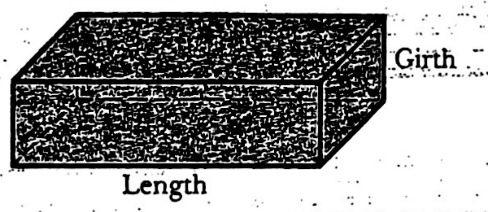
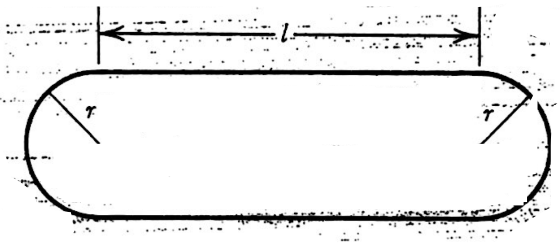
Optimization

# ▶ EXERCISE SET 3.2

1. Find the positive number with the property that the sum of the number and its reciprocal is a minimum.
2. Find the positive number with the property that the sum of the square of the number and the square of its reciprocal is a minimum.
3. A rectangular plot of ground containing 432 $ft^2$ is fenced off in a large lot. Find the dimensions of the plot that requires the least amount of fence.
4. A rectangular plot of ground containing 432 $ft^2$ is fenced off in a large lot, and a fence is constructed down the middle of the lot to separate it into equal parts. Find the dimensions of the plot that requires the minimal amount of fencing.
5. Suppose the fence used to enclose the plot of ground described in Exercise 4 costs \$10 per foot and the fence used to divide the plot into parts costs \$5 per foot. Find the dimensions of the plot that requires the least expense for fencing.
6. A rectangular dog run is to contain 864 $ft^2$.
   a) If the dog's owner must pay for the fencing, what should be the dimensions of the run to minimize cost?
   b) Suppose a neighbor has agreed to let the owner use an already constructed fence for one side of the run. What should the dimensions of the run be in this situation if the owner's cost is to be a minimum?
7. A rectangular box with no top is to contain 2250 $in^3$. Find the dimensions to minimize the amount of material used to construct the box if the length of the base is three times the width.
8. Suppose the box described in Exercise 7 is constructed with a top. What dimensions would minimize the amount of material required?
9. The United States Postal Service has recently decreed that no rectangular-shaped parcel can be mailed if the total of its length and girth (perimeter of a cross section) exceeds 108 in. (see the figure). Find the maximum volume in a rectangular parcel that can be mailed if a cross section of the parcel is a square.

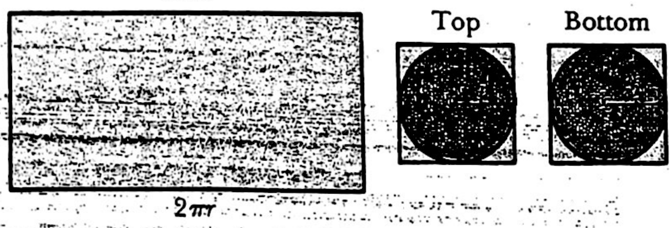



10. The speed of a point on the rim of a flywheel $t$ sec after the flywheel has started to turn is given by the formula $v = 36t^2 - t^3$ ft/sec.
    a) Find its greatest speed.
    b) How long does it run before it reaches this speed?
11. The turning effect of a ship's rudder is found to be $T = k \cos\theta (\sin\theta)^2$, where $k$ is a positive constant and $\theta$ is the angle that the direction of the rudder makes with the keel line of the ship ($0 \le \theta \le \pi/2$). For what value of $\theta$ is the rudder most effective?
12. A 1-mi race track is to be built with two straight sides and semicircles at the ends (see the figure). What is the maximum amount of area needed to construct the track?




13. A standard can contains a volume of 900 $cm^3$. The can is in the shape of a right circular cylinder with a top and bottom. Find the dimensions of the can that minimize the amount of material needed for construction.
14. In constructing a can in the shape of a right circular cylinder, no waste is produced when the side of the can is cut, but the top and bottom are each stamped from a square sheet and the remainder is wasted (see

the figure). Find the relative dimensions of the can that uses the least amount of material with this construction method.




15. An open rectangular box is to be made from a piece of cardboard 8 in. wide and 8 in. long by cutting a square from each corner and bending up the sides. Find the dimensions of the box with the largest volume.

16. A rectangular plot that will contain a vineyard of one acre in area (43,560 $ft^2$) is to be laid out. The vineyard must have a boundary of 8 ft on all sides in order for equipment to pass and an 8-ft pathway down the middle. What is the minimal acreage required for this plot?

17. A charter bus company charges $10 per person for a round-trip to a ball game with a discount given for group fares. A group purchasing more than 10 tickets at one time receives a reduction per ticket of $0.25 times the number of tickets purchased in excess of 10. Determine the maximum revenue that can be received by the bus company.

18. A hotel with 25 rooms normally charges $40 for a room; however, special group rates are advertised: If the group requires more than 5 rooms, the price for each room is decreased by $1 times the number of rooms exceeding 5. Find the maximum revenue that the hotel can receive from a group.

19. A rectangle is placed inside a circle of radius $r$ with its corners on the boundary of the circle (see the figure). Of all such rectangles, find the dimensions of the one that encloses the maximum area.

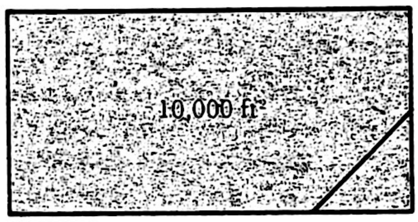
20. A rectangle is placed inside a circle of radius $r$ with its corners on the boundary of the circle. What dimensions should be given to the rectangle to maximize the sum of its perimeter and the length of its two diagonals?

21. An isosceles triangle is placed inside a circle of radius $r$ with its vertices on the boundary of the circle (see the figure). How should this be accomplished if the area of the triangle is to be maximized?

22. "The isosceles triangle with two fixed equal sides and maximum area is not an equilateral triangle." Show that this statement is true by finding the length of the base of an isosceles triangle that maximizes the area over all such triangles.

23. The area of the print on a book page is 42.5 $in^2$. The margins are 1 in. on the sides and bottom and 1/2 in. at the top. What should be the dimensions of a page of this book if the only object is to use the minimal amount of paper?

24. A field is fenced off in the form of a rectangle containing 10,000 $ft^2$. In addition to the fencing required for the perimeter of the field, an isosceles triangle is fenced off in one corner by running a fence from the midpoint on the shortest side to the adjacent side enclosing the corner of the rectangle (see the figure). Find the dimensions of the field that minimizes the amount of fencing required.




25. A wire 1 ft long is cut in two pieces: One piece is used to construct a square, the other to construct an equilateral triangle. Where should the cut be made in order to minimize the sum of the areas of the figures?

26. A wire 1 ft long is cut in two pieces: One piece is used to construct a square, the other to construct a circle. Where should the cut be made in order to minimize the sum of the areas of the figures?

27. Find the volume of the largest right circular cylinder that can be placed inside a sphere of radius 1.

28. Find the volume of the largest right circular cone that can be placed inside a sphere of radius 1.

29. The strength of a rectangular beam is directly proportional to the product of the width of the beam and the square of its depth. Find the dimensions of the strongest beam that can be cut from a log with radius $r$.

30. The stiffness of a rectangular beam is directly proportional to the product of the width of the beam and the cube of its depth. Find the dimensions of the stiffest beam that can be cut from a log with radius $r$.

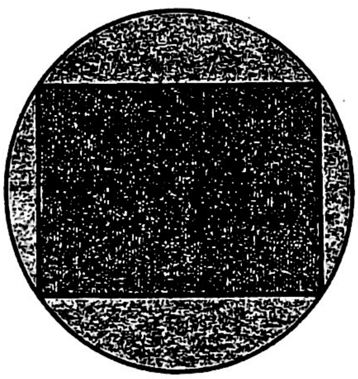


**Figure for Exercise 19.**

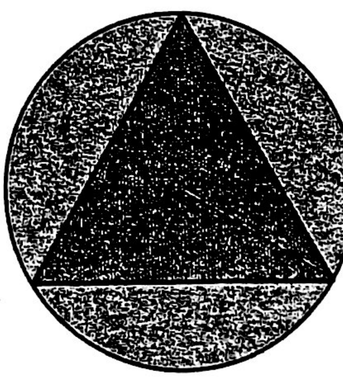


**Figure for Exercise 21.**

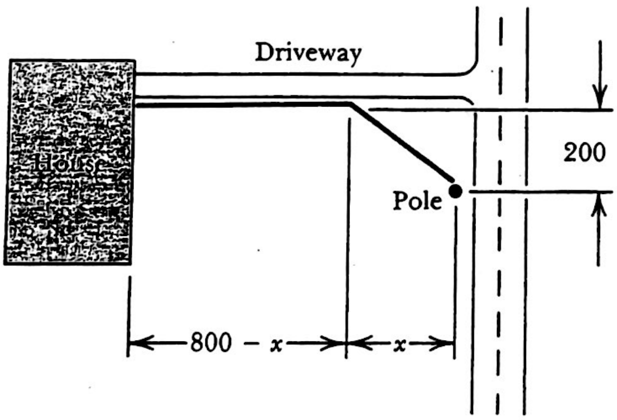
**31.** Suppose $(x_0, y_0)$ is a point that does not lie on the circle $x^2 + y^2 = 1$. Show that the shortest distance from $(x_0, y_0)$ to the circle is along a line that passes through the center of the circle.

**32.** Show that the shortest distance from the point $(x_0, y_0)$ to the line with equation $Ax + By + C = 0$ is

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

**33.** The maximum volume in the problem in Example 7 can be found without using trigonometry by expressing the area $A$ totally as a function of $x$. Show that this method of solution gives the same result as that found in Example 7.

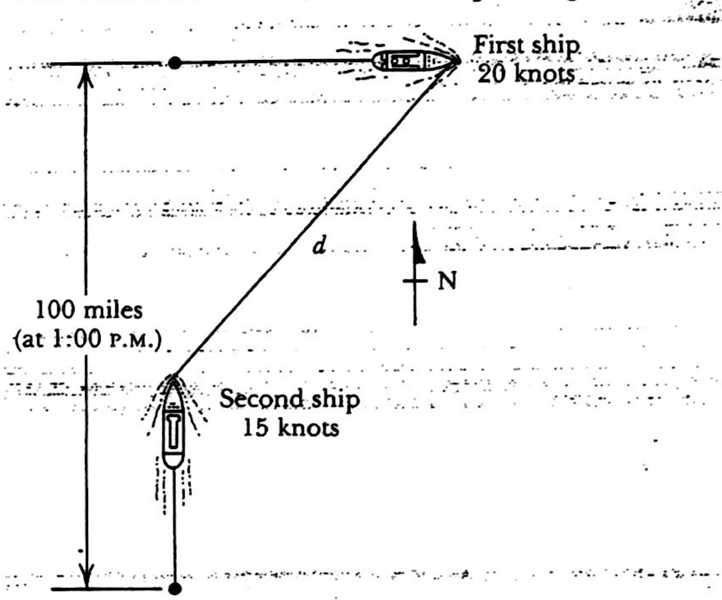
**34.** A house is built with a straight driveway 800 ft long, as shown in the figure. A utility pole on a line perpendicular to the driveway and 200 ft from the end of the driveway is the closest point from which electricity can be furnished. The utility company will furnish power with underground cable at \$2 per foot and with overhead lines at no charge. However, for overhead lines the company requires that a strip of 30 ft wide be cleared. The owner of the house estimates that to clear a strip this wide will cost \$3 for each foot of overhead wire used. At what point on the driveway should the switch from overhead to underground be made in order to minimize the cost?




**35.** A crew of painters is assigned to paint a second-floor wall on the outside of a building along a busy sidewalk. They must leave a corridor, for unsuperstitious pedestrians, between the wall and their ladders. The corridor is 6 ft wide and 8 ft high. What is the minimal length of ladder they can use to reach the wall, and how far from the base of the wall should it be placed?

**36.** A bully armed with a knowledge of calculus is planning an attack on his next victim. The attack must be made on a sidewalk between two lights that are 200 ft apart, one of which is twice as bright as the other. Before dropping out of high school the bully took a physics course and recalls that the intensity of illumination from a light varies inversely as the square of the distance from the light. Where will he attack if he always attacks at the darkest point between the lights?

**37.** Two tankers are traveling in the midst of the Atlantic Ocean. The first tanker is 100 nautical miles due north of the second at 1:00 P.M. GMT (Greenwich Mean Time) and traveling due east at the rate of 20 knots. The second tanker is traveling due north at 15 knots. At what time are the tankers closest together, and what is the minimal distance separating them?




**38.** A warehouse is to be built beside a long straight highway running north and south (see the figure at the top of page 203). This warehouse will house equipment produced in two factories and sent there by air for storage. The northern factory lies 80 mi east of the highway; the other lies on the highway. The point on the highway that is closest to the northern factory is 100 mi north of the second factory. Where should the

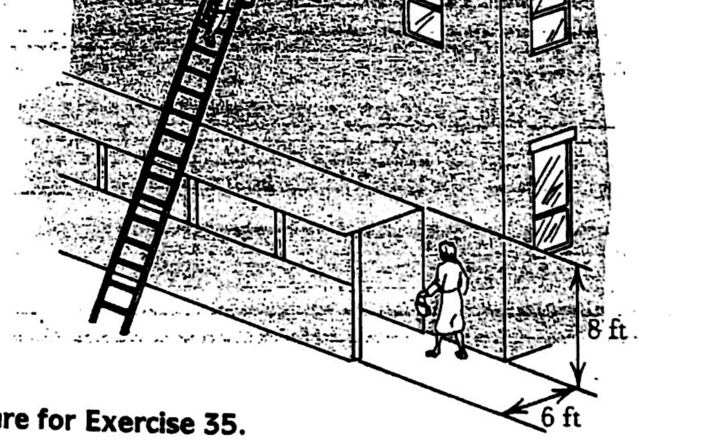



**Figure for Exercise 35.**

Related Rates

## EXERCISE SET 3.3

1. A metal sphere is heated so that its radius increases at the rate of 1 mm/sec. How fast is its volume changing when its radius is 30 mm?

2. The radius of a circle is increasing at the rate of 3 cm/sec. At what rate is the area of the circle increasing when the radius is 20 cm?

3. The sides of a square are increasing at the rate of 2 in./min. At what rate is the area of the square increasing when the sides are 4 in.?

4. The length of a rectangle is three times its width and the length is increasing at the rate of 9 in./sec. How fast is the area of the rectangle changing?

5. The edges of a cube are increasing at the rate of 2 in./min. At what rate is the volume of the cube increasing when the edges are 4 in.?

6. Reconsider the cube described in Exercise 5. At what rate is the total surface area of the cube increasing when the edges are 4 in.?

7. The sides of an equilateral triangle are increasing at the rate of 1 cm/sec. At what rate is the area of the triangle increasing when the sides are 4 cm?

8. The two equal sides of an isosceles triangle are increasing at the rate of 1 cm/sec while the third side is held at 4 cm. At what rate is the area of the triangle increasing when the sides are all 4 cm?

9. A stone is dropped into a pool of still water from a height of 150 ft. Circular ripples radiate at the rate of 3 in./sec from the spot where the stone hits the water.
   a) What is the area of the disturbed water 4 sec after the stone hits?
   b) How fast is the area changing at this time?

10. A certain yeast culture grows in a circular colony. As it grows the surface area it covers is directly proportional to its population and contains $10^5$ members when the area is 1 $cm^2$. How fast is the population increasing when the radius of the circle is 12 cm if the radius of the circle is increasing at the rate of 3 cm/hr?

11. Gas is pumped into a spherical balloon at the rate of 1 $ft^3$/min. How fast is the diameter of the balloon increasing when the balloon contains 36 $ft^3$ of gas?

12. Flour sifted onto waxed paper forms a conical pile whose radius and height are always equal, although both increase with time (see the figure). The volume of flour on the waxed paper is increasing at the rate of 7.26 $in^3$/sec. How fast is the height of the flour increasing when the volume is 29 $in^3$?

13. Find the rate of change of the area of a circle with respect to its radius. Compare this with the circumference of the circle.

14. Find the rate of change of the volume of a sphere with respect to its radius. Compare this with the surface area of a sphere.

15. Work the problem in Example 5 by first expressing the distance $z$ between the ships explicitly as a function of $t$.

16. A 15-ft ladder is leaning against a wall, and its base is pushed toward the wall at the rate of 2 ft/sec. How fast is the top of the ladder moving up the wall when the top is 9 ft from the ground?

17. An 8-ft $2 \times 4$ is leaning against a 10-ft wall. The lower end of the $2 \times 4$ is pulled away from the wall at the rate of 1 ft/sec. How fast is the top of the $2 \times 4$ moving toward the ground (a) when it is 5 ft from the ground and (b) when it is 4 ft from the ground?

18. An 8-ft $2 \times 4$ is leaning against a 5-ft wall with the remainder of the $2 \times 4$ hanging over the wall. The lower end of the $2 \times 4$ is pulled away from the wall at a rate of 1 ft/sec. How fast is the top of the $2 \times 4$ moving toward the ground when this end is 5 ft from the ground, that is, when the upper end just reaches the wall?

19. A rectangular swimming pool 50 ft long and 30 ft wide is being filled with water to a depth of 8 ft at the rate of 3 $ft^3$/min.
    a) How long does it take to fill the pool?
    b) At what rate is the depth of water in the pool increasing when the pool is half full of water?

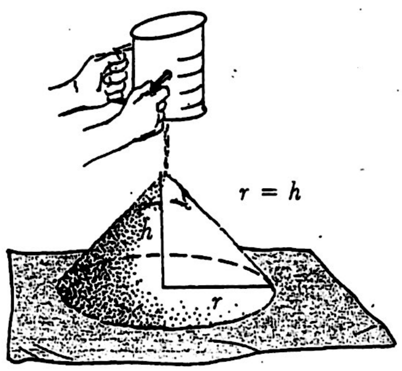
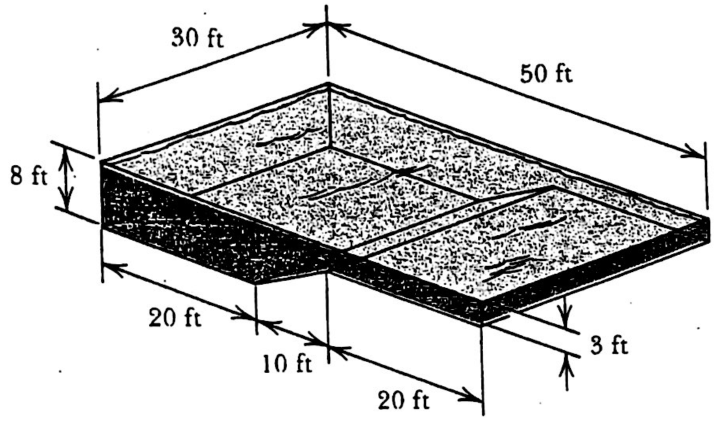



**Figure for Exercise 12.**

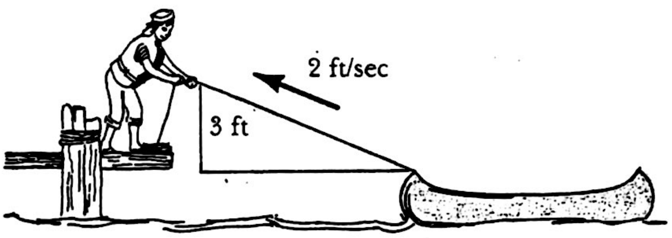
**20.** A rectangular swimming pool 50 ft long and 30 ft wide has a depth of 8 ft for the first 20 ft of its length and a depth of 3 ft on the last 20 ft of its length, and tapers linearly for the 10 ft in the middle of its length (see the figure). The pool is being filled with water at the rate of 3 $ft^3$/min.

a) How long does it take to fill the pool?

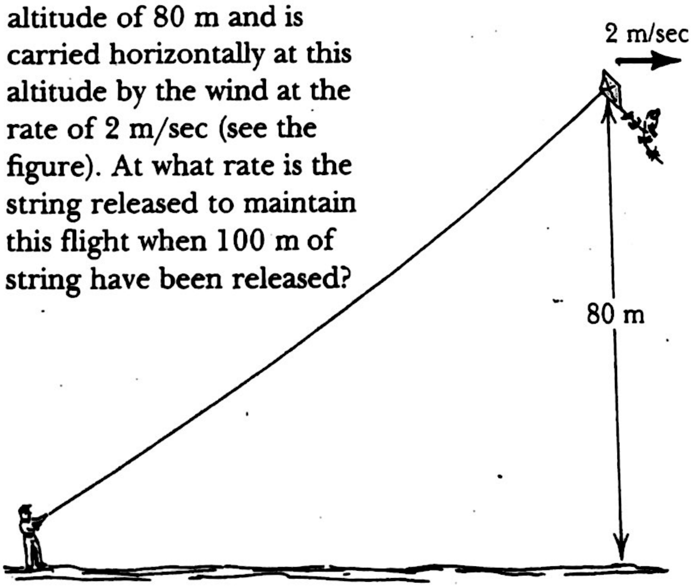
b) At what rate is the depth of water in the pool increasing when the pool is half full of water?




**21.** A woman on a dock is using a rope to pull in a canoe. The rope is pulled at the rate of 2 ft/sec, 3 ft above the point level with the connection of the rope to the canoe (see the figure). How fast is the canoe approaching the dock when the length of rope from her hands to the canoe is 10 ft?




**22.** A kite is flying at an altitude of 80 m and is carried horizontally at this altitude by the wind at the rate of 2 m/sec (see the figure). At what rate is the string released to maintain this flight when 100 m of string have been released?




**23.** A kite is carried horizontally at the rate of 1.5 m/sec and is rising at 2.0 m/sec. How fast is the string released to maintain this flight when 100 m of string have been released and the kite is at an altitude of 80 m?

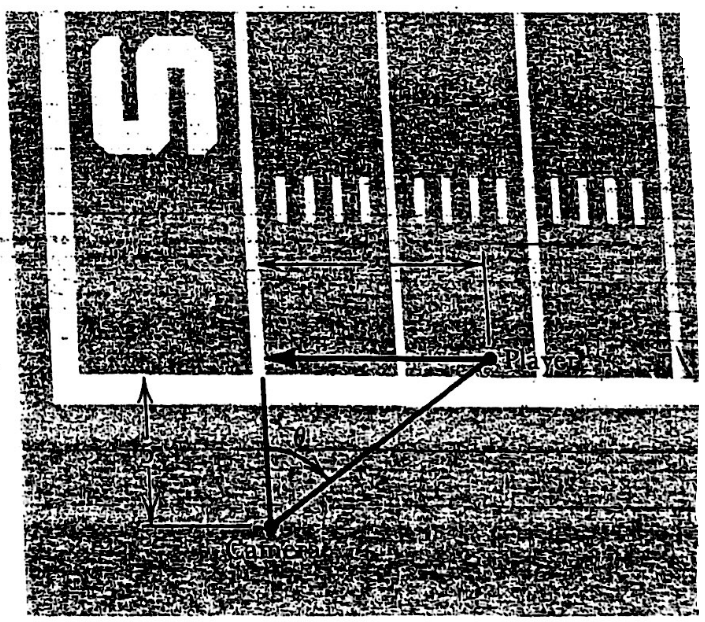
**24.** A single-engine airplane passes over a beacon and heads east at the rate of 100 mph. Two hours later a jet passes over the beacon at the same altitude traveling north at 400 mph. Assuming that the planes stay on these courses, how fast are they separating 1 hr after the jet has passed over the beacon?

**25.** Two ships meet at a point in the ocean with one of the ships traveling south at the rate of 15 mph and the other traveling west at 20 mph. At what rate are the ships separating 2 hr after they meet?

**26.** Two cars approach an intersection at right angles. One car is traveling at the rate of 50 mph and the other is traveling 40 mph. How fast are the cars approaching each other when the first car is 30 ft from the intersection and the second is 40 ft from the intersection?

**27.** A fisherman sitting on the end of a pier with his pole 3 m above the water snags what he assumes to be a large fish and reels in his line at the steady rate of 1 m/sec. He does not realize that the object is actually an old log lying just below the surface until the log is 5 m from the end of the rod. How fast is the log approaching the pier at this time?

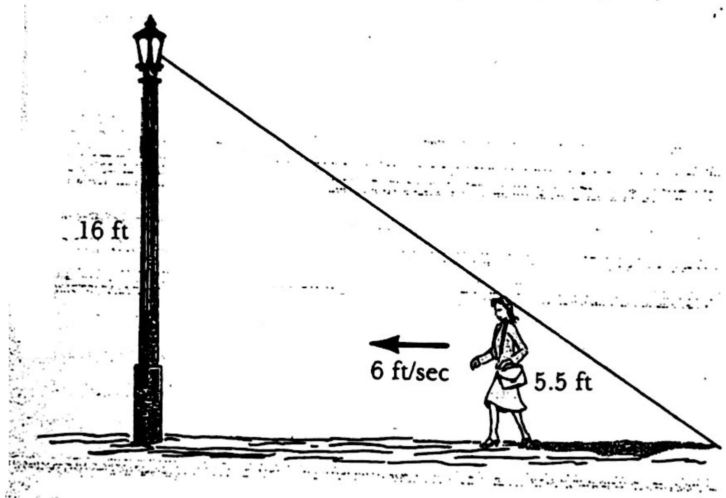
**28.** A camera televising the return of the opening kickoff of a football game is located 5 yd from the east edge of the field and in line with the goal line. The player with the football runs down the east edge (just in bounds) for a touchdown. When he is 10 yd from the goal line, the camera is turning at a rate of 0.5 radian/sec. How fast is the player running?



29. A revolving beacon located 1 mi from a straight shoreline turns at 1 revolution per minute. Find the speed of the spot of light along the shore when it is 2 mi away from the point on the shore nearest the light.

30. A metal cylinder contracts as it cools, the height of the cylinder decreasing at $4.5 \times 10^{-4}$ cm/sec and the radius decreasing at $3.75 \times 10^{-5}$ cm/sec. At what rate is the volume of the cylinder decreasing when its height is 200 cm and its radius is 10 cm?

31. A woman 5 ft 6 in. tall walks at the rate of 6 ft/sec toward a street light that is 16 ft above the ground.
   a) At what rate is the tip of her shadow moving?
   b) At what rate is the length of her shadow changing when she is 10 ft from the base of the light?




32. Suppose Farmer MacDonald constructs a hog trough to maximize capacity in the manner described in Example 7 of Section 3.2. If the hogs continuously consume the slop at the rate of 1.2 ft$^3$ per hour, how fast is the height of the slop decreasing when the height of the slop in the trough is 6 in.?

33. A horse trough 10 ft long has a cross section in the shape of an inverted equilateral triangle with an altitude of 2 ft. The trough leaks water through a crack in the bottom at the rate of 1 ft$^3$/hour.
   a) At what rate is the height of the water in the trough decreasing when the depth of the water is 1 ft?
   b) At what rate is the height of the water in the trough decreasing when the trough contains 10 ft$^3$ of water?

34. A picture with height 3 ft is placed on a wall with its base 3 ft above an observer's eye level. The observer approaches the wall at the rate of 1 ft/sec. How fast is the angle $\theta$, shown in the figure, changing when the observer is 10 ft from the wall?

35. An object that weighs $w_0$ lb on the surface of the earth weighs approximately

$$w(r) = w_0\left(\frac{3960}{3960 + r}\right)^2 \text{ lb}$$

when lifted a distance of $r$ mi from the earth's surface. Find the rate at which the weight of an object weighing 2000 lb on the earth's surface is changing when it is 100 mi above the earth's surface and is being lifted at the rate of 10 mi/sec.

36. The owner of a dog kennel reads in *Dog's Life* that the surface area of a dog is approximately related to its weight by the equation

$$s = 0.1w^{2/3},$$

where the weight $w$ of the dog is measured in kilograms and the surface area $s$ of the dog is measured in square meters. The amount of flea powder the owner must purchase is directly proportional to the surface area of the dogs. If the average pup in the kennel gains weight at the approximate rate of 0.8 kg/wk, at what rate is the purchase of powder increasing when there are 23 dogs, the average dog weighs 20 kg, and a can of powder covers 3 m$^2$?

37. Oil is leaking from an ocean tanker at the rate of 5000 L/sec. The leakage results in a circular oil slick with a depth of 5 cm. (*Note:* 1 liter = 1000 cm$^3$.)
   a) How fast is the radius of the oil slick increasing when the radius is 300 m?
   b) How fast is the radius of the oil slick increasing 4 hr after the leakage has begun?

38. In actual practice an oil slick like the one described in Exercise 37 does not have a constant depth; the depth of the slick decreases as the oil moves from the point of spillage and depends primarily on the turbulence of the water and the viscosity of the oil. Suppose the

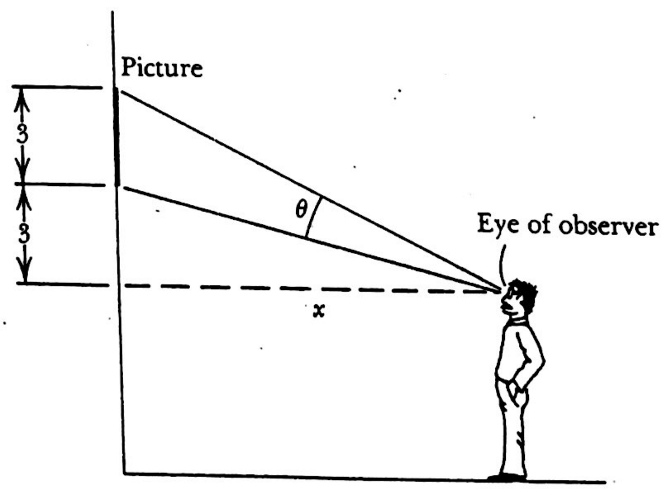



**Figure for Exercise 34.**