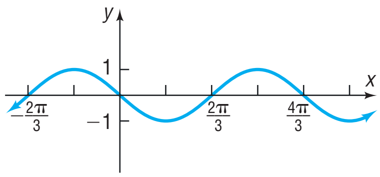

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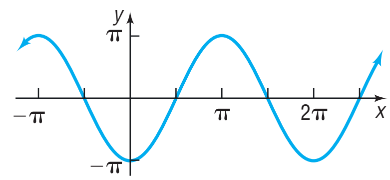

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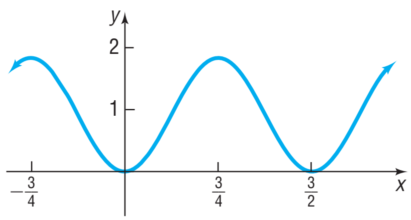
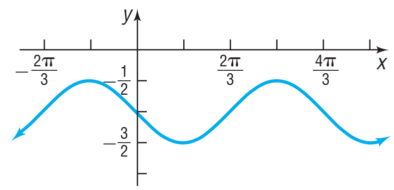


72.



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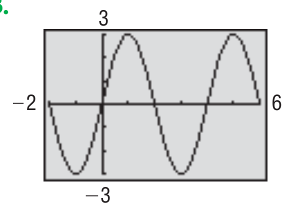

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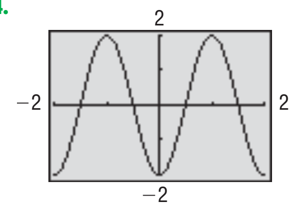

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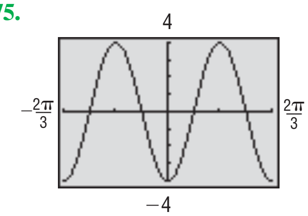

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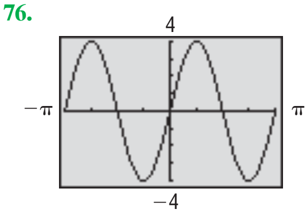


Mixed Practice

In Problems 77–80, find the average rate of change of f from 0 *to* $\frac{\pi}{2}$.

77. $f(x) = \sin x$ **78.** $f(x) = \cos x$ **79.** $f(x) = \sin\left(\frac{x}{2}\right)$ **80.** $f(x) = \cos(2x)$

In Problems 81–84, find $(f \circ g)(x)$ *and* $(g \circ f)(x)$ *and graph each of these functions.*

81. $f(x) = \sin x$, $g(x) = 4x$

82. $f(x) = \cos x$, $g(x) = \frac{1}{2}x$

83. $f(x) = -2x$, $g(x) = \cos x$

84. $f(x) = -3x$, $g(x) = \sin x$

In Problems 85 and 86, graph each function.

85. $f(x) = \begin{cases} \sin x & 0 \le x < \frac{5\pi}{4} \\ \cos x & \frac{5\pi}{4} \le x \le 2\pi \end{cases}$

86. $g(x) = \begin{cases} 2\sin x & 0 \le x \le \pi \\ \cos x + 1 & \pi < x \le 2\pi \end{cases}$

Applications and Extensions

87. Alternating Current (ac) Circuits The current I, in amperes, flowing through an ac (alternating current) circuit at time t in seconds, is

$$I(t) = 220\sin(60\pi t) \qquad t \ge 0$$

What is the period? What is the amplitude? Graph this function over two periods.

88. Alternating Current (ac) Circuits The current I, in amperes, flowing through an ac (alternating current) circuit at time t in seconds, is

$$I(t) = 120\sin(30\pi t) \qquad t \ge 0$$

What is the period? What is the amplitude? Graph this function over two periods.

89. Alternating Current (ac) Generators The voltage V, in volts, produced by an ac generator at time t, in seconds, is

$$V(t) = 220\sin(120\pi t)$$

(a) What is the amplitude? What is the period?
(b) Graph V over two periods, beginning at $t = 0$.
(c) If a resistance of $R = 10$ ohms is present, what is the current I?
[**Hint:** Use Ohm's Law, $V = IR$.]
(d) What is the amplitude and period of the current I?
(e) Graph I over two periods, beginning at $t = 0$.

90. Alternating Current (ac) Generators The voltage V, in volts, produced by an ac generator at time t, in seconds, is

$$V(t) = 120\sin(120\pi t)$$

(a) What is the amplitude? What is the period?
(b) Graph V over two periods, beginning at $t = 0$.
(c) If a resistance of $R = 20$ ohms is present, what is the current I?
[**Hint:** Use Ohm's Law, $V = IR$.]
(d) What is the amplitude and period of the current I?
(e) Graph I over two periods, beginning at $t = 0$.

91. Alternating Current (ac) Generators The voltage V produced by an ac generator is sinusoidal. As a function of time, the voltage V is

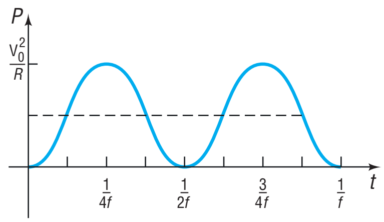
$$V(t) = V_0 \sin(2\pi f t)$$

where f is the **frequency,** the number of complete oscillations (cycles) per second. [In the United States and Canada, f is 60 hertz (Hz).] The **power** P delivered to a resistance R at any time t is defined as

$$P(t) = \frac{[V(t)]^2}{R}$$

(a) Show that $P(t) = \frac{V_0^2}{R}\sin^2(2\pi f t)$.

(b) The graph of P is shown in the figure. Express P as a sinusoidal function.

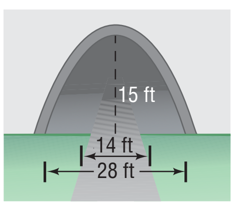


Power in an ac generator

(c) Deduce that

$$\sin^2(2\pi f t) = \frac{1}{2}[1 - \cos(4\pi f t)]$$

92. Bridge Clearance A one-lane highway runs through a tunnel in the shape of one-half a sine curve cycle. The opening is 28 feet wide at road level and is 15 feet tall at its highest point.



(a) Find an equation for the sine curve that fits the opening. Place the origin at the left end of the sine curve.

(b) If the road is 14 feet wide with 7-foot shoulders on each side, what is the height of the tunnel at the edge of the road?

Source: en.wikipedia.org/wiki/Interstate_Highway_standards and *Ohio Revised Code*

93. Biorhythms In the theory of biorhythms, a sine function of the form

$$P(t) = 50\sin(\omega t) + 50$$

is used to measure the percent P of a person's potential at time t, where t is measured in days and $t = 0$ is the person's birthday. Three characteristics are commonly measured:

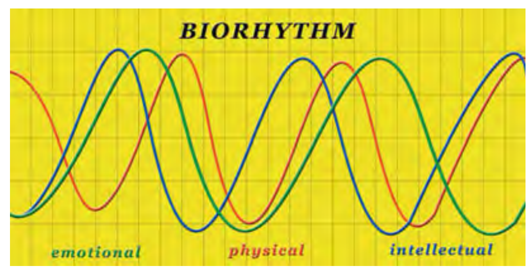
Physical potential: period of 23 days
Emotional potential: period of 28 days
Intellectual potential: period of 33 days

(a) Find ω for each characteristic.

(b) Using a graphing utility, graph all three functions on the same screen.

(c) Is there a time t when all three characteristics have 100% potential? When is it?

(d) Suppose that you are 20 years old today ($t = 7305$ days). Describe your physical, emotional, and intellectual potential for the next 30 days.



94. Graph $y = |\cos x|$, $-2\pi \le x \le 2\pi$.

95. Graph $y = |\sin x|$, $-2\pi \le x \le 2\pi$.

In Problems 96–99, the graphs of the given pairs of functions intersect infinitely many times. Find four of these points of intersection.

96. $y = \sin x$, $y = \frac{1}{2}$

97. $y = \cos x$, $y = \frac{1}{2}$

98. $y = 2\sin x$, $y = -2$

99. $y = \tan x$, $y = 1$

Explaining Concepts: Discussion and Writing

100. Explain how you would scale the x-axis and y-axis before graphing $y = 3\cos(\pi x)$.

101. Explain the term *amplitude* as it relates to the graph of a sinusoidal function.

102. Explain the term *period* as it relates to the graph of a sinusoidal function.

103. Explain how the amplitude and period of a sinusoidal graph are used to establish the scale on each coordinate axis.

104. Find an application in your major field that leads to a sinusoidal graph. Write a paper about your findings.

Interactive Exercises

Ask your instructor if the applet exercises below are of interest to you.

105. *Open the Trace Sine Curve applet.* On the screen you will see the graph of the unit circle with a point C labeled. Use your mouse and move point C around the unit circle in the counter-clockwise direction. What do you notice? In particular, what is the relation between the angle and the y-coordinate of point C?

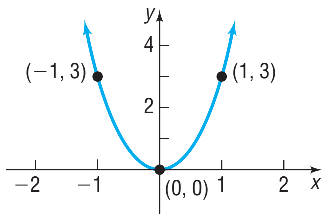
106. *Open the Trace Cosine Curve applet.* On the screen you will see the graph of the unit circle with a point C labeled. Use your mouse and move point C around the unit circle in the counter-clockwise direction. What do you notice? In particular, what is the relation between the angle and the x-coordinate of point C?

107. *Open the Amplitude applet.* On the screen you will see a slider. Move the point along the slider to see the role a plays in the graph of $f(x) = a \sin x$.

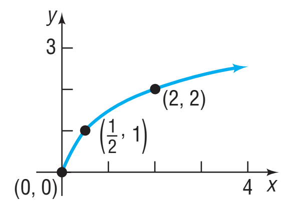
108. *Open the Period applet.* On the screen you will see a slider. Move the point along the slider to see the role ω plays in the graph of $f(x) = \sin(\omega x)$. Pay particular attention to the key points matched by color on each graph. For convenience the graph of $g(x) = \sin x$ is shown as a dashed, gray curve.

'Are You Prepared?' Answers

1. Vertical stretch by a factor of 3



2. Horizontal compression by a factor of $\frac{1}{2}$



6.5 Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Vertical Asymptotes (Section 4.2, pp. 191–192)

Now Work the **'Are You Prepared?'** problems on page 413.

OBJECTIVES
1. Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$ (p. 410)
2. Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$ (p. 412)

The Graph of the Tangent Function

Because the tangent function has period π, we only need to determine the graph over some interval of length π. The rest of the graph will consist of repetitions of that graph. Because the tangent function is not defined at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, we will concentrate on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, of length π, and construct Table 8, which lists some points on the graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We plot the points in the table and connect them with a smooth curve. See Figure 62 for a partial graph of $y = \tan x$, where $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

To complete one period of the graph of $y = \tan x$, we need to investigate the behavior of the function as x approaches $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We must be careful, though, because $y = \tan x$ is not defined at these numbers. To determine this behavior, we use the identity

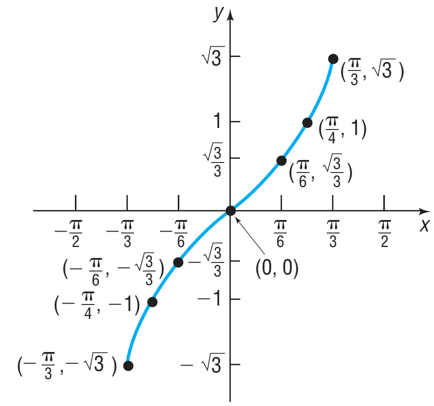
$$\tan x = \frac{\sin x}{\cos x}$$

See Table 9. If x is close to $\frac{\pi}{2} \approx 1.5708$, but remains less than $\frac{\pi}{2}$, then $\sin x$ will be close to 1 and $\cos x$ will be positive and close to 0. (To see this, refer back to the graphs of the sine function and the cosine function.) So the ratio $\frac{\sin x}{\cos x}$ will be

Table 8

x	$y = \tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.73$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -0.58$	$\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx 0.58$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$

Figure 62
$y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$



positive and large. In fact, the closer x gets to $\frac{\pi}{2}$, the closer $\sin x$ gets to 1 and $\cos x$ gets to 0, so $\tan x$ approaches $\infty \left(\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty\right)$. In other words, the vertical line $x = \frac{\pi}{2}$ is a vertical asymptote to the graph of $y = \tan x$.

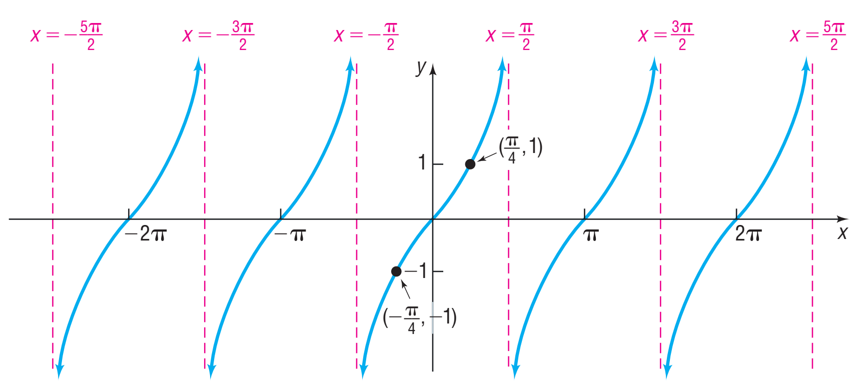
Table 9

x	$\sin x$	$\cos x$	$y = \tan x$
$\frac{\pi}{3} \approx 1.05$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3} \approx 1.73$
1.5	0.9975	0.0707	14.1
1.57	0.9999	7.96×10^{-4}	1255.8
1.5707	0.9999	9.6×10^{-5}	10,381
$\frac{\pi}{2} \approx 1.5708$	1	0	Undefined

If x is close to $-\frac{\pi}{2}$, but remains greater than $-\frac{\pi}{2}$, then $\sin x$ will be close to -1 and $\cos x$ will be positive and close to 0. The ratio $\frac{\sin x}{\cos x}$ approaches $-\infty$ $\left(\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty\right)$. In other words, the vertical line $x = -\frac{\pi}{2}$ is also a vertical asymptote to the graph.

With these observations, we can complete one period of the graph. We obtain the complete graph of $y = \tan x$ by repeating this period, as shown in Figure 63.

Figure 63
$y = \tan x, -\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$



Check: Graph $Y_1 = \tan x$ and compare the result with Figure 63. Use TRACE to see what happens as x gets close to $\frac{\pi}{2}$, but is less than $\frac{\pi}{2}$.

The graph of $y = \tan x$ in Figure 63 on page 409 illustrates the following properties.

Properties of the Tangent Function

1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The tangent function is periodic, with period π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

Now Work PROBLEMS 7 AND 15

1 Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$

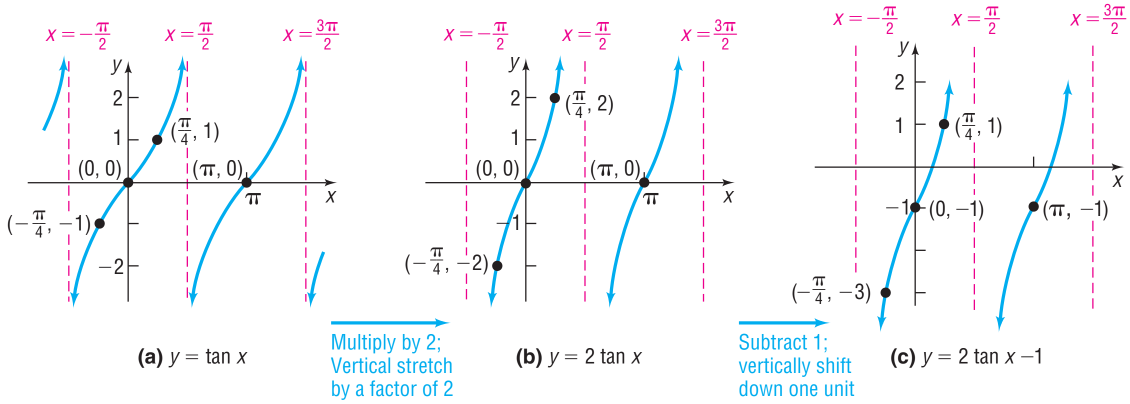
For tangent functions, there is no concept of amplitude since the range of the tangent function is $(-\infty, \infty)$. The role of A in $y = A\tan(\omega x) + B$ is to provide the magnitude of the vertical stretch. The period of $y = \tan x$ is π, so the period of $y = A\tan(\omega x) + B$ is $\frac{\pi}{\omega}$, caused by the horizontal compression of the graph by a factor of $\frac{1}{\omega}$. Finally, the presence of B indicates that a vertical shift is required.

EXAMPLE 1 **Graphing Functions of the Form $y = A\tan(\omega x) + B$**

Graph: $y = 2\tan x - 1$. Use the graph to determine the domain and the range of $y = 2\tan x - 1$.

Solution Figure 64 shows the steps using transformations.

Figure 64



✓Check: Graph $Y_1 = 2\tan x - 1$ to verify the graph shown in Figure 64(c).

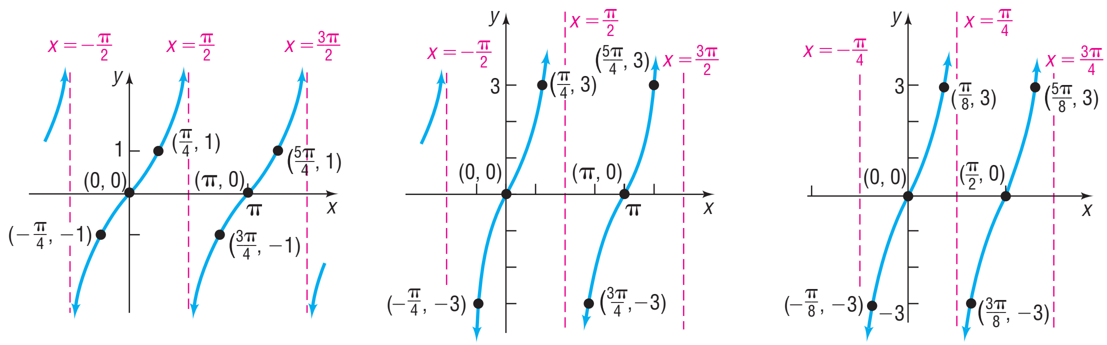
The domain of $y = 2\tan x - 1$ is $\left\{x \middle| x \neq \frac{k\pi}{2}, k \text{ is an odd integer}\right\}$, and the range is the set of all real numbers, or $(-\infty, \infty)$.

EXAMPLE 2 **Graphing Functions of the Form $y = A\tan(\omega x) + B$**

Graph $y = 3\tan(2x)$. Use the graph to determine the domain and the range of $y = 3\tan(2x)$.

Solution Figure 65 shows the steps using transformations.

Figure 65



(a) $y = \tan x$ — Multiply by 3; Vertical stretch by a factor of 3 — **(b)** $y = 3\tan x$ — Replace x by $2x$; Horizontal compression by a factor of $\frac{1}{2}$ — **(c)** $y = 3\tan(2x)$

The domain of $y = 3\tan(2x)$ is $\left\{x \,\middle|\, x \neq \frac{k\pi}{4}, k \text{ is an odd integer}\right\}$, and the range is the set of all real numbers or $(-\infty, \infty)$.

✓Check: Graph $Y_1 = 3\tan(2x)$ to verify the graph in Figure 65(c).

Notice in Figure 65(c) that the period of $y = 3\tan(2x)$ is $\frac{\pi}{2}$ due to the compression of the original period π by a factor of $\frac{1}{2}$. Notice that the asymptotes are $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, and so on, also due to the compression.

Now Work PROBLEM 21

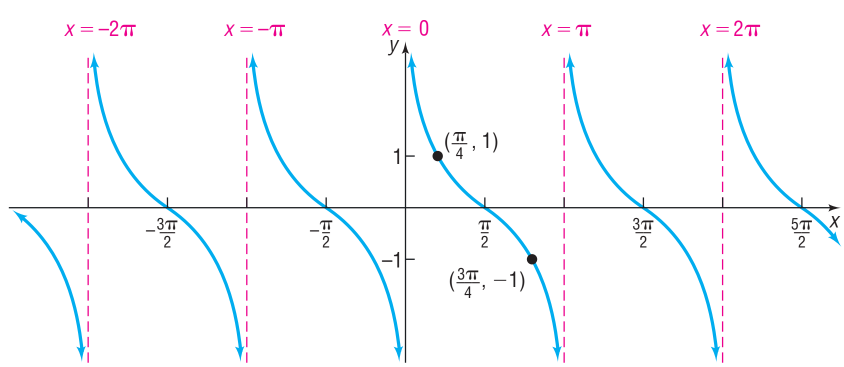
The Graph of the Cotangent Function

We obtain the graph of $y = \cot x$ as we did the graph of $y = \tan x$. The period of $y = \cot x$ is π. Because the cotangent function is not defined for integer multiples of π, we will concentrate on the interval $(0, \pi)$. Table 10 lists some points on the graph of $y = \cot x$, $0 < x < \pi$. As x approaches 0, but remains greater than 0, the value of $\cos x$ will be close to 1 and the value of $\sin x$ will be positive and close to 0. Hence, the ratio $\frac{\cos x}{\sin x} = \cot x$ will be positive and large; so as x approaches 0, with $x > 0$, $\cot x$ approaches $\infty \left(\lim_{x \to 0^+} \cot x = \infty\right)$. Similarly, as x approaches π, but remains less than π, the value of $\cos x$ will be close to -1, and the value of $\sin x$ will be positive and close to 0. So the ratio $\frac{\cos x}{\sin x} = \cot x$ will be negative and will approach $-\infty$ as x approaches $\pi \left(\lim_{x \to \pi^-} \cot x = -\infty\right)$. Figure 66 shows the graph.

Table 10

x	$y = \cot x$	(x, y)
$\frac{\pi}{6}$	$\sqrt{3}$	$\left(\frac{\pi}{6}, \sqrt{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3}$	$\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3}$	$\left(\frac{2\pi}{3}, -\frac{\sqrt{3}}{3}\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\left(\frac{5\pi}{6}, -\sqrt{3}\right)$

Figure 66
$y = \cot x$, $-\infty < x < \infty$, x not equal to integer multiples of π, $-\infty < y < \infty$



The graph of $y = A\cot(\omega x) + B$ has similar characteristics to those of the tangent function. The cotangent function $y = A\cot(\omega x) + B$ has period $\frac{\pi}{\omega}$. The cotangent function has no amplitude. The role of A is to provide the magnitude of the vertical stretch; the presence of B indicates a vertical shift is required.

Now Work PROBLEM 23

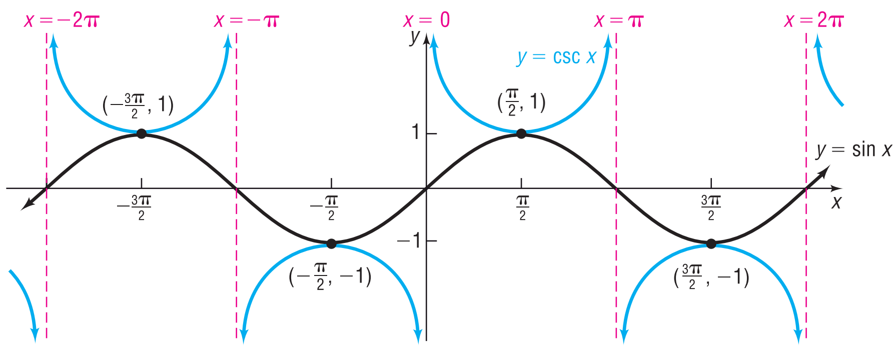
The Graphs of the Cosecant Function and the Secant Function

The cosecant and secant functions, sometimes referred to as **reciprocal functions,** are graphed by making use of the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

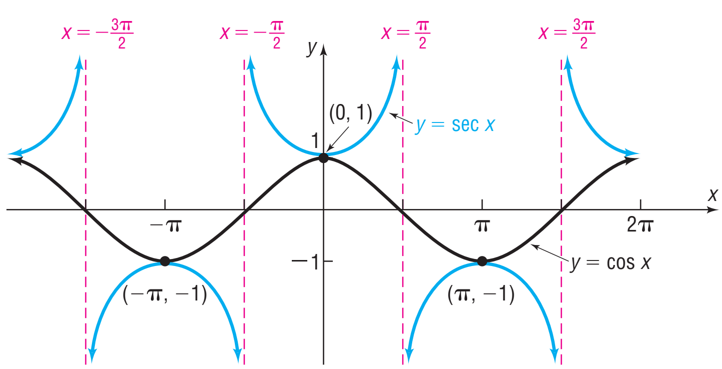
For example, the value of the cosecant function $y = \csc x$ at a given number x equals the reciprocal of the corresponding value of the sine function, provided that the value of the sine function is not 0. If the value of $\sin x$ is 0, then x is an integer multiple of π. At such numbers, the cosecant function is not defined. In fact, the graph of the cosecant function has vertical asymptotes at integer multiples of π. Figure 67 shows the graph.

Figure 67
$y = \csc x$, $-\infty < x < \infty$, x not equal to integer multiples of π, $|y| \geq 1$



Using the idea of reciprocals, we can similarly obtain the graph of $y = \sec x$. See Figure 68.

Figure 68
$y = \sec x$, $-\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $|y| \geq 1$



2 Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$

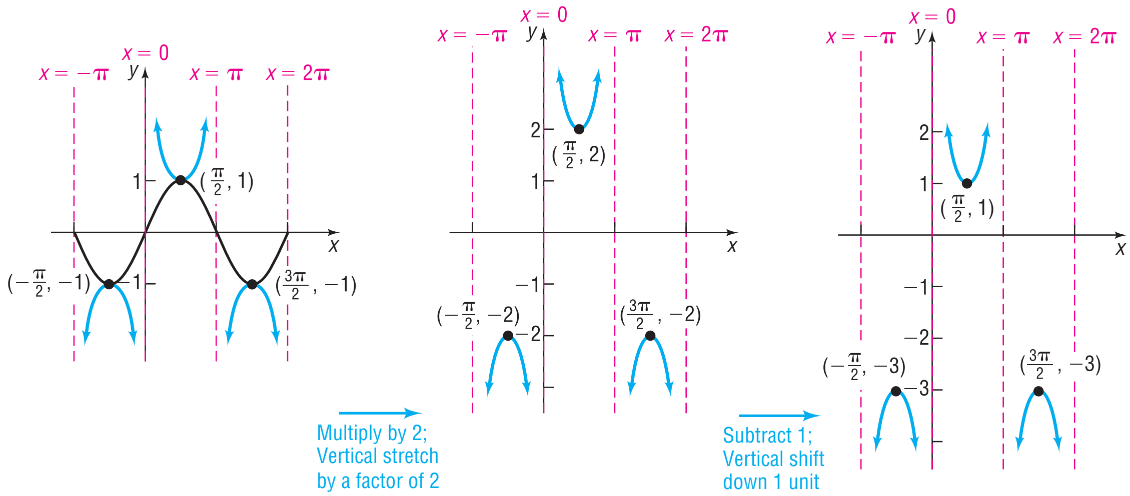
The role of A in these functions is to set the range. The range of $y = \csc x$ is $\{y | y \leq -1 \text{ or } y \geq 1\}$ or $\{y | |y| \geq 1\}$; the range of $y = A\csc x$ is $\{y | |y| \geq |A|\}$, due to the vertical stretch of the graph by a factor of $|A|$. Just as with the sine and cosine functions, the period of $y = \csc(\omega x)$ and $y = \sec(\omega x)$ becomes $\dfrac{2\pi}{\omega}$, due to the horizontal compression of the graph by a factor of $\dfrac{1}{\omega}$. The presence of B indicates that a vertical shift is required.

EXAMPLE 3 Graphing Functions of the Form $y = A\csc(\omega x) + B$

Graph $y = 2\csc x - 1$. Use the graph to determine the domain and the range of $y = 2\csc x - 1$.

Solution We use transformations. Figure 69 shows the required steps.

Figure 69



(a) $y = \csc x$ **(b)** $y = 2\csc x$ **(c)** $y = 2\csc x - 1$

The domain of $y = 2\csc x - 1$ is $\{x \mid x \neq k\pi, k \text{ is an integer}\}$ and the range is $\{y \mid y \leq -3 \text{ or } y \geq 1\}$ or, using interval notation, $(-\infty, -3] \cup [1, \infty)$.

✓**Check:** Graph $Y_1 = 2\csc x - 1$ to verify the graph shown in Figure 69.

Now Work PROBLEM 29

6.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The graph of $y = \dfrac{3x - 6}{x - 4}$ has a vertical asymptote. What is it? (pp. 191–192)

2. ***True or False*** If $x = 3$ is a vertical asymptote of a rational function R, then $\lim_{x \to 3} |R(x)| = \infty$. (pp. 191–192)

Concepts and Vocabulary

3. The graph of $y = \tan x$ is symmetric with respect to the _______ and has vertical asymptotes at ______________.

4. The graph of $y = \sec x$ is symmetric with respect to the _______ and has vertical asymptotes at ______________.

5. It is easiest to graph $y = \sec x$ by first sketching the graph of __________.

6. ***True or False*** The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ each have infinitely many vertical asymptotes.

Skill Building

In Problems 7–16, if necessary, refer to the graphs to answer each question.

7. What is the y-intercept of $y = \tan x$?

8. What is the y-intercept of $y = \cot x$?

9. What is the y-intercept of $y = \sec x$?

10. What is the y-intercept of $y = \csc x$?

11. For what numbers x, $-2\pi \le x \le 2\pi$, does $\sec x = 1$? For what numbers x does $\sec x = -1$?

12. For what numbers x, $-2\pi \le x \le 2\pi$, does $\csc x = 1$? For what numbers x does $\csc x = -1$?

13. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \sec x$ have vertical asymptotes?

14. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \csc x$ have vertical asymptotes?

15. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \tan x$ have vertical asymptotes?

16. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \cot x$ have vertical asymptotes?

In Problems 17–40, graph each function. Be sure to label key points and show at least two cycles. Use the graph to determine the domain and the range of each function.

17. $y = 3\tan x$

18. $y = -2\tan x$

19. $y = 4\cot x$

20. $y = -3\cot x$

21. $y = \tan\left(\frac{\pi}{2}x\right)$

22. $y = \tan\left(\frac{1}{2}x\right)$

23. $y = \cot\left(\frac{1}{4}x\right)$

24. $y = \cot\left(\frac{\pi}{4}x\right)$

25. $y = 2\sec x$

26. $y = \frac{1}{2}\csc x$

27. $y = -3\csc x$

28. $y = -4\sec x$

29. $y = 4\sec\left(\frac{1}{2}x\right)$

30. $y = \frac{1}{2}\csc(2x)$

31. $y = -2\csc(\pi x)$

32. $y = -3\sec\left(\frac{\pi}{2}x\right)$

33. $y = \tan\left(\frac{1}{4}x\right) + 1$

34. $y = 2\cot x - 1$

35. $y = \sec\left(\frac{2\pi}{3}x\right) + 2$

36. $y = \csc\left(\frac{3\pi}{2}x\right)$

37. $y = \frac{1}{2}\tan\left(\frac{1}{4}x\right) - 2$

38. $y = 3\cot\left(\frac{1}{2}x\right) - 2$

39. $y = 2\csc\left(\frac{1}{3}x\right) - 1$

40. $y = 3\sec\left(\frac{1}{4}x\right) + 1$

Mixed Practice

In Problems 41–44, find the average rate of change of f from 0 to $\frac{\pi}{6}$.

41. $f(x) = \tan x$

42. $f(x) = \sec x$

43. $f(x) = \tan(2x)$

44. $f(x) = \sec(2x)$

In Problems 45–48, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and graph each of these functions.

45. $f(x) = \tan x$, $g(x) = 4x$

46. $f(x) = 2\sec x$, $g(x) = \frac{1}{2}x$

47. $f(x) = -2x$, $g(x) = \cot x$

48. $f(x) = \frac{1}{2}x$, $g(x) = 2\csc x$

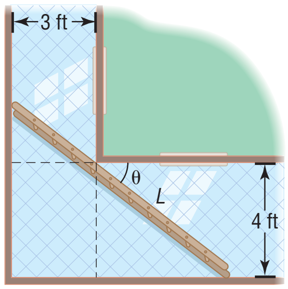
In Problems 49 and 50, graph each function.

49. $$f(x) = \begin{cases} \tan x & 0 \le x < \frac{\pi}{2} \\ 0 & x = \frac{\pi}{2} \\ \sec x & \frac{\pi}{2} < x \le \pi \end{cases}$$

50. $$g(x) = \begin{cases} \csc x & 0 < x < \pi \\ 0 & x = \pi \\ \cot x & \pi < x < 2\pi \end{cases}$$

Applications and Extensions

51. Carrying a Ladder around a Corner Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.



(a) Show that the length L of the line segment shown as a function of the angle θ is

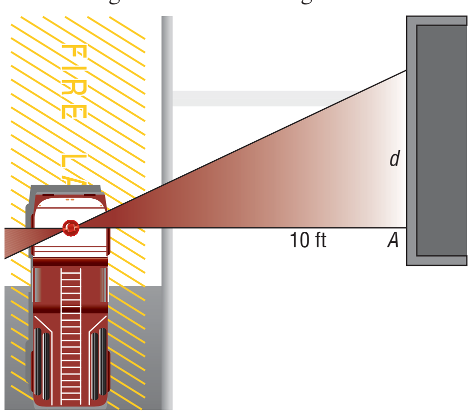
$$L(\theta) = 3\sec\theta + 4\csc\theta$$

(b) Graph $L = L(\theta)$, $0 < \theta < \frac{\pi}{2}$.

(c) For what value of θ is L the least?

(d) What is the length of the longest ladder that can be carried around the corner? Why is this also the least value of L?

52. A Rotating Beacon Suppose that a fire truck is parked in front of a building as shown in the figure.



The beacon light on top of the fire truck is located 10 feet from the wall and has a light on each side. If the beacon light rotates 1 revolution every 2 seconds, then a model for determining the distance d, in feet, that the beacon of light is from point A on the wall after t seconds is given by

$$d(t) = |10\tan(\pi t)|$$

(a) Graph $d(t) = |10\tan(\pi t)|$ for $0 \le t \le 2$.

(b) For what values of t is the function undefined? Explain what this means in terms of the beam of light on the wall.

(c) Fill in the following table.

t	0	0.1	0.2	0.3	0.4
$d(t) = 10\tan(\pi t)$					

(d) Compute $\dfrac{d(0.1) - d(0)}{0.1 - 0}$, $\dfrac{d(0.2) - d(0.1)}{0.2 - 0.1}$, and so on, for each consecutive value of t. These are called **first differences.**

(e) Interpret the first differences found in part (d). What is happening to the speed of the beam of light as d increases?

53. Exploration Graph

$$y = \tan x \quad \text{and} \quad y = -\cot\left(x + \frac{\pi}{2}\right)$$

Do you think that $\tan x = -\cot\left(x + \frac{\pi}{2}\right)$?

'Are You Prepared?' Answers

1. $x = 4$ **2.** True

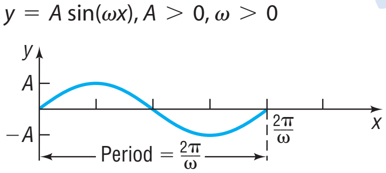
6.6 Phase Shift; Sinusoidal Curve Fitting

OBJECTIVES 1 Graph Sinusoidal Functions of the Form $y = A\sin(\omega x - \phi) + B$ (p. 415)
2 Build Sinusoidal Models from Data (p. 419)

1 Graph Sinusoidal Functions of the Form $y = A\sin(\omega x - \phi) + B$

We have seen that the graph of $y = A\sin(\omega x)$, $\omega > 0$, has amplitude $|A|$ and period $T = \dfrac{2\pi}{\omega}$. One cycle can be drawn as x varies from 0 to $\dfrac{2\pi}{\omega}$ or, equivalently, as ωx varies from 0 to 2π. See Figure 70.

Figure 70 One cycle of $y = A\sin(\omega x)$, $A > 0$, $\omega > 0$



We now want to discuss the graph of

$$y = A \sin(\omega x - \phi)$$

which may also be written as

$$y = A \sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$$

where $\omega > 0$ and ϕ (the Greek letter phi) are real numbers. The graph will be a sine curve with amplitude $|A|$. As $\omega x - \phi$ varies from 0 to 2π, one period will be traced out. This period will begin when

$$\omega x - \phi = 0 \quad \text{or} \quad x = \frac{\phi}{\omega}$$

and will end when

$$\omega x - \phi = 2\pi \quad \text{or} \quad x = \frac{\phi}{\omega} + \frac{2\pi}{\omega}$$

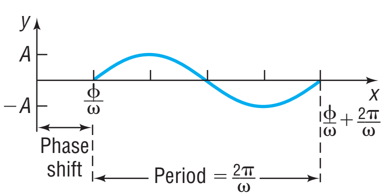
NOTE We can also find the beginning and end of the period by solving the inequality:

$$0 \le \omega x - \phi \le 2\pi$$
$$\phi \le \omega x \le 2\pi + \phi$$
$$\frac{\phi}{\omega} \le x \le \frac{2\pi}{\omega} + \frac{\phi}{\omega}$$

■

See Figure 71.

Figure 71 One cycle of $y = A\sin(\omega x - \phi), A > 0, \omega > 0, \phi > 0$



We see that the graph of $y = A \sin(\omega x - \phi) = A \sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$ is the same as the graph of $y = A \sin(\omega x)$, except that it has been shifted $\left|\frac{\phi}{\omega}\right|$ units (to the right if $\phi > 0$ and to the left if $\phi < 0$). This number $\frac{\phi}{\omega}$ is called the **phase shift** of the graph of $y = A \sin(\omega x - \phi)$.

For the graphs of $y = A \sin(\omega x - \phi)$ or $y = A \cos(\omega x - \phi)$, $\omega > 0$,

$$\boxed{\text{Amplitude} = |A| \qquad \text{Period} = T = \frac{2\pi}{\omega} \qquad \text{Phase shift} = \frac{\phi}{\omega}}$$

The phase shift is to the left if $\phi < 0$ and to the right if $\phi > 0$.

EXAMPLE 1 Finding the Amplitude, Period, and Phase Shift of a Sinusoidal Function and Graphing It

Find the amplitude, period, and phase shift of $y = 3\sin(2x - \pi)$ and graph the function.

Solution We use the same four steps used to graph sinusoidal functions of the form $y = A \sin(\omega x)$ or $y = A \cos(\omega x)$ given on page 400.

Step 1: Comparing

$$y = 3\sin(2x - \pi) = 3\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$$

to

$$y = A\sin(\omega x - \phi) = A \sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$$

we find that $A = 3, \omega = 2$, and $\phi = \pi$. The graph is a sine curve with amplitude $|A| = 3$, period $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$, and phase shift $= \frac{\phi}{\omega} = \frac{\pi}{2}$.

COMMENT We can also find the interval defining one cycle by solving the inequality

$$0 \le 2x - \pi \le 2\pi$$

Then

$$\pi \le 2x \le 3\pi$$

$$\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

STEP 2: The graph of $y = 3\sin(2x - \pi)$ will lie between -3 and 3 on the y-axis. One cycle will begin at $x = \frac{\phi}{\omega} = \frac{\pi}{2}$ and end at $x = \frac{\phi}{\omega} + \frac{2\pi}{\omega} = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$. To find the five key points, divide the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ into four subintervals, each of length $\pi \div 4 = \frac{\pi}{4}$, by finding the following values of x:

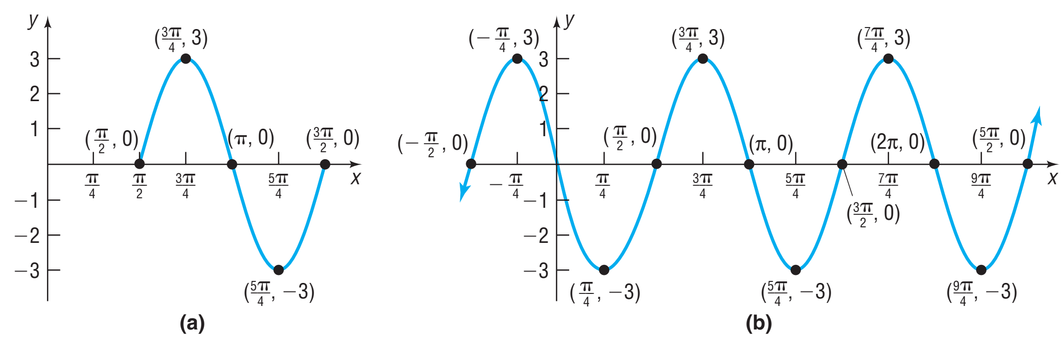
$\frac{\pi}{2}$	$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$	$\frac{3\pi}{4} + \frac{\pi}{4} = \pi$	$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$	$\frac{5\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{2}$
1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate

STEP 3: Use these values of x to determine the five key points on the graph:

$$\left(\frac{\pi}{2}, 0\right) \quad \left(\frac{3\pi}{4}, 3\right) \quad (\pi, 0) \quad \left(\frac{5\pi}{4}, -3\right) \quad \left(\frac{3\pi}{2}, 0\right)$$

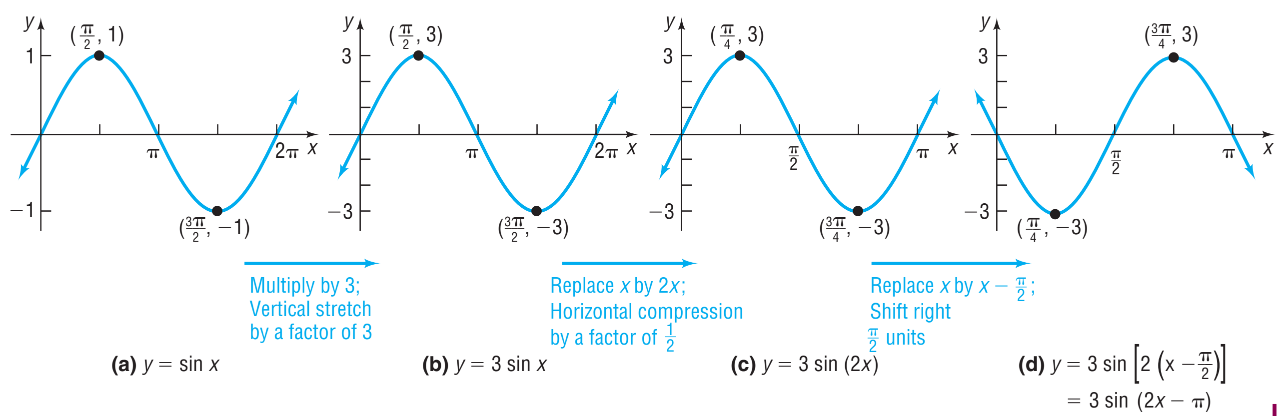
STEP 4: Plot these five points and fill in the graph of the sine function as shown in Figure 72(a). Extending the graph in each direction, we obtain Figure 72(b).

Figure 72



The graph of $y = 3\sin(2x - \pi) = 3\sin\left[2\left(x - \frac{\pi}{2}\right)\right]$ may also be obtained using transformations. See Figure 73.

Figure 73



To graph a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$, first graph the function $y = A\sin(\omega x - \phi)$ and then apply a vertical shift.

EXAMPLE 2 Finding the Amplitude, Period, and Phase Shift of a Sinusoidal Function and Graphing It

Find the amplitude, period, and phase shift of $y = 2\cos(4x + 3\pi) + 1$ and graph the function.

Solution

Step 1: Begin by graphing $y = 2\cos(4x + 3\pi)$. Comparing

$$y = 2\cos(4x + 3\pi) = 2\cos\left[4\left(x + \frac{3\pi}{4}\right)\right]$$

to

$$y = A\cos(\omega x - \phi) = A\cos\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$$

we see that $A = 2$, $\omega = 4$, and $\phi = -3\pi$. The graph is a cosine curve with amplitude $|A| = 2$, period $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$, and phase shift $= \frac{\phi}{\omega} = -\frac{3\pi}{4}$.

Step 2: The graph of $y = 2\cos(4x + 3\pi)$ will lie between -2 and 2 on the y-axis. One cycle will begin at $x = \frac{\phi}{\omega} = -\frac{3\pi}{4}$ and end at $x = \frac{\phi}{\omega} + \frac{2\pi}{\omega} = -\frac{3\pi}{4} + \frac{\pi}{2} = -\frac{\pi}{4}$. To find the five key points, divide the interval $\left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]$ into four subintervals, each of the length $\frac{\pi}{2} \div 4 = \frac{\pi}{8}$, by finding the following values.

COMMENT We can also find the interval defining one cycle by solving the inequality

$$0 \le 4x + 3\pi \le 2\pi$$

Then

$$-3\pi \le 4x \le -\pi$$

$$-\frac{3\pi}{4} \le x \le -\frac{\pi}{4}$$

$-\frac{3\pi}{4}$	$-\frac{3\pi}{4} + \frac{\pi}{8} = -\frac{5\pi}{8}$	$-\frac{5\pi}{8} + \frac{\pi}{8} = -\frac{\pi}{2}$	$-\frac{\pi}{2} + \frac{\pi}{8} = -\frac{3\pi}{8}$	$-\frac{3\pi}{8} + \frac{\pi}{8} = -\frac{\pi}{4}$
1st x-coordinate	2nd x-coordinate	3rd x-coordinate	4th x-coordinate	5th x-coordinate

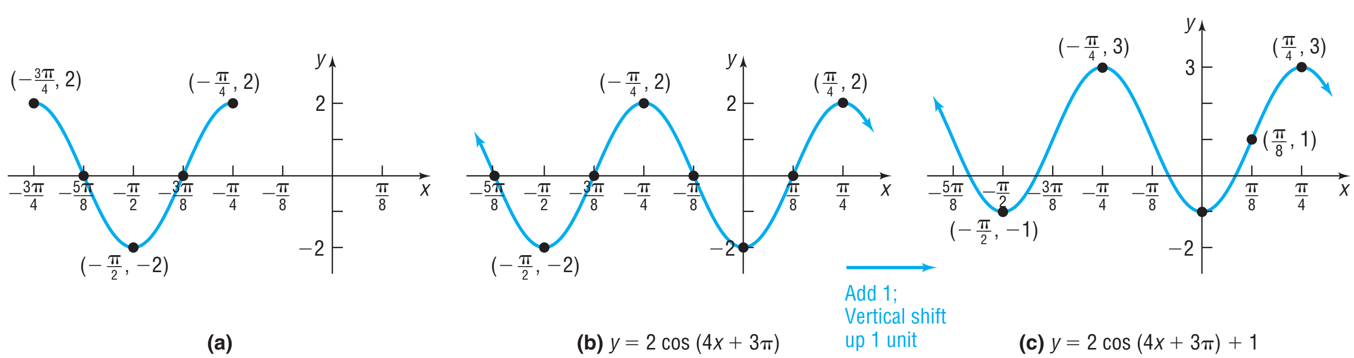
Step 3: The five key points on the graph of $y = 2\cos(4x + 3\pi)$ are

$$\left(-\frac{3\pi}{4}, 2\right)\quad \left(-\frac{5\pi}{8}, 0\right)\quad \left(-\frac{\pi}{2}, -2\right)\quad \left(-\frac{3\pi}{8}, 0\right)\quad \left(-\frac{\pi}{4}, 2\right)$$

Step 4: Plot these five points and fill in the graph of the cosine function as shown in Figure 74(a). Extending the graph in each direction, we obtain Figure 74(b), the graph of $y = 2\cos(4x + 3\pi)$.

Step 5: A vertical shift up 1 unit gives the final graph. See Figure 74(c).

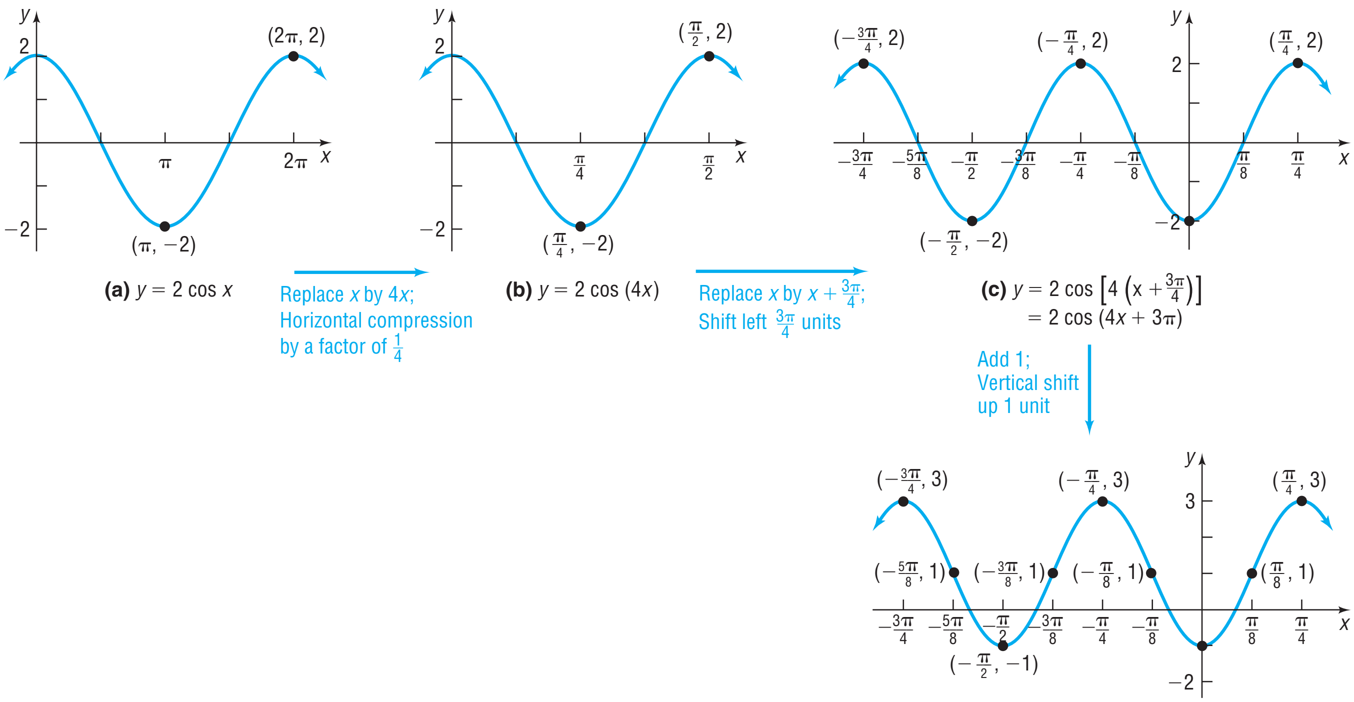
Figure 74



(a) **(b)** $y = 2\cos(4x + 3\pi)$ **(c)** $y = 2\cos(4x + 3\pi) + 1$

The graph of $y = 2\cos(4x + 3\pi) + 1 = 2\cos\left[4\left(x + \frac{3\pi}{4}\right)\right] + 1$ may also be obtained using transformations. See Figure 75.

Figure 75



(a) $y = 2\cos x$

(b) $y = 2\cos(4x)$

(c) $y = 2\cos\left[4\left(x + \frac{3\pi}{4}\right)\right] = 2\cos(4x + 3\pi)$

(d) $y = 2\cos(4x + 3\pi) + 1$

Now Work PROBLEM 3

SUMMARY Steps for Graphing Sinusoidal Functions $y = A\sin(\omega x - \phi) + B$ or $y = A\cos(\omega x - \phi) + B$

STEP 1: Determine the amplitude $|A|$, period $T = \dfrac{2\pi}{\omega}$, and phase shift $\dfrac{\phi}{\omega}$.

STEP 2: Determine the starting point of one cycle of the graph, $\dfrac{\phi}{\omega}$. Determine the ending point of one cycle of the graph, $\dfrac{\phi}{\omega} + \dfrac{2\pi}{\omega}$. Divide the interval $\left[\dfrac{\phi}{\omega}, \dfrac{\phi}{\omega} + \dfrac{2\pi}{\omega}\right]$ into four subintervals, each of length $\dfrac{2\pi}{\omega} \div 4$.

STEP 3: Use the endpoints of the subintervals to find the five key points on the graph.

STEP 4: Plot the five key points and connect them with a sinusoidal graph to obtain one cycle of the graph. Extend the graph in each direction to make it complete.

STEP 5: If $B \neq 0$, apply a vertical shift.

2 Build Sinusoidal Models from Data

Scatter diagrams of data sometimes take the form of a sinusoidal function. Let's look at an example.

The data given in Table 11 on page 420 represent the average monthly temperatures in Denver, Colorado. Since the data represent *average* monthly temperatures collected over many years, the data will not vary much from year to year and so will essentially repeat each year. In other words, the data are periodic. Figure 76 shows the scatter diagram of these data repeated over 2 years, where $x = 1$ represents January, $x = 2$ represents February, and so on.

Notice that the scatter diagram looks like the graph of a sinusoidal function. We choose to fit the data to a sine function of the form

$$y = A\sin(\omega x - \phi) + B$$

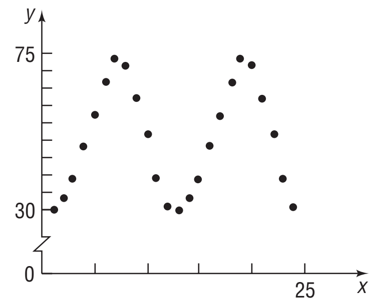
where A, B, ω, and ϕ are constants.

Table 11

Month, x	Average Monthly Temperature, °F
January, 1	29.7
February, 2	33.4
March, 3	39.0
April, 4	48.2
May, 5	57.2
June, 6	66.9
July, 7	73.5
August, 8	71.4
September, 9	62.3
October, 10	51.4
November, 11	39.0
December, 12	31.0

Source: U.S. National Oceanic and Atmospheric Administration

Figure 76



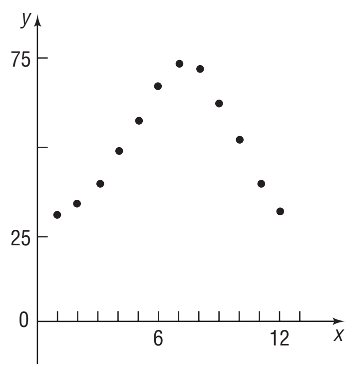
EXAMPLE 3 Finding a Sinusoidal Function from Temperature Data

Fit a sine function to the data in Table 11.

Solution Begin with a scatter diagram of the data for one year. See Figure 77. The data will be fitted to a sine function of the form

$$y = A\sin(\omega x - \phi) + B$$

Figure 77



STEP 1: To find the amplitude A, we compute

$$\text{Amplitude} = \frac{\text{largest data value} - \text{smallest data value}}{2}$$

$$= \frac{73.5 - 29.7}{2} = 21.9$$

To see the remaining steps in this process, superimpose the graph of the function $y = 21.9\sin x$, where x represents months, on the scatter diagram.

Figure 78 shows the two graphs. To fit the data, the graph needs to be shifted vertically, shifted horizontally, and stretched horizontally.

STEP 2: Determine the vertical shift by finding the average of the highest and lowest data values.

$$\text{Vertical shift} = \frac{73.5 + 29.7}{2} = 51.6$$

Now superimpose the graph of $y = 21.9\sin x + 51.6$ on the scatter diagram. See Figure 79.

Figure 78

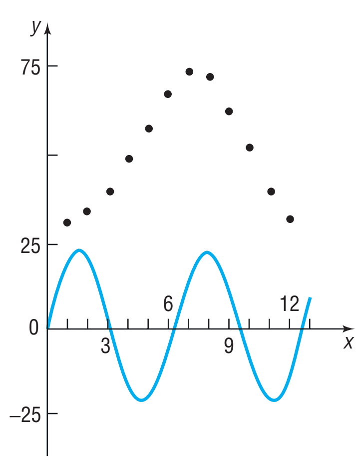


Figure 79

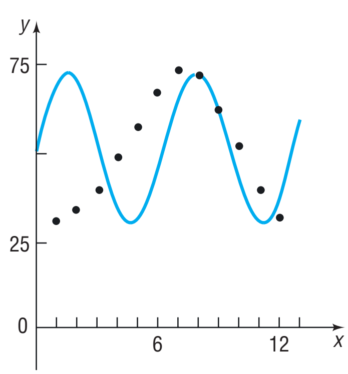
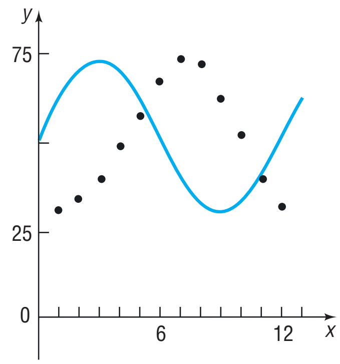


Figure 80



We see that the graph needs to be shifted horizontally and stretched horizontally.

STEP 3: It is easier to find the horizontal stretch factor first. Since the temperatures repeat every 12 months, the period of the function is $T = 12$. Since $T = \frac{2\pi}{\omega} = 12$, we find

$$\omega = \frac{2\pi}{12} = \frac{\pi}{6}$$

Now superimpose the graph of $y = 21.9 \sin\left(\frac{\pi}{6}x\right) + 51.6$ on the scatter diagram. See Figure 80. We see that the graph still needs to be shifted horizontally.

STEP 4: To determine the horizontal shift, use the period $T = 12$ and divide the interval $[0, 12]$ into four subintervals of length $12 \div 4 = 3$:

$$[0, 3], \quad [3, 6], \quad [6, 9], \quad [9, 12]$$

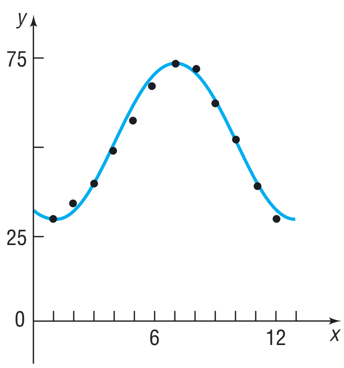
The sine curve is increasing on the interval $(0, 3)$ and is decreasing on the interval $(3, 9)$, so a local maximum occurs at $x = 3$. The data indicate that a maximum occurs at $x = 7$ (corresponding to July's temperature), so we must shift the graph of the function 4 units to the right by replacing x by $x - 4$. Doing this, we obtain

$$y = 21.9 \sin\left(\frac{\pi}{6}(x - 4)\right) + 51.6$$

Multiplying out, we find that a sine function of the form $y = A \sin(\omega x - \phi) + B$ that fits the data is

$$y = 21.9 \sin\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right) + 51.6$$

Figure 81



The graph of $y = 21.9 \sin\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right) + 51.6$ and the scatter diagram of the data are shown in Figure 81.

The steps to fit a sine function

$$y = A \sin(\omega x - \phi) + B$$

to sinusoidal data follow:

Steps for Fitting a Sine Function $y = A \sin(\omega x - \phi) + B$ to Data

STEP 1: Determine A, the amplitude of the function.

$$\text{Amplitude} = \frac{\text{largest data value} - \text{smallest data value}}{2}$$

STEP 2: Determine B, the vertical shift of the function.

$$\text{Vertical shift} = \frac{\text{largest data value} + \text{smallest data value}}{2}$$

STEP 3: Determine ω. Since the period T, the time it takes for the data to repeat, is $T = \frac{2\pi}{\omega}$, we have

$$\omega = \frac{2\pi}{T}$$

STEP 4: Determine the horizontal shift of the function by using the period of the data. Divide the period into four subintervals of equal length. Determine the x-coordinate for the maximum of the sine function and the x-coordinate for the maximum value of the data. Use this information to determine the value of the phase shift, $\frac{\phi}{\omega}$.

Now Work PROBLEMS 29(a)–(c)

Let's look at another example. Since the number of hours of sunlight in a day cycles annually, the number of hours of sunlight in a day for a given location can be modeled by a sinusoidal function.

The longest day of the year (in terms of hours of sunlight) occurs on the day of the summer solstice. For locations in the northern hemisphere, the summer solstice is the time when the sun is farthest north. In 2010, the summer solstice occurred on June 21 (the 172nd day of the year) at 6:28 AM EDT. The shortest day of the year occurs on the day of the winter solstice. The winter solstice is the time when the Sun is farthest south (again, for locations in the northern hemisphere). In 2010, the winter solstice occurred on December 21 (the 355th day of the year) at 6:38 PM (EST).

EXAMPLE 4 Finding a Sinusoidal Function for Hours of Daylight

According to the *Old Farmer's Almanac,* the number of hours of sunlight in Boston on the summer solstice is 15.30 and the number of hours of sunlight on the winter solstice is 9.08.

(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that fits the data.

(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.

(c) Draw a graph of the function found in part (a).

(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac* and compare it to the results found in part (b).

Source: The Old Farmer's Almanac, www.almanac.com/rise

Solution (a) **STEP 1:** Amplitude $= \dfrac{\text{largest data value} - \text{smallest data value}}{2}$

$$= \frac{15.30 - 9.08}{2} = 3.11$$

STEP 2: Vertical shift $= \dfrac{\text{largest data value} + \text{smallest data value}}{2}$

$$= \frac{15.30 + 9.08}{2} = 12.19$$

STEP 3: The data repeat every 365 days. Since $T = \dfrac{2\pi}{\omega} = 365$, we find

$$\omega = \frac{2\pi}{365}$$

So far, we have $y = 3.11\sin\left(\dfrac{2\pi}{365}x - \phi\right) + 12.19$.

STEP 4: To determine the horizontal shift, we use the period $T = 365$ and divide the interval $[0, 365]$ into four subintervals of length $365 \div 4 = 91.25$:

$$[0, 91.25], \quad [91.25, 182.5], \quad [182.5, 273.75], \quad [273.75, 365]$$

The sine curve is increasing on the interval $(0, 91.25)$ and is decreasing on the interval $(91.25, 273.75)$, so a local maximum occurs at $x = 91.25$.

Since the maximum occurs on the summer solstice at $x = 172$, we must shift the graph of the function $172 - 91.25 = 80.75$ units to the right by replacing x by $x - 80.75$. Doing this, we obtain

$$y = 3.11 \sin\left(\frac{2\pi}{365}(x - 80.75)\right) + 12.19$$

Multiplying out, we find that a sine function of the form $y = A\sin(\omega x - \phi) + B$ that fits the data is

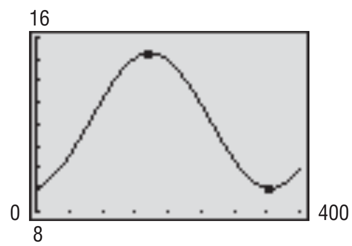
$$y = 3.11 \sin\left(\frac{2\pi}{365}x - \frac{323\pi}{730}\right) + 12.19$$

(b) To predict the number of hours of daylight on April 1, we let $x = 91$ in the function found in part (a) and obtain

$$y = 3.11 \sin\left(\frac{2\pi}{365}\cdot 91 - \frac{323}{730}\pi\right) + 12.19$$
$$\approx 12.74$$

So we predict that there will be about 12.74 hours = 12 hours, 44 minutes of sunlight on April 1 in Boston.

Figure 82



(c) The graph of the function found in part (a) is given in Figure 82.

(d) According to the *Old Farmer's Almanac*, there will be 12 hours 45 minutes of sunlight on April 1 in Boston.

Now Work PROBLEM 35

Certain graphing utilities (such as a TI-83, TI-84 Plus, and TI-86) have the capability of finding the sine function of best fit for sinusoidal data. At least four data points are required for this process.

EXAMPLE 5 Finding the Sine Function of Best Fit

Use a graphing utility to find the sine function of best fit for the data in Table 11. Graph this function with the scatter diagram of the data.

Solution Enter the data from Table 11 and execute the SINe REGression program. The result is shown in Figure 83.

The output that the utility provides shows the equation

$$y = a\sin(bx + c) + d$$

The sinusoidal function of best fit is

$$y = 21.15\sin(0.55x - 2.35) + 51.19$$

where x represents the month and y represents the average temperature.

Figure 84 shows the graph of the sinusoidal function of best fit on the scatter diagram.

Figure 83

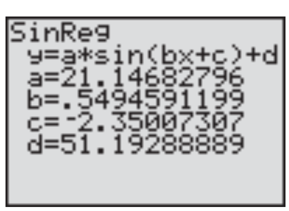
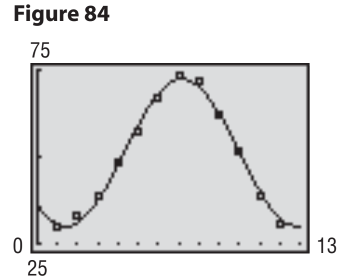


Figure 84



Now Work PROBLEMS 29(d) AND (e)

6.6 Assess Your Understanding

Concepts and Vocabulary

1. For the graph of $y = A\sin(\omega x - \phi)$, the number $\frac{\phi}{\omega}$ is called the ________ ________.

2. ***True or False*** Only two data points are required by a graphing utility to find the sine function of best fit.

Skill Building

In Problems 3–14, find the amplitude, period, and phase shift of each function. Graph each function. Be sure to label key points. Show at least two periods.

3. $y = 4\sin(2x - \pi)$
4. $y = 3\sin(3x - \pi)$
5. $y = 2\cos\left(3x + \frac{\pi}{2}\right)$
6. $y = 3\cos(2x + \pi)$
7. $y = -3\sin\left(2x + \frac{\pi}{2}\right)$
8. $y = -2\cos\left(2x - \frac{\pi}{2}\right)$
9. $y = 4\sin(\pi x + 2) - 5$
10. $y = 2\cos(2\pi x + 4) + 4$
11. $y = 3\cos(\pi x - 2) + 5$
12. $y = 2\cos(2\pi x - 4) - 1$
13. $y = -3\sin\left(-2x + \frac{\pi}{2}\right)$
14. $y = -3\cos\left(-2x + \frac{\pi}{2}\right)$

In Problems 15–18, write the equation of a sine function that has the given characteristics.

15. Amplitude: 2
Period: π
Phase shift: $\frac{1}{2}$

16. Amplitude: 3
Period: $\frac{\pi}{2}$
Phase shift: 2

17. Amplitude: 3
Period: 3π
Phase shift: $-\frac{1}{3}$

18. Amplitude: 2
Period: π
Phase shift: -2

Mixed Practice

In Problems 19–26, apply the methods of this and the previous section to graph each function. Be sure to label key points and show at least two periods.

19. $y = 2\tan(4x - \pi)$
20. $y = \frac{1}{2}\cot(2x - \pi)$
21. $y = 3\csc\left(2x - \frac{\pi}{4}\right)$
22. $y = \frac{1}{2}\sec(3x - \pi)$
23. $y = -\cot\left(2x + \frac{\pi}{2}\right)$
24. $y = -\tan\left(3x + \frac{\pi}{2}\right)$
25. $y = -\sec(2\pi x + \pi)$
26. $y = -\csc\left(-\frac{1}{2}\pi x + \frac{\pi}{4}\right)$

Applications and Extensions

27. **Alternating Current (ac) Circuits** The current I, in amperes, flowing through an ac (alternating current) circuit at time t, in seconds, is

$$I(t) = 120\sin\left(30\pi t - \frac{\pi}{3}\right) \qquad t \geq 0$$

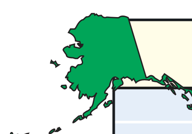
What is the period? What is the amplitude? What is the phase shift? Graph this function over two periods.

28. **Alternating Current (ac) Circuits** The current I, in amperes, flowing through an ac (alternating current) circuit at time t, in seconds, is

$$I(t) = 220\sin\left(60\pi t - \frac{\pi}{6}\right) \qquad t \geq 0$$

What is the period? What is the amplitude? What is the phase shift? Graph this function over two periods.

29. **Monthly Temperature** The following data represent the average monthly temperatures for Juneau, Alaska.

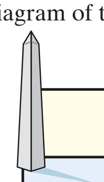


Month, x	Average Monthly Temperature, °F
January, 1	24.2
February, 2	28.4
March, 3	32.7
April, 4	39.7
May, 5	47.0
June, 6	53.0
July, 7	56.0
August, 8	55.0
September, 9	49.4
October, 10	42.2
November, 11	32.0
December, 12	27.1

Source: U.S. National Oceanic and Atmospheric Administration

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Draw the sinusoidal function of best fit on a scatter diagram of the data.

30. Monthly Temperature The following data represent the average monthly temperatures for Washington, D.C.
(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on a scatter diagram of the data.



Month, x	Average Monthly Temperature, °F
January, 1	34.6
February, 2	37.5
March, 3	47.2
April, 4	56.5
May, 5	66.4
June, 6	75.6
July, 7	80.0
August, 8	78.5
September, 9	71.3
October, 10	59.7
November, 11	49.8
December, 12	39.4

Source: U.S. National Oceanic and Atmospheric Administration

31. Monthly Temperature The following data represent the average monthly temperatures for Indianapolis, Indiana.

Month, x	Average Monthly Temperature, °F
January, 1	25.5
February, 2	29.6
March, 3	41.4
April, 4	52.4
May, 5	62.8
June, 6	71.9
July, 7	75.4
August, 8	73.2
September, 9	66.6
October, 10	54.7
November, 11	43.0
December, 12	30.9

Source: U.S. National Oceanic and Atmospheric Administration

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on a scatter diagram of the data.

32. Monthly Temperature The following data represent the average monthly temperatures for Baltimore, Maryland.
(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on a scatter diagram of the data.

Month, x	Average Monthly Temperature, °F
January, 1	31.8
February, 2	34.8
March, 3	44.1
April, 4	53.4
May, 5	63.4
June, 6	72.5
July, 7	77.0
August, 8	75.6
September, 9	68.5
October, 10	56.6
November, 11	46.8
December, 12	36.7

Source: U.S. National Oceanic and Atmospheric Administration

33. Tides The length of time between consecutive high tides is 12 hours and 25 minutes. According to the National Oceanic and Atmospheric Administration, on Saturday, July 25, 2009, in Charleston, South Carolina, high tide occurred at 11:30 AM (11.5 hours) and low tide occurred at 5:31 PM (17.5167 hours). Water heights are measured as the amounts above or below the mean lower low water. The height of the water at high tide was 5.84 feet, and the height of the water at low tide was −0.37 foot.
(a) Approximately when will the next high tide occur?
(b) Find a sinusoidal function of the form
$y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Use the function found in part (b) to predict the height of the water at 3 PM on July 25, 2009.

34. Tides The length of time between consecutive high tides is 12 hours and 25 minutes. According to the National Oceanic and Atmospheric Administration, on Saturday, July 25, 2009, in Sitka Sound, Alaska, high tide occurred at 2:37 AM (2.6167 hours) and low tide occurred at 9:12 PM (9.2 hours). Water heights are measured as the amounts above or below the mean lower low water. The height of the water at high tide was 11.09 feet, and the height of the water at low tide was −2.49 feet.

(a) Approximately when will the next high tide occur?
(b) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that models the data.
(c) Use the function found in part (b) to predict the height of the water at 6 PM.

35. **Hours of Daylight** According to the *Old Farmer's Almanac*, in Miami, Florida, the number of hours of sunlight on the summer solstice of 2010 was 13.75, and the number of hours of sunlight on the winter solstice was 10.55.
(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that models the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac*, and compare the actual hours of daylight to the results found in part (c).

36. **Hours of Daylight** According to the *Old Farmer's Almanac*, in Detroit, Michigan, the number of hours of sunlight on the summer solstice of 2010 was 15.30, and the number of hours of sunlight on the winter solstice was 9.10.
(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that models the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac*, and compare the actual hours of daylight to the results found in part (c).

37. **Hours of Daylight** According to the *Old Farmer's Almanac*, in Anchorage, Alaska, the number of hours of sunlight on the summer solstice of 2010 was 19.42 and the number of hours of sunlight on the winter solstice was 5.48.
(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that models the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
*(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac*, and compare the actual hours of daylight to the results found in part (c).

38. **Hours of Daylight** According to the *Old Farmer's Almanac*, in Honolulu, Hawaii, the number of hours of sunlight on the summer solstice of 2010 was 13.43 and the number of hours of sunlight on the winter solstice was 10.85.
(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that models the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac*, and compare the actual hours of daylight to the results found in part (c).

Explaining Concepts: Discussion and Writing

39. Explain how the amplitude and period of a sinusoidal graph are used to establish the scale on each coordinate axis.

40. Find an application in your major field that leads to a sinusoidal graph. Write a paper about your findings.

CHAPTER REVIEW

Things to Know

Definitions

Angle in standard position (p. 350)	Vertex is at the origin; initial side is along the positive x-axis.
1 Degree (1°) (p. 351)	$1° = \frac{1}{360}$ revolution
1 Radian (p. 353)	The measure of a central angle of a circle whose rays subtend an arc whose length is the radius of the circle
Trigonometric functions (pp. 364–365, 366)	$P = (x, y)$ is the point on the unit circle corresponding to $\theta = t$ radians. $\sin t = \sin\theta = y$ $\cos t = \cos\theta = x$ $\tan t = \tan\theta = \frac{y}{x}$ $x \neq 0$ $\csc t = \csc\theta = \frac{1}{y}$ $y \neq 0$ $\sec t = \sec\theta = \frac{1}{x}$ $x \neq 0$ $\cot t = \cot\theta = \frac{x}{y}$ $y \neq 0$
Trigonometric functions using a circle of radius r (p. 374)	For an angle θ in standard position, $P = (x, y)$ is the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. $\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$ $x \neq 0$ $\csc\theta = \frac{r}{y}$ $y \neq 0$ $\sec\theta = \frac{r}{x}$ $x \neq 0$ $\cot\theta = \frac{x}{y}$ $y \neq 0$
Periodic function (p. 382)	$f(\theta + p) = f(\theta)$, for all θ, $p > 0$, where the smallest such p is the fundamental period.

Formulas

1 revolution $= 360°$ (p. 352)

$= 2\pi$ radians (p. 355)

$1° = \dfrac{\pi}{180}$ radian (p. 352) 1 radian $= \dfrac{180}{\pi}$ degrees (p. 352)

$s = r\theta$ (p. 354) — θ is measured in radians; s is the length of the arc subtended by the central angle θ of the circle of radius r.

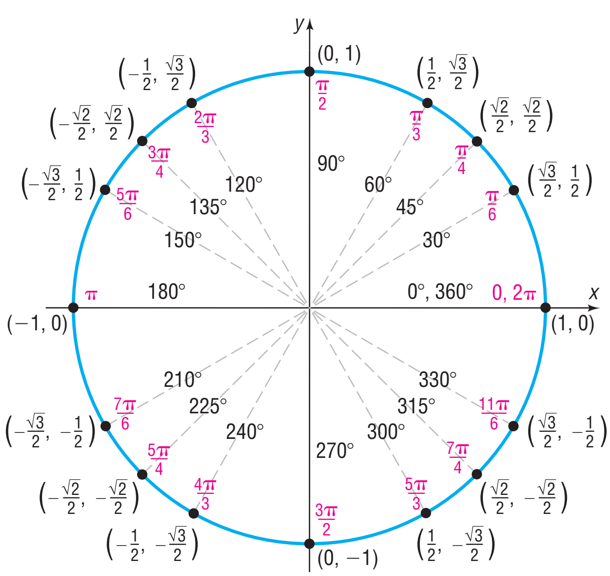
$A = \dfrac{1}{2}r^2\theta$ (p. 357) — A is the area of the sector of a circle of radius r formed by a central angle of θ radians.

$v = r\omega$ (p. 358) — v is the linear speed along the circle of radius r; ω is the angular speed (measured in radians per unit time).

Table of Values (pp. 368 and 371)

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0°	0	1	0	Not defined	1	Not defined
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	90°	1	0	Not defined	1	Not defined	0
π	180°	0	-1	0	Not defined	-1	Not defined
$\frac{3\pi}{2}$	270°	-1	0	Not defined	-1	Not defined	0

The Unit Circle (pp. 372–373)



Fundamental identities (p. 385)

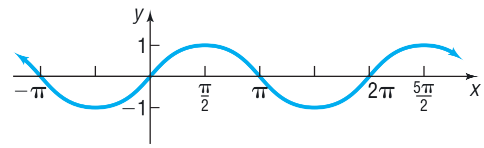
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

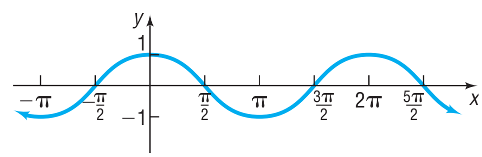
$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Properties of the trigonometric functions

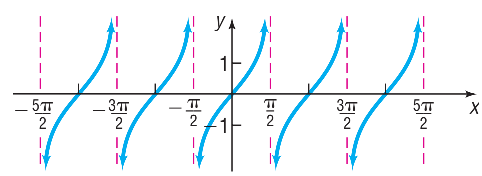
$y = \sin x$ (p. 394)
- Domain: $-\infty < x < \infty$
- Range: $-1 \le y \le 1$
- Periodic: period $= 2\pi\,(360°)$
- Odd function



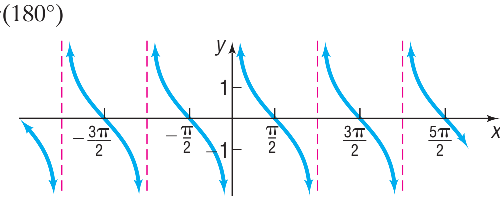
$y = \cos x$ (p. 396)
- Domain: $-\infty < x < \infty$
- Range: $-1 \le y \le 1$
- Periodic: period $= 2\pi\,(360°)$
- Even function



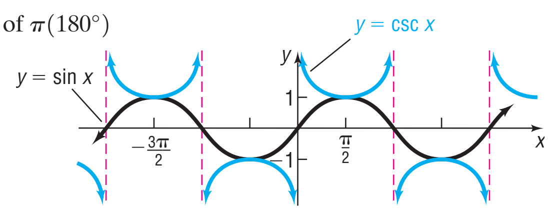
$y = \tan x$ (pp. 408–410)
- Domain: $-\infty < x < \infty$, except odd integer multiples of $\frac{\pi}{2}$ (90°)
- Range: $-\infty < y < \infty$
- Periodic: period $= \pi\,(180°)$
- Odd function
- Vertical asymptotes at odd integer multiples of $\frac{\pi}{2}$



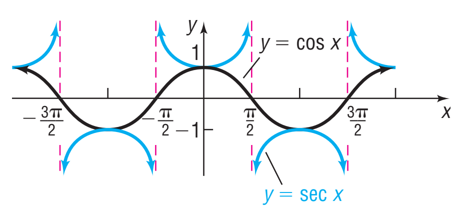
$y = \cot x$ (p. 411)
- Domain: $-\infty < x < \infty$, except integer multiples of π (180°)
- Range: $-\infty < y < \infty$
- Periodic: period $= \pi\,(180°)$
- Odd function
- Vertical asymptotes at integer multiples of π



$y = \csc x$ (p. 412)
- Domain: $-\infty < x < \infty$, except integer multiples of π (180°)
- Range: $|y| \ge 1$ ($y \le -1$ or $y \ge 1$)
- Periodic: period $= 2\pi\,(360°)$
- Odd function
- Vertical asymptotes at integer multiples of π



$y = \sec x$ (p. 412)
- Domain: $-\infty < x < \infty$, except odd integer multiples of $\frac{\pi}{2}$ (90°)
- Range: $|y| \ge 1$ ($y \le -1$ or $y \ge 1$)
- Periodic: period $= 2\pi\,(360°)$
- Even function
- Vertical asymptotes at odd integer multiples of $\frac{\pi}{2}$



Sinusoidal graphs

$y = A\sin(\omega x) + B, \quad \omega > 0$ | Period $= \frac{2\pi}{\omega}$ (pp. 398, 416)

$y = A\cos(\omega x) + B, \quad \omega > 0$ | Amplitude $= |A|$ (pp. 398, 416)

$y = A\sin(\omega x - \phi) + B = A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right] + B$ | Phase shift $= \frac{\phi}{\omega}$ (p. 416)

$y = A\cos(\omega x - \phi) + B = A\cos\left[\omega\left(x - \frac{\phi}{\omega}\right)\right] + B$

Objectives

Section		You should be able to:	Example(s)	Review Exercises
6.1	1	Convert between decimals and degrees, minutes, seconds measures for angles (p. 352)	2	86
	2	Find the length of an arc of a circle (p. 354)	3	87, 88
	3	Convert from degrees to radians and from radians to degrees (p. 354)	4–6	1–8
	4	Find the area of a sector of a circle (p. 357)	7	87
	5	Find the linear speed of an object traveling in circular motion (p. 358)	8	89–92
6.2	1	Find the exact values of the trigonometric functions using a point on the unit circle (p. 365)	1	83, 97
	2	Find the exact values of the trigonometric functions of quadrantal angles (p. 366)	2, 3	10, 17, 18, 20, 97
	3	Find the exact values of the trigonometric functions of $\frac{\pi}{4} = 45°$ (p. 368)	4, 5	9, 11, 13, 15, 16
	4	Find the exact values of the trigonometric functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$ (p. 369)	6–8	9–15
	5	Find the exact values of the trigonometric functions for integer multiples of $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$ (p. 372)	9, 10	13–16, 19, 97
	6	Use a calculator to approximate the value of a trigonometric function (p. 373)	11	79, 80
	7	Use a circle of radius r to evaluate the trigonometric functions (p. 374)	12	84
6.3	1	Determine the domain and the range of the trigonometric functions (p. 380)	pp. 380–381	85
	2	Determine the period of the trigonometric functions (p. 381)	1	85
	3	Determine the signs of the trigonometric functions in a given quadrant (p. 383)	2	81, 82
	4	Find the values of the trigonometric functions using fundamental identities (p. 384)	3, 4	21–30
	5	Find the exact values of the trigonometric functions of an angle given one of the functions and the quadrant of the angle (p. 386)	5, 6	31–46
	6	Use even–odd properties to find the exact values of the trigonometric functions (p. 389)	7	27–30
6.4	1	Graph functions of the form $y = A\sin(\omega x)$ using transformations (p. 394)	1, 2	47
	2	Graph functions of the form $y = A\cos(\omega x)$ using transformations (p. 396)	3	48
	3	Determine the amplitude and period of sinusoidal functions (p. 397)	4	63–68
	4	Graph sinusoidal functions using key points (p. 398)	5–7	47, 48, 67, 68, 93
	5	Find an equation for a sinusoidal graph (p. 402)	8, 9	75–78
6.5	1	Graph functions of the form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$ (p. 410)	1, 2	53, 54, 56
	2	Graph functions of the form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$ (p. 412)	3	57
6.6	1	Graph sinusoidal functions of the form $y = A\sin(\omega x - \phi) + B$ (p. 415)	1, 2	49, 50, 59, 60, 69–74, 94
	2	Build sinusoidal models from data (p. 419)	3–5	95, 96

Review Exercises

In Problems 1–4, convert each angle in degrees to radians. Express your answer as a multiple of π.

1. $135°$ **2.** $210°$ **3.** $18°$ **4.** $15°$

In Problems 5–8, convert each angle in radians to degrees.

5. $\frac{3\pi}{4}$ **6.** $\frac{2\pi}{3}$ **7.** $-\frac{5\pi}{2}$ **8.** $-\frac{3\pi}{2}$

In Problems 9–30, find the exact value of each expression. Do not use a calculator.

9. $\tan\frac{\pi}{4} - \sin\frac{\pi}{6}$ **10.** $\cos\frac{\pi}{3} + \sin\frac{\pi}{2}$ **11.** $3\sin 45° - 4\tan\frac{\pi}{6}$

12. $4\cos 60° + 3\tan\frac{\pi}{3}$ **13.** $6\cos\frac{3\pi}{4} + 2\tan\left(-\frac{\pi}{3}\right)$ **14.** $3\sin\frac{2\pi}{3} - 4\cos\frac{5\pi}{2}$

15. $\sec\left(-\frac{\pi}{3}\right) - \cot\left(-\frac{5\pi}{4}\right)$ **16.** $4\csc\frac{3\pi}{4} - \cot\left(-\frac{\pi}{4}\right)$ **17.** $\tan\pi + \sin\pi$

18. $\cos\frac{\pi}{2} - \csc\left(-\frac{\pi}{2}\right)$ **19.** $\cos 540° - \tan(-405°)$ **20.** $\sin 270° + \cos(-180°)$

21. $\sin^2 20° + \frac{1}{\sec^2 20°}$ **22.** $\frac{1}{\cos^2 40°} - \frac{1}{\cot^2 40°}$ **23.** $\sec 50° \cos 50°$

24. $\tan 10° \cot 10°$ **25.** $\frac{\sin 50°}{\cos 40°}$ **26.** $\frac{\tan 20°}{\cot 70°}$

27. $\frac{\sin(-40°)}{\cos 50°}$ **28.** $\tan(-20°)\cot 20°$ **29.** $\sin 400° \sec(-50°)$ **30.** $\cot 200° \cot(-70°)$

In Problems 31–46, find the exact value of each of the remaining trigonometric functions.

31. $\sin\theta = \frac{4}{5}$, θ is acute **32.** $\tan\theta = \frac{1}{4}$, θ is acute **33.** $\tan\theta = \frac{12}{5}$, $\sin\theta < 0$

34. $\cot\theta = \frac{12}{5}$, $\cos\theta < 0$ **35.** $\sec\theta = -\frac{5}{4}$, $\tan\theta < 0$ **36.** $\csc\theta = -\frac{5}{3}$, $\cot\theta < 0$

37. $\sin\theta = \frac{12}{13}$, θ in quadrant II **38.** $\cos\theta = -\frac{3}{5}$, θ in quadrant III **39.** $\sin\theta = -\frac{5}{13}$, $\frac{3\pi}{2} < \theta < 2\pi$

40. $\cos\theta = \frac{12}{13}$, $\frac{3\pi}{2} < \theta < 2\pi$ **41.** $\tan\theta = \frac{1}{3}$, $180° < \theta < 270°$ **42.** $\tan\theta = -\frac{2}{3}$, $90° < \theta < 180°$

43. $\sec\theta = 3$, $\frac{3\pi}{2} < \theta < 2\pi$ **44.** $\csc\theta = -4$, $\pi < \theta < \frac{3\pi}{2}$ **45.** $\cot\theta = -2$, $\frac{\pi}{2} < \theta < \pi$

46. $\tan\theta = -2$, $\frac{3\pi}{2} < \theta < 2\pi$

In Problems 47–62, graph each function. Each graph should contain at least two periods. Use the graph to determine the domain and the range of each function.

47. $y = 2\sin(4x)$ **48.** $y = -3\cos(2x)$ **49.** $y = -2\cos\left(x + \frac{\pi}{2}\right)$ **50.** $y = 3\sin(x - \pi)$

51. $y = \tan(x + \pi)$ **52.** $y = -\tan\left(x - \frac{\pi}{2}\right)$ **53.** $y = -2\tan(3x)$ **54.** $y = 4\tan(2x)$

55. $y = \cot\left(x + \frac{\pi}{4}\right)$ **56.** $y = -4\cot(2x)$ **57.** $y = 4\sec(2x)$ **58.** $y = \csc\left(x + \frac{\pi}{4}\right)$

59. $y = 4\sin(2x + 4) - 2$ **60.** $y = 3\cos(4x + 2) + 1$ **61.** $y = 4\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ **62.** $y = 5\cot\left(\frac{x}{3} - \frac{\pi}{4}\right)$

In Problems 63–66, determine the amplitude and period of each function without graphing.

63. $y = 4\cos x$ **64.** $y = \sin(2x)$ **65.** $y = -8\sin\left(\frac{\pi}{2}x\right)$ **66.** $y = -2\cos(3\pi x)$

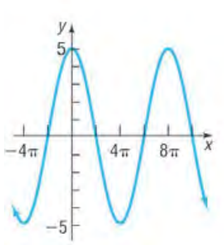
In Problems 67–74, find the amplitude, period, and phase shift of each function. Graph each function. Show at least two periods.

67. $y = 4\sin(3x)$ **68.** $y = 2\cos\left(\frac{1}{3}x\right)$ **69.** $y = 2\sin(2x - \pi)$ **70.** $y = -\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

71. $y = \frac{1}{2}\sin\left(\frac{3}{2}x - \pi\right)$ **72.** $y = \frac{3}{2}\cos(6x + 3\pi)$ **73.** $y = -\frac{2}{3}\cos(\pi x - 6)$ **74.** $y = -7\sin\left(\frac{\pi}{3}x - \frac{4}{3}\right)$

In Problems 75–78, find a function whose graph is given.

75.



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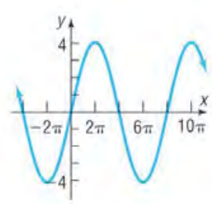

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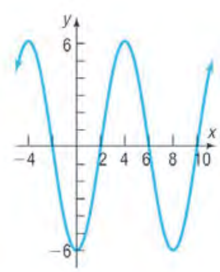

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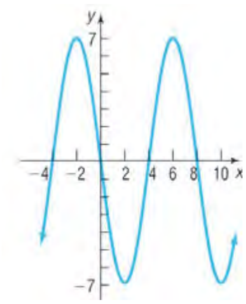

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79. Use a calculator to approximate $\sin\frac{\pi}{8}$. Round the answer to two decimal places.

80. Use a calculator to approximate sec 10°. Round the answer to two decimal places.

81. Determine the signs of the six trigonometric functions of an angle θ whose terminal side is in quadrant III.

82. Name the quadrant θ lies in if $\cos\theta > 0$ and $\tan\theta < 0$.

83. Find the exact values of the six trigonometric functions of t if $P = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ is the point on the unit circle that corresponds to t.

84. Find the exact value of $\sin t$, $\cos t$, and $\tan t$ if $P = (-2, 5)$ is the point on the circle that corresponds to t.

85. What is the domain and the range of the secant function? What is the period?

86. (a) Convert the angle $32°20'35''$ to a decimal in degrees. Round the answer to two decimal places.
(b) Convert the angle 63.18° to D°M′S″ form. Express the answer to the nearest second.

87. Find the length of the arc subtended by a central angle of 30° on a circle of radius 2 feet. What is the area of the sector?

88. The minute hand of a clock is 8 inches long. How far does the tip of the minute hand move in 30 minutes? How far does it move in 20 minutes?

89. Angular Speed of a Race Car A race car is driven around a circular track at a constant speed of 180 miles per hour. If the diameter of the track is $\frac{1}{2}$ mile, what is the angular speed of the car? Express your answer in revolutions per hour (which is equivalent to laps per hour).

90. Merry-Go-Rounds A neighborhood carnival has a merry-go-round whose radius is 25 feet. If the time for one revolution is 30 seconds, how fast is the merry-go-round going?

91. Lighthouse Beacons The Montauk Point Lighthouse on Long Island has dual beams (two light sources opposite each other). Ships at sea observe a blinking light every 5 seconds. What rotation speed is required to do this?

92. Spin Balancing Tires The radius of each wheel of a car is 16 inches. At how many revolutions per minute should a spin balancer be set to balance the tires at a speed of 90 miles per hour? Is the setting different for a wheel of radius 14 inches? If so, what is this setting?

93. Alternating Voltage The electromotive force E, in volts, in a certain ac (alternating circuit) circuit obeys the function

$$E(t) = 120\sin(120\pi t), \qquad t \ge 0$$

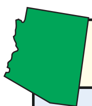
where t is measured in seconds.
(a) What is the maximum value of E?
(b) What is the period?
(c) Graph this function over two periods.

94. Alternating Current The current I, in amperes, flowing through an ac (alternating current) circuit at time t is

$$I(t) = 220\sin\left(30\pi t + \frac{\pi}{6}\right), \qquad t \ge 0$$

(a) What is the period?
(b) What is the amplitude?
(c) What is the phase shift?
(d) Graph this function over two periods.

95. Monthly Temperature The following data represent the average monthly temperatures for Phoenix, Arizona.
(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on the scatter diagram.

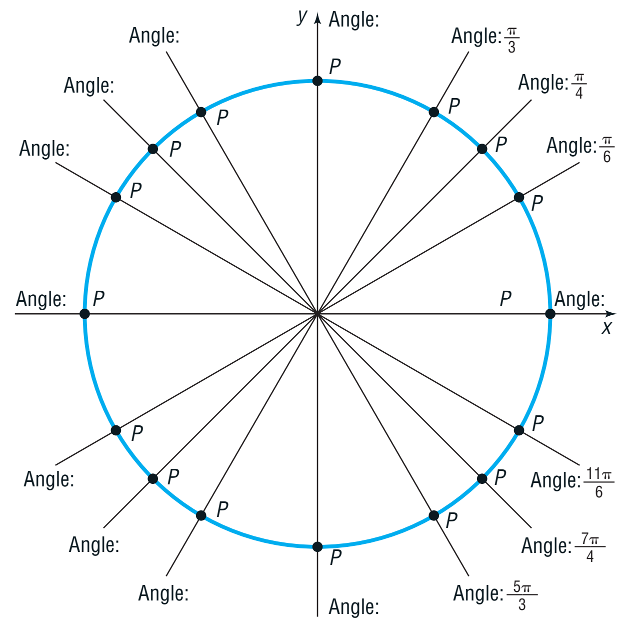


Month, *m*	Average Monthly Temperature, *T*
January, 1	51
February, 2	55
March, 3	63
April, 4	67
May, 5	77
June, 6	86
July, 7	90
August, 8	90
September, 9	84
October, 10	71
November, 11	59
December, 12	52

SOURCE: U.S. National Oceanic and Atmospheric Administration

96. Hours of Daylight According to the *Old Farmer's Almanac*, in Las Vegas, Nevada, the number of hours of sunlight on the summer solstice is 14.63 and the number of hours of sunlight on the winter solstice is 9.72.
(a) Find a sinusoidal function of the form $y = A\sin(\omega x - \phi) + B$ that fits the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer's Almanac* and compare the actual hours of daylight to the results found in part (c).

97. Unit Circle On the given unit circle, fill in the missing angles (in radians) and the corresponding points P of each angle.



CHAPTER TEST

Chapter Test Prep Videos

The Chapter Test Prep Videos are step-by-step test solutions available in the Video Resources DVD, in MyMathLab, or on this text's YouTube Channel. Flip back to the Student Resources page to see the exact web address for this text's YouTube channel.

In Problems 1–3, convert each angle in degrees to radians. Express your answer as a multiple of π.

1. $260°$ **2.** $-400°$ **3.** $13°$

In Problems 4–6 convert each angle in radians to degrees.

4. $-\frac{\pi}{8}$ **5.** $\frac{9\pi}{2}$ **6.** $\frac{3\pi}{4}$

In Problems 7–12, find the exact value of each expression.

7. $\sin\frac{\pi}{6}$ **8.** $\cos\left(-\frac{5\pi}{4}\right) - \cos\frac{3\pi}{4}$

9. $\cos(-120°)$ **10.** $\tan 330°$

11. $\sin\frac{\pi}{2} - \tan\frac{19\pi}{4}$ **12.** $2\sin^2 60° - 3\cos 45°$

In Problems 13–16, use a calculator to evaluate each expression. Round your answer to three decimal places.

13. $\sin 17°$ **14.** $\cos\frac{2\pi}{5}$ **15.** $\sec 229°$ **16.** $\cot\frac{28\pi}{9}$

17. Fill in each table entry with the sign of each function.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
θ in QI						
θ in QII						
θ in QIII						
θ in QIV						

18. If $f(x) = \sin x$ and $f(a) = \frac{3}{5}$, find $f(-a)$.

In Problems 19–21 find the value of the remaining five trigonometric functions of θ.

19. $\sin\theta = \frac{5}{7}$, θ in quadrant II **20.** $\cos\theta = \frac{2}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

21. $\tan\theta = -\frac{12}{5}$, $\frac{\pi}{2} < \theta < \pi$

In Problems 22–24, the point (x, y) is on the terminal side of angle θ in standard position. Find the exact value of the given trigonometric function.

22. $(2, 7)$, $\sin\theta$

23. $(-5, 11)$, $\cos\theta$

24. $(6, -3)$, $\tan\theta$

In Problems 25 and 26, graph the function.

25. $y = 2\sin\left(\frac{x}{3} - \frac{\pi}{6}\right)$

26. $y = \tan\left(-x + \frac{\pi}{4}\right) + 2$

27. Write an equation for a sinusoidal graph with the following properties:

$$A = -3 \qquad \text{period} = \frac{2\pi}{3} \qquad \text{phase shift} = -\frac{\pi}{4}$$

28. Logan has a garden in the shape of a sector of a circle; the outer rim of the garden is 25 feet long and the central angle of the sector is 50°. She wants to add a 3-foot-wide walk to the outer rim; how many square feet of paving blocks will she need to build the walk?

29. Hungarian Adrian Annus won the gold medal for the hammer throw at the 2004 Olympics in Athens with a winning distance of 83.19 meters.* The event consists of swinging a 16-pound weight attached to a wire 190 centimeters long in a circle and then releasing it. Assuming his release is at a 45° angle to the ground, the hammer will travel a distance of $\frac{v_0^2}{g}$ meters, where $g = 9.8$ meters/second^2 and v_0 is the linear speed of the hammer when released. At what rate (rpm) was he swinging the hammer upon release?

*Annus was stripped of his medal after refusing to cooperate with postmedal drug testing.

CUMULATIVE REVIEW

1. Find the real solutions, if any, of the equation $2x^2 + x - 1 = 0$.

2. Find an equation for the line with slope -3 containing the point $(-2, 5)$.

3. Find an equation for a circle of radius 4 and center at the point $(0, -2)$.

4. Discuss the equation $2x - 3y = 12$. Graph it.

5. Discuss the equation $x^2 + y^2 - 2x + 4y - 4 = 0$. Graph it.

6. Use transformations to graph the function $y = (x - 3)^2 + 2$.

7. Sketch a graph of each of the following functions. Label at least three points on each graph.

(a) $y = x^2$ (b) $y = x^3$ (c) $y = e^x$
(d) $y = \ln x$ (e) $y = \sin x$ (f) $y = \tan x$

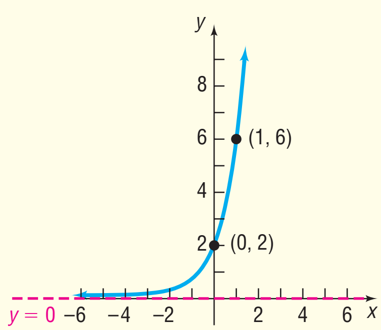
8. Find the inverse function of $f(x) = 3x - 2$.

9. Find the exact value of $(\sin 14°)^2 + (\cos 14°)^2 - 3$.

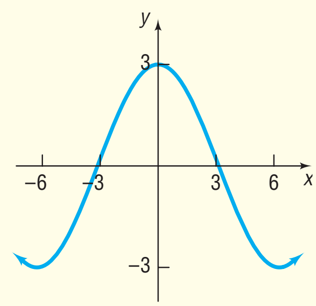
10. Graph $y = 3\sin(2x)$.

11. Find the exact value of $\tan\frac{\pi}{4} - 3\cos\frac{\pi}{6} + \csc\frac{\pi}{6}$.

12. Find an exponential function for the following graph. Express your answer in the form $y = Ab^x$.



13. Find a sinusoidal function for the following graph.



14. (a) Find a linear function that contains the points $(-2, 3)$ and $(1, -6)$. What is the slope? What are the intercepts of the function? Graph the function. Be sure to label the intercepts.

(b) Find a quadratic function that contains the point $(-2, 3)$ with vertex $(1, -6)$. What are the intercepts of the function? Graph the function.

(c) Show that there is no exponential function of the form $f(x) = ae^x$ that contains the points $(-2, 3)$ and $(1, -6)$.

15. (a) Find a polynomial function of degree 3 whose y-intercept is 5 and whose x-intercepts are -2, 3, and 5. Graph the function.

(b) Find a rational function whose y-intercept is 5 and whose x-intercepts are -2, 3, and 5 that has the line $x = 2$ as a vertical asymptote. Graph the function.