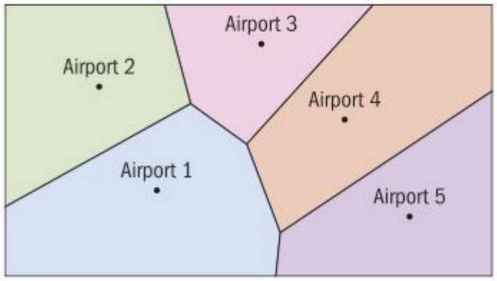
4.5 Voronoi diagrams and the toxic waste dump problem

There are five airports in a state, represented by dots on the diagram. The coloured regions on the diagram indicate the points which are closest to a given airport. For example, Airport 5 is the closest one to all points in the purple region, and Airport 2 is the closest one to the points that are in the green region. What do you think this diagram can be used for? What kinds of questions can it help answer?



This is an example of a Voronoi diagram, named after the mathematician Georgy Voronoy (1868–1908). Voronoi diagrams have a large number of applications nowadays, such as in science, technology and visual art.

Reflect What does a Voronoi diagram show?

The regions that include the closest points to a given point are called **cells**, and the given points are called **sites**. The diagram above shows five sites—the five airports—and five cells—the five coloured regions. The lines separating the regions are called **boundaries** or **edges**.

Investigation 13

Three hospitals A, B and C lie inside a square of side length 10 units, which marks the boundaries of the town. The sides of the square are formed by the lines $x=0$, $x=10$, $y=0$ and $y=10$.

Use technology or graph paper to complete the investigation.

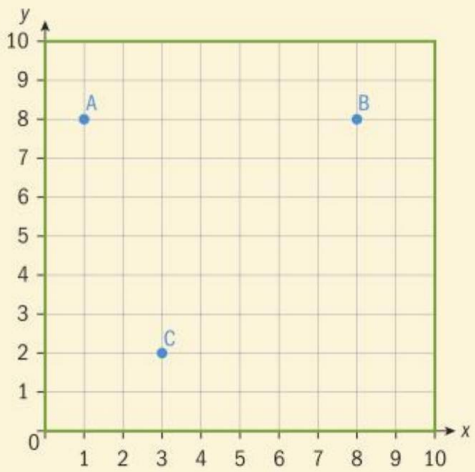
1 On a coordinate system, draw the lines with equations $x=0$, $x=10$, $y=0$ and $y=10$ to mark the boundaries of the town.

Hospitals A and B have coordinates (1, 8) and (8, 8).

2 Locate them on the coordinate system and label them with A and B.

3 Locate the midpoint of [AB] and label it with M1. Draw the perpendicular bisector of [AB].

4 Hospital C has coordinates (3 , 2). Locate the midpoint of [AC] and label it with M2. Draw the perpendicular bisector for A and C of [AC].



Hospitals A, B and C located within the town's boundaries

5 Locate the midpoint of [BC] and label it with M3. Draw the perpendicular bisector of [BC].

6 Would the perpendicular bisector of [AB] intersect the perpendicular bisector of [AC]? Explain why or why not.

Would the perpendicular bisector of [AB] intersect the perpendicular bisector of [BC]? Explain why or why not.

Would the perpendicular bisector of [AC] intersect the perpendicular bisector of [BC]? Explain why or why not.

Would all three perpendicular bisectors intersect in one point or not? Explain.

7 Where would the boundaries of the Voronoi diagram be?

8 **Conceptual** What does the boundary of a Voronoi diagram represent?

9 Use the pentagon tool to shade in a different colour each region separated by the Voronoi diagram boundaries.

10 Which point is equally distant from the three hospitals A, B and C? Is there more than one such point?

11 Samar's house is located at point (4, 5). Determine which hospital is closest to her house.

12 **Conceptual** What does the Voronoi diagram show?

13 **Conceptual** Why is a vertex always equidistant from three sites?

14 **Conceptual** How many edges meet at a vertex of a Voronoi diagram?

The boundaries of the **cells** in a Voronoi diagram are formed by the perpendicular bisectors of the line segments joining the sites. Usually all other lines that do not form boundaries to the cells are erased when the construction of the diagram is complete.

A point at which cell boundaries meet in a Voronoi diagram is called a **vertex**. A vertex is equidistant from the three surrounding sites.

Investigation 14

There are three fire stations, A, B and C, in a town. The coordinates of the fire stations are A(1, 4), B(1, 0) and C(4, 3).

In order to improve response times, the township has installed a new centralized fire response system, which allows a dispatcher to send a fire truck from the nearest fire station to the location of the fire. How should the township be divided into regions so that there is one fire station in each region and this fire station is the closest one to each house in the region?

Use technology or graph paper to complete the investigation.

1 Plot the fire stations A and B on coordinate axes, either by hand or using a software package.

2 Draw the perpendicular bisector of [AB] and gently shade those points nearest to A. How would you divide the township into two regions if fire station C is not yet in operation?

3 Add fire station C to the diagram, then find the equations of the perpendicular bisectors of [AC] and [BC] and add these to the diagram. How would you divide the township into three sections when the three fire stations A, B and C are fully operational?

4 The township is relocating fire station D within its territory. Station D will have location (5, 1). Divide the township into four regions for the four fire stations A, B, C and D.

5 Why does adding the fire stations one by one make sense for creating a town map?

6 What is the incremental algorithm?

TOK

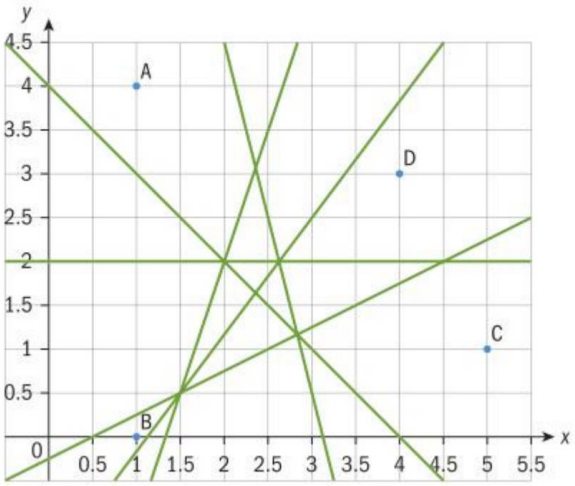
John Snow used Voronoi diagrams to show that the Broad Street cholera outbreak that killed 616 people in London in 1852–60 was due to contaminated water and not air.

HINT

Many software packages allow you to draw and obtain the equations of perpendicular bisectors directly. If not using software, a perpendicular bisector can be constructed using a compass and a ruler or by finding its equation and sketching it.

EXAM HINT

In an exam you will not be asked to draw a perpendicular bisector unless you already have its equation.



It can be difficult to draw all the perpendicular bisectors at the same time and then create a Voronoi diagram, especially if the number of sites is more than three. The diagram shows that even with just four points (sites) there are difficulties in deciding which perpendicular bisectors to use to form boundaries of the cells.

The **incremental algorithm** described below avoids this problem by adding each of the sites one at a time.

Incremental algorithm for constructing Voronoi diagrams

Method

1 Plot the points A and B on coordinate axes, either by hand or using a software package. Draw the perpendicular bisector of line segment [AB] and gently shade the region of points nearest to A.

2 Add point C to the diagram, then find the equations of the perpendicular bisectors of [AC] and [BC] and add these to the diagram.

3 The incremental algorithm:

- **i** Begin with the perpendicular bisector which lies between the new site C and the site in whose cell this vertex currently lies.
- **ii** Move along this line until you reach an intersection with another of the perpendicular bisectors between the new site and an existing one. (This will also be on a boundary of the previous Voronoi diagram.)
- **iii** Leave the intersection along this other perpendicular bisector in the direction that lies entirely in the cell surrounding another of the sites. (This will be the direction that creates a convex polygon around the new site.)
- **iv** The algorithm stops either when you return to your starting point (if the cell is bounded) or if there are no more intersections (if the cell is unbounded). In this case you may need to reverse the direction of the algorithm to ensure all sides have been found.

4 Shade the three regions making up the new Voronoi diagram.

5 Having completed the diagram for the three sites A, B and C, you can now add the next site, D. Repeat steps 2 and 3. This time there will be two intersection points with the perpendicular bisectors of [BD] and [AD].

TOK

Is mathematics independent of culture?

HINT

Usually a final version of the Voronoi diagram will have the perpendicular bisectors removed so that only the edges of the regions remain.

Reflect What is the incremental algorithm?

Exercise 4I

In these questions, unless told to calculate the equations of the perpendicular bisectors, you can construct the lines with a pair of compasses, by eye or by using software.

1 By finding the perpendicular bisectors between each pair of points, use the incremental algorithm to complete the Voronoi diagrams for the given sites.

a $(1, 1), (3, 1), (2, 3)$

b $(1.1), (3, 1), (3, 5)$

2 A neighbourhood has three bookstores located at $A(1, 1)$, $B(5, 1)$ and $C(5, 5)$.

a Construct the Voronoi diagram for the three bookstores A, B and C either by hand or using a software package.

b Determine the closest bookstore to Kari, whose house is located at $(1, 4)$.

A fourth bookstore was built recently at point $D(3, 5)$.

c Construct the Voronoi diagram for the four bookstores A, B, C and D either by hand or using a software package.

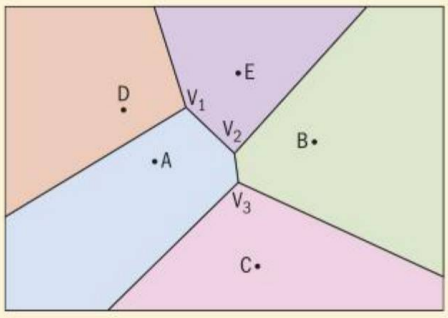
d Determine whether there is another bookstore that is now closest to Kari's house. If so, determine which one.

The toxic waste dump problem

This problem is to find the point on the Voronoi diagram that is as far as possible from any of the sites. It is called the **toxic waste dump problem** because one application might be to find where waste can be deposited so that it is as far as possible from habitation.

Investigation 15

Toxic waste is to be deposited between three towns A, B and C. Find the point that is the farthest from all three towns.



1 On a copy of the Voronoi diagram draw a circle that will pass through points A, E and D. Identify the centre of this circle.

2 Draw another circle through points A, B and E. Identify the centre of this circle.

3 Draw a third circle through points A, B and C. Identify the centre of this circle.

4 Write down what you notice.

5 Explain using the diagram below why there will always be three points (sites) equidistant from a vertex of a Voronoi diagram which has three edges incident to it.

6 Explain why the circle with V_3 as a centre which passes through A will also pass through B and C.

7 Determine the radius of each of the three circles from parts **1**, **2** and **3**. Identify the circle with the largest radius.

8 Determine the location of the toxic waste deposit so that it is farthest from the surrounding towns A, B, C, D and E.

Continued on next page

9 Explain why another site (town) cannot be inside the largest circle, assuming that no new sites are added.

10 **Conceptual** Where is the point that gives the solution to the toxic waste problem?

11 **Conceptual** What applications are there for the solutions of the toxic waste problem?

12 **Conceptual** Why are vertices of a Voronoi diagram the only points that give the solution to the toxic waste problem (assuming a location between the towns is sought)?

EXAM HINT

Within a Voronoi diagram the solution to the toxic waste problem will be at an intersection of cell boundaries or on the boundary of the diagram. In exams the solution will always be one of the internal vertices rather than a boundary edge.

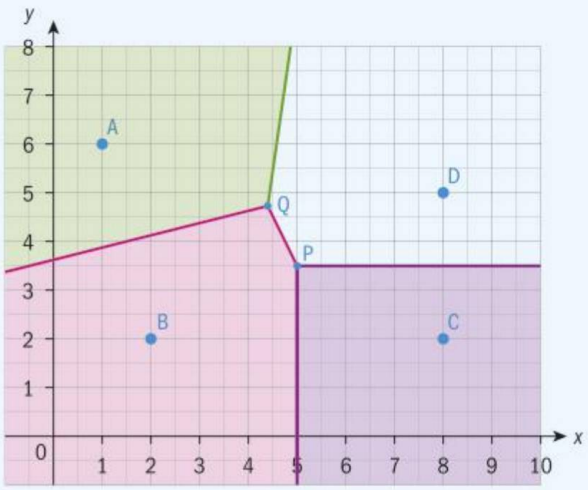
Example 31

A town has four coffee shops. An entrepreneur wishes to open a new shop in the town but would like it to be as far as possible from all the other four coffee shops. Where should he put it?

Consider the Voronoi diagram below showing the positions of the four coffee shops on a set of coordinate axes: A(1, 6), B(2, 2), C(8, 2) and D(8, 5), where one unit represents 1 km.

a Find the coordinates of the vertices P and Q in the Voronoi diagram.

b Determine where a fifth shop should be sited so as to be as far as possible from any other shop, and how far this will be.



a P is the point where $x = 5$ and $y = 3.5$ meet: (4, 3.5)	Three perpendicular bisectors meet at each of the vertices. Finding the intersection of any two perpendicular bisectors will be sufficient to find the vertex.
The perpendicular bisector of [AB] is $-x + 4y = 14.5$.	Any two of these equations need to be calculated by finding the midpoint and gradient.

The perpendicular bisector of [AD] is $7x - y = 26$.

The perpendicular bisector of [BD] is $2x + y = 13.5$.

The coordinates of Q are (4.39, 4.72).

The coordinates can then be found algebraically or by using a GDC.

b Centred at P: $PD = \sqrt{(5-3.5)^2 + (8-5)^2}$

$= 3.35$

The solution will be at whichever of the points P and Q is farthest from the three sites nearest to them.

Centred at Q:

$QA = \sqrt{(6-4.72)^2 + (4.39-1)^2} = 3.62$

Only one length for each vertex needs to be checked as the other two sites will be an equal distance from the vertex.

The new coffee shop should therefore be built at point Q.

Reflect How do you find the distance from a site to a vertex?

Where is the point that gives the solution to the toxic waste problem?

How do you find the point that gives the solution to the toxic waste problem?

International-mindedness

Which do you think is superior: the Bourbaki group analytical approach or the Mandelbrot visual approach to mathematics?

Exercise 4J

1 In a town three schools A, B and C are located at the points with coordinates A(1, 3), B(6, 4) and C(6, 1).

It is decided that a new school should be built as close as possible to a point which is farthest from all three existing schools.

a Explain why this point will be at the intersection of the perpendicular bisectors of line segments [AB], [BC] and [AC].

b Find the equations of the perpendicular bisectors of line segments [AB] and [BC].

c Hence find the coordinates of the point where the new school should be built.

d Determine how far the new school will be from each of the other schools.

2 At a fair there are three hamburger stands, A, B and C. The fairground is in the shape of a rectangle with dimensions 100 m by 50 m. The bottom-left corner of the field can be regarded as the origin of a coordinate system, with the diagonally opposite corner as (100, 50). The hamburger stands are at the points A(20, 30), B(80, 30) and C(40, 10).

a Find the equations of the perpendicular bisectors of:

i A and C **ii** B and C.

People will always go to the hamburger stand that is closest to them.

b Draw the Voronoi diagram that represents this situation.

c Find the region of the fairground which is closest to:

i stand C **ii** stand A.

d A fourth hamburger stand is to be added to the fairground at a point as far as possible from the other three stands.

i State the coordinates of the position at which it should be built.

ii Determine how far the new stand will be from the other hamburger stands.

3 A town can be considered as a rectangle which runs 10 km east to west and 8 km north to south. A coordinate grid is placed on a map of the town with the origin in the south-west corner. There are four schools in the town, A, B, C and D, whose coordinates are A(2, 5), B(3, 3), C(8, 6) and D(8, 1).

Children go to the school that is closest to their home when measured in a direct line.

An estate agent wishes to construct a diagram that shows in which school's catchment area a house lies.

a Find the perpendicular bisector of [AB].

b Show the positions of A and B and their perpendicular bisector on a diagram of the town.

The perpendicular bisectors of [AC] and [BC] are $y = -6x + 35.5$ and $y = -\frac{5}{3}x + \frac{41}{3}$.

c Construct the Voronoi diagram, using the incremental algorithm, for the three schools A, B and C.

d **i** Find the perpendicular bisectors of [BD] and [CD] and show these on the diagram.

ii Complete the diagram required by the estate agent.

e Find the coordinates of the two vertices where three edges meet.

f A fifth school is to be built in the town as far as possible from the other schools. Give the coordinates of the point at which it should be built if it was to meet this requirement.

g On the diagram already drawn, add the fifth school at the position found in part **f** and **sketch** the new Voronoi diagram. There is no need to find equations for the new perpendicular bisectors.

Chapter summary

- In two-dimensional (2D) space, the midpoint, M, of the line segment joining points $A(x_1, y_1)$ and $B(x_1, y_1)$ is $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- In three-dimensional (3D) space, the location of any point A is determined by an ordered triple of coordinates (x, y, z), where each coordinate determines the location along the x-, y- and z-axis respectively.
- In 3D space, the midpoint, M, of the line segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
- In 2D, the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- In 3D, the distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Developing your toolkit

Now do the Modelling and investigation activity on page 202.