Calculus BC Bible

(3rd most important book in the world)

(To be used in conjunction with the Calculus AB Bible)

APPLICATIONS OF THE INTEGRAL

Rectangular Equations $(y = f(x))$

Length of a Curve (Arclength)

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$

Surface Area

$$S.A. = \int_a^b 2\pi\, f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx$$

Parametric Equations

Parametric equations are set of equations, both functions of t such that:

$x = f(t)$ $\qquad$ $y = g(t)$

Length of a Curve

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt \quad \text{or} \quad \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Surface Area

$$S.A. = \int_a^b 2\pi\, g(t)\sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt$$

Speed Equation:

$$speed = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$$

If a smooth curve c is given by the equation, $x = f(t)$ and $y = g(t)$, then the slope of c at (x, y) is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}}$$

EX#1:

$x = \sqrt{t} \quad y = \frac{1}{4}\left(t^2 - 4\right) \quad t \geq 0$

$$\frac{dx}{dt} = \frac{1}{2}t^{-1/2} \qquad \frac{dy}{dt} = \frac{1}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2t^{1/2}}} = 3t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}}$$

$$\frac{d^3y}{dx^3} = \frac{3}{\frac{1}{2t^{1/2}}} = 6t^{1/2}$$

Find slope and concavity at (2, 3)

Must find t

$2 = t^{1/2}$ $\qquad$ $3 = \frac{1}{4}(t^2 - 4)$

$t = 4$ $\qquad$ $t = 4$

$$\left.\frac{dy}{dx}\right|_{(2,3)} = 8$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 12 \quad \textit{Concave up}$$

Polar Coordinates

Polar coordinates are defined as such:

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\dfrac{y}{x}$

Area Under a Curve

$$A = \int_a^b \frac{1}{2}\,(f(\theta))^2\,d\theta$$

or

$$A = \int_a^b \frac{1}{2}\,r^2\,d\theta$$

Area Between Two Curves

$$A = \int_a^b \frac{1}{2}\left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right]d\theta$$

or

$$A = \int_a^b \frac{1}{2}\left[(R)^2 - (r)^2\right]d\theta$$

EX#1: Find the Area of one leaf of the three-leaf rose $r = 3\cos 3\theta$. (One leaf is formed from $-\pi/6$ to $\pi/6$)

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2\,d\theta = \frac{9}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta = \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1-\cos 6\theta}{2}\,d\theta = \frac{9}{4}\theta - \frac{3\sin 6\theta}{8}\Bigg|_{-\pi/6}^{\pi/6}$$

$$= \left(\frac{3\pi}{8} - 0\right) - \left(\frac{-3\pi}{8} - 0\right) = \frac{3\pi}{4}$$

EX#2: Find the Inner Loop of $r = 1 - 2\sin\theta$. ($\sin\theta = \frac{1}{2}$ at $\pi/6$ and $5\pi/6$ to form the inner loop)

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1-2\sin\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(1 - 4\sin\theta + 4\sin^2\theta\right)d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(1 - 4\sin\theta + 4\cdot\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(3 - 4\sin\theta - 2\cos 2\theta\right)d\theta = \frac{3}{2}\theta + 2\cos\theta - \frac{\sin 2\theta}{2}\Bigg|_{\pi/6}^{5\pi/6} = \left(\frac{5\pi}{4} - \sqrt{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{4} + \sqrt{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = 0.543516$$

EX#3: Find the Total Area of $r = 1 - 2\sin\theta$. (Half the figure is between $5\pi/6$ and $3\pi/2$, so we double that area)

$$A = 2\cdot\frac{1}{2}\int_{5\pi/6}^{3\pi/2} (1-2\sin\theta)^2\,d\theta = \int_{5\pi/6}^{3\pi/2} \left(1 - 4\sin\theta + 4\sin^2\theta\right)d\theta = \int_{5\pi/6}^{3\pi/2} \left(1 - 4\sin\theta + 4\cdot\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \int_{5\pi/6}^{3\pi/2} \left(3 - 4\sin\theta - 2\cos 2\theta\right)d\theta = 3\theta + 4\cos\theta - \sin 2\theta\Bigg|_{5\pi/6}^{3\pi/2} = \left(\frac{9\pi}{2} - 0 + 0\right) - \left(\frac{5\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$$

$$= 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} = 8.8812614$$

EX#4: Find the Area of the Outer Loop of $r = 1 - 2\sin\theta$. (Whole figure $-$ Inner Loop)

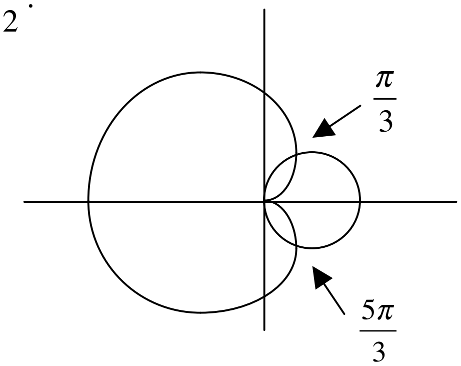
$A = 8.881261 - 0.543516 = 8.337745$

EX#5 : Find the Area of the enclosed region between $r_1 = 2 - 2\cos\theta$ and $r_2 = 2\cos\theta$.

$2 - 2\cos\theta = 2\cos\theta$ when $\cos\theta = \frac{1}{2}$ which is $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

They form two identical enclosures. Find the Area of one enclosure and double it. They meet at $\theta = \frac{\pi}{3}$ in the first enclosure. You have to figure out which equation outlines your encosure. r_1 outlines the enclosure from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and r_2 outlines the enclosure from $\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{2}$.

$$A = 2 \cdot \left[\frac{1}{2} \int_0^{\pi/3} (2 - 2\cos\theta)^2 \, d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (2\cos\theta)^2 \, d\theta \right] = 0.402180$$



INTEGRATION TECHNIQUES

Integration by Parts

Integration by Parts is done when taking the integral of a product in which the terms have nothing to do with each other.

$$\int u \, dv = uv - \int v \, du$$

Refer to the **Calculus AB Bible** for the general technique.
If the derivative of one product will eventually reach zero, use the tabular method.

Tabular method :
List the term that will reach zero on the left and keep taking the derivative of that term until it reaches zero.
List the other term on the right and take the integral for as many times as it takes for the left side to reach zero.

$$EX: \int x\cos x \, dx$$

x	$+$	$\cos x$
1	$-$	$\sin x$
0		$-\cos x$

Multiply the terms as shown. The first one on the left multiplies with the second one on the right and the sign is positive, the second term on the left matches with the third one on the right and is negative, and continue on until the last term on the right is matched up, alternating signs as you go.

$$\therefore \int x\cos x \, dx = x\sin x + \cos x + C$$

Special cases (Integration by Parts)

($\ln x$ must be $u =$ when doing integration by parts)

$$\int \ln x\, dx \quad = \quad x\ln x - \int dx \quad = \quad x\ln x - x + C$$

$u = \ln x \quad dv = dx$

$du = \dfrac{1}{x} \quad v = x$

(When neither function goes away, you may pick either function to be $u =$, but you must pick the same kind of function throughout) (i.e. If you pick the trig. function to be $u =$ then you must pick the trig. function to be $u =$ the next time also. You will perform integration by parts twice.)

1st time

$u = \sin x \qquad dv = e^x$

$du = \cos x\, dx \quad v = e^x$

2nd time

$u = \cos x \qquad dv = e^x$

$du = -\sin x\, dx \quad v = e^x$

$$\begin{aligned}
\int e^x \sin x\, dx &= e^x \sin x - \int e^x \cos x\, dx \\
&= e^x \sin x - \left[e^x \cos x - \int -e^x \sin x\, dx \right] \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \qquad \text{(Add to other side)} \\
2\int e^x \sin x\, dx &= e^x \sin x - e^x \cos x \\
\int e^x \sin x\, dx &= \frac{e^x \sin x - e^x \cos x}{2} + C
\end{aligned}$$

Trigonometric Integrals

$\int \sin^m x \cos^n x\, dx$

if **n is odd**, substitute $u = \sin x$ Useful Identity: $\cos^2 x = 1 - \sin^2 x$

if **m is odd**, substitute $u = \cos x$ Useful identity: $\sin^2 x = 1 - \cos^2 x$

If **m and n are both even**, then reduce so that either m or n are odd, using trigonometric (double angle) identities.

Double angle identities : $\sin 2x = 2\sin x \cos x \quad \cos 2x = \cos^2 x - \sin^2 x$

Half - Angle Identities : $\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

EX#1 : $\int \sin^3 x \cos^4 x\, dx$

$$\begin{aligned}
&= \int \sin x \left(1 - \cos^2 x\right) \cos^4 x\, dx \\
&= \int \sin x \left(\cos^4 x - \cos^6 x\right) dx \\
&= -\int \left(u^4 - u^6\right) du
\end{aligned}$$

$(u = \cos x \quad du = \text{-}\sin x\, dx)$

$$\begin{aligned}
&= \frac{-u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{-(\cos x)^5}{5} + \frac{(\cos x)^7}{7} + C
\end{aligned}$$

EX#2 : $\int \sin^2 x \cos^5 x\, dx$

$$\begin{aligned}
&= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x\, dx \quad (\textit{Each } \cos^2 x = 1 - \sin^2 x) \\
&= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x\, dx \\
&= \int \left(u^2 - 2u^4 + u^6\right) du
\end{aligned}$$

$(u = \sin x \quad du = \cos x\, dx)$

$$\begin{aligned}
&= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{(\sin x)^3}{3} - \frac{2(\sin x)^5}{5} + \frac{(\sin x)^7}{7} + C
\end{aligned}$$

EX#3 : $\int \sin^2 x \cos^4 x \, dx$

$$= \int (\sin x \cos x)^2 \cos^2 x \, dx$$

$$= \int \left(\frac{\sin 2x}{2}\right)^2 \bullet \left(\frac{1+\cos 2x}{2}\right) dx \qquad \left(\sin 2x = 2\sin x \cos x \;\; and \;\; \cos^2 x = \frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{8}\int \sin^2 2x \, dx + \frac{1}{8}\int \sin^2 2x \;\; \cos 2x \, dx$$

$$= \frac{1}{8}\cdot\int \frac{1-\cos 4x}{2} dx \quad + \frac{1}{8}\cdot\frac{1}{2}\int u^2 \, du \quad (u = \sin 2x \quad du = 2\cos 2x) \; \left(\sin^2 2x = \frac{1-\cos 4x}{2}\right)$$

$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{16}\frac{u^3}{3} + C \qquad = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{(\sin 2x)^3}{48} + C$$

$\int \tan^m x \sec^n x \, dx$

If **n is even**, substitute $u = \tan x$ Useful identity: $\tan^2 x = \sec^2 x - 1$

If **m is odd**, substitute $u = \sec x$ Useful identity: $\tan^2 x = \sec^2 x - 1$

If **m is even and n is odd**, reduce powers of $\sec x$ alone Useful identity: $\tan^2 x = \sec^2 x - 1$

EX#1 : $\int \sec^4 x \tan^2 x \, dx$

$$= \int \sec^2 x (1 + \tan^2 x) \tan^2 x \, dx$$

$$(\sec^2 x = 1 + \tan^2 x)$$

$$= \int \sec^2 x (\tan^2 x + \tan^4 x) \, dx$$

$$= \int (u^2 + u^4) \, du$$

$$(u = \tan x \quad du = \sec^2 x \, dx)$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{(\tan x)^3}{3} + \frac{(\tan x)^5}{5} + C$$

EX#2 : $\int \sec^5 x \tan^3 x \, dx$

$$= \int \sec x \tan x \left(\sec^4 x \tan^2 x\right) dx$$

$$= \int \sec x \tan x \left(\sec^4 x \left(1 - \sec^2 x\right)\right) dx$$

$$(\tan^2 x = 1 - \sec^2 x)$$

$$= \int \sec x \tan x \left(\sec^4 x - \sec^6 x\right) dx$$

$$(u = \sec x \quad du = \sec x \tan x \, dx)$$

$$= \int (u^4 - u^6) \, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C \qquad = \frac{(\sec x)^5}{5} - \frac{(\sec x)^7}{7} + C$$

Trigonometric Substitutions

If the integrand contains $\sqrt{a^2 - x^2}$ then substitute $x = a\sin u \quad dx = a\cos u \; du$

The radicand becomes $\sqrt{(a^2 - a^2\sin^2 u)} = \sqrt{a^2(1 - \sin^2 u)} = \sqrt{a^2\cos^2 u} = a\cos u$

If the integrand contains $\sqrt{a^2 + x^2}$ then substitute $x = a\tan u \quad dx = a\sec^2 u \; du$

The radicand becomes $\sqrt{(a^2 + a^2\tan^2 u)} = \sqrt{a^2(1 + \tan^2 u)} = \sqrt{a^2\sec^2 u} = a\sec u$

If the integrand contains $\sqrt{x^2 - a^2}$ then substitute $x = a\sec u \quad dx = a\sec u \tan u \; du$

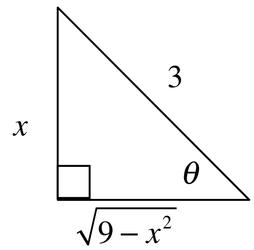
The radicand becomes $\sqrt{(a^2\sec^2 u - a^2)} = \sqrt{a^2(\sec^2 u - 1)} = \sqrt{a^2\tan^2 u} = a\tan u$

EX#1: $\int \sqrt{9-x^2}\ dx \quad = \int \sqrt{9-(3\sin\theta)^2}\ 3\cos\theta\ d\theta \quad = \int \sqrt{9\left(1-\sin^2\theta\right)}\ 3\cos\theta\ d\theta$

$x = 3\sin\theta \qquad = \int \sqrt{9\cos^2\theta}\ 3\cos\theta\ d\theta \quad = \int 9\cos^2\theta\ dx \quad = 9\int \frac{1+\cos 2\theta}{2}\ d\theta$

$dx = 3\cos\theta\ d\theta \qquad = \frac{9\theta}{2} + \frac{9\sin 2\theta}{4} + C = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C$$



$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

EX#2: $\int \frac{x}{\sqrt{16+x^2}}\ dx$

$\left(\text{Substitute } \ x = 4\tan\theta \quad dx = 4\sec^2\theta\, d\theta\right)$

$$= \int \frac{4\tan\theta}{\sqrt{16+16\tan^2\theta}} \cdot 4\sec^2\theta\, d\theta$$

$$= \int \frac{4\tan\theta}{\sqrt{16\left(1+\tan^2\theta\right)}} \cdot 4\sec^2\theta\, d\theta$$

$$= \int \frac{4\tan\theta}{\sqrt{16\sec^2\theta}} \cdot 4\sec^2\theta\, d\theta$$

$$= \int 4\tan\theta\sec\theta\ d\theta$$

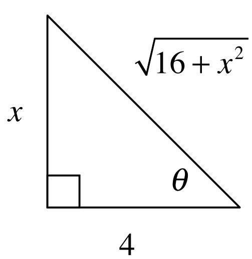
$$= 4\sec\theta + C$$

$$= 4\frac{\sqrt{16+x^2}}{4} + C$$

$$= \sqrt{16+x^2} + C$$

$x = 4\tan\theta$

$\frac{x}{4} = \tan\theta$



EX#3: $\int \frac{\sqrt{x^2-25}}{x} dx$

$\left(\text{Substitute } \ x = 5\sec\theta \quad dx = 5\sec\theta\tan\theta\, d\theta\right)$

$$= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta} \cdot 5\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sqrt{25\left(\sec^2\theta - 1\right)}}{5\sec\theta} \cdot 5\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sqrt{25\tan^2\theta}}{5\sec\theta} \cdot 5\sec\theta\tan\theta\, d\theta$$

$$= \int 5\tan^2\theta d\theta \qquad = \int 5\left(\sec^2\theta - 1\right) d\theta$$

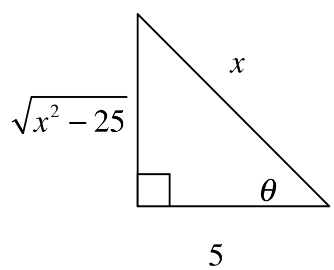
$$= \int \left(5\sec^2\theta - 5\right) d\theta \quad = \quad 5\tan\theta - 5\theta + C$$

$$= 5\left(\frac{\sqrt{x^2-25}}{5}\right) - 5\,arc\sec\left(\frac{x}{5}\right) + C$$

$$= \sqrt{x^2-25} - 5\,arc\sec\left(\frac{x}{5}\right) + C$$

$x = 5\sec\theta$

$\frac{x}{5} = \sec\theta$



Integration Techniques

Long Division (Must use long division if the top function has the same or greater power as the bottom function)

Step 1 - Divide the numerator by the denominator. Separate the integrand into separate integrals for each term

EX. $\int \frac{2x^3}{x^2+3}\,dx \quad = \int 2x - \frac{6x}{x^2+3}\,dx \quad = x^2 - 3\ln\left(x^2+3\right) + C$

$$\begin{array}{r l} & \quad 2x \\ x^2+3 \,\big) & \overline{\;2x^3\qquad} \\ & \underline{2x^3+6x} \\ & \qquad -6x \end{array}$$

Add and subtract terms in the numerator

$$\int \frac{2x}{x^2+6x+13}\,dx \quad = \int \frac{2x+6}{x^2+6x+13} - \frac{6}{x^2+6x+13}\,dx \quad = \int \frac{2x+6}{x^2+6x+13} - \frac{6}{x^2+6x+9+13-9}\,dx$$

$$= \int \frac{2x+6}{x^2+6x+13} - \frac{6}{(x+3)^2+4}\,dx \quad = \ln\left|x^2+6x+13\right| + 3\arctan\left(\frac{x+3}{2}\right) + C$$

Partial Fractions (Used when the denominator is factorable)

Before performing partial fractions, you must use long division if the degree of the numerator is greater or equal to the denominator.

Step 1 - Factor the denominator of the function

Step 2 - Separate into partial fractions

Rewrite as fractional terms - arbitrary constants divided by each factor of the denominator.

If a factor is repeated more than once, write that many terms for that factor

EX#1: $\int \frac{4x-1}{x^2-x-12}\,dx = \int \frac{4x-1}{(x-4)(x+3)}\,dx = \int \frac{A}{x-4} + \frac{B}{x+3}\,dx$

Multiply by the common denominator.

$4x - 1 = A(x+3) + B(x-4)$

Plug in the roots of the denominator and solve for the constants

Let $x = 4 \quad \Rightarrow \quad 15 = 7A \qquad \Rightarrow \quad A = \frac{15}{7}$

Let $x = -3 \quad \Rightarrow \quad -13 = -7B \quad \Rightarrow \quad B = \frac{13}{7}$

The integral has been reduced down to something much more reasonable and now we are able to integrate.

Therefore,

$$\int \frac{4x-1}{x^2-x-12}\,dx \quad = \int \frac{15/7}{x-4} + \frac{13/7}{x+3}\,dx \quad = \frac{15}{7}\ln|x-4| + \frac{13}{7}\ln|x+3| + C$$

EX#2 : $\int \frac{2x+3}{x^3+2x^2+x}\,dx = \int \frac{2x+3}{x(x+1)^2}\,dx = \int \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\,dx$

Multiply by the common denominator.

$2x+3 = A(x+1)^2 + Bx(x+1) + Cx$

Plug in the roots of the denominator and solve for the constants

Let $x=0 \quad \Rightarrow \quad 3=A$

Let $x=-1 \quad \Rightarrow \quad 1=-C \quad \Rightarrow \quad C=-1$

Let $x=1 \quad \Rightarrow \quad 5=4A+2B+C \quad \Rightarrow \quad B=-3$

The integral has been reduced down to something much more reasonable and now we are able to integrate. Therefore,

$$\int \frac{2x+3}{x(x+1)^2}\,dx \quad = \quad \int \frac{3}{x}-\frac{3}{x+1}-\frac{1}{(x+1)^2}\,dx \quad = \quad 3\ln|x|-3\ln|x+1|+\frac{1}{x+1}+C$$

Simple partial fractions (Other method)

EX#3 : $\int \frac{8}{(x+3)(x-5)}\,dx \quad = \int \frac{A}{x+3}+\frac{B}{x-5}\,dx \qquad \Rightarrow \int \frac{-1}{x+3}+\frac{1}{x-5}\,dx \quad = -\ln|x+3|+\ln|x-5|+C$

$8 = A(x-5)+B(x+3) \qquad 8 = Ax-5A+Bx+3B \qquad\qquad = \ln\left|\frac{x-5}{x+3}\right|+C$

$A+B=0 \qquad\qquad 3A+3B=0$

$-5A+3B=8 \qquad -5A+3B=8 \qquad 8A=-8 \qquad A=-1 \qquad B=1$

Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad\qquad \int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

In the first case:

If $f(x)$ converges then the integral exists

If $f(x)$ diverges then the integral is ∞ or $-\infty$

EX #1 : $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left(-e^{-x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-e^{-b}\right)+e^{-1} = 0+e^{-1} = \frac{1}{e}$ So $\int_1^\infty e^{-x}\,dx$ converges

EX #2 : $\int_1^\infty \frac{1}{x}\,dx = \int_1^{b=\infty} \frac{1}{x}\,dx = \ln x\Big|_1^{b=\infty} = \lim_{b\to\infty}(\ln b)-\ln 1 = \infty$ So $\int_1^\infty \frac{1}{x}\,dx$ diverges.

EX #3 : $\int_1^\infty \frac{1}{x^2}\,dx = \int_1^{b=\infty} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{b=\infty} = \lim_{b\to\infty}\left(\frac{-1}{b}\right)-\frac{-1}{1} = 0+1 = 1$ So $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

EX #4 : $\int_0^1 \frac{1}{x^2}\,dx = \int_{a=0}^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{a=0}^1 = \frac{-1}{1}-\lim_{a\to0^-}\left(\frac{-1}{a}\right) = -1+\infty$ So $\int_0^1 \frac{1}{x^2}\,dx$ diverges.

General Series and Sequences

For the sequence $\{a_n\}_{n=m}^{\infty}$, if $\lim_{n\to\infty} a_n$ exists, then the sequence converge to L.

If $\lim_{n\to\infty} a_n$ does not exist, then the sequence diverges to ∞ or $-\infty$.

Boundness refers to whether or not a sequence has a finite range

If $\{a_n\}_{n=m}^{\infty}$ converges, then $\{a_n\}_{n=m}^{\infty}$ is bounded

If $\{a_n\}_{n=m}^{\infty}$ diverges, then $\{a_n\}_{n=m}^{\infty}$ is unbounded

The nth partial sum (S_n) refers to the sum of the first n terms of a sequence.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ does also and $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n + b_n)$

Convergence Tests (PARTING CC)

nth term test (Used to show immediate divergence)

If top power is the same or greater than the bottom power then the series is divergent.

If bottom power is greater we must proceed and use a different test, because this test cannot prove convergence.

EX#1: $\sum_{n=2}^{\infty} \frac{n^3+2}{n^3-5}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n^5}{3n^3+2}$ These are divergent by the nth term test.

P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1 \text{ and diverges if } p \le 1$$

EX#1: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ Converge by p-series

EX#2: $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ Diverge by p-series

Integral Test for Convergence (for nonnegative sequences) (Used if you know the integral of the series)

$$\sum_{n=1}^{\infty} a_n \text{ converges iff } \int_1^{\infty} f(x)\,dx \text{ converges} \qquad \sum_{n=1}^{\infty} a_n \text{ diverges iff } \int_1^{\infty} f(x)\,dx \text{ diverges}$$

EX#1: $\sum_{n=0}^{\infty} \frac{1}{n^2+1} \Rightarrow \int_0^{\infty} \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^{b=\infty} = \lim_{b\to\infty}(\arctan b) - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ so series Converges

EX#2: $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \int_1^{\infty} \frac{1}{x}\,dx = \ln x\Big|_1^{b=\infty} = \lim_{b\to\infty}(\ln b) - \ln 1 = \infty - 0 = \infty$ so series Diverges

Alternating Series Test for Convergence (for alternating sequences)

The series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges if $\{a_n\}_{n=1}^{\infty}$ is a positive decreasing sequence and $\lim_{n\to\infty} a_n = 0$.

EX#1: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$ Converges because the denominator is larger in the alternating series.

Alternating series can not be used to prove divergence.

Direct Comparison Test for Convergence and Divergence (for positive sequences)

If $\sum_{n=1}^{\infty} b_n$ converges and $0 \le a_n \le b_n$ then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} b_n$ diverges and $0 \le b_n \le a_n$ then $\sum_{n=1}^{\infty} a_n$ diverges

EX#1: $\sum_{n=0}^{\infty}\frac{1}{n-1} \Rightarrow$ Diverges since $\frac{1}{n-1} > \frac{1}{n}$ and we know $\frac{1}{n}$ diverges by p-series.

EX#2: $\sum_{n=0}^{\infty}\frac{2^n}{7^n+2} \Rightarrow$ Converges since $\frac{2^n}{7^n+2} < \left(\frac{2}{7}\right)^n$ and we know $\left(\frac{2}{7}\right)^n$ converges by geometric series.

Limit Comparison Test (for nonnegative sequences) (Used on messy algebraic series)

Assume $\lim_{n\to\infty}\frac{a_n}{b_n} > 0$ If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

EX#1: $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n-1}} \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{1}{\sqrt{n-1}}}{\frac{1}{n^{1/2}}}\right| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{n-1}}\cdot\frac{n^{1/2}}{1}\right| = 1$ Limit is finite and positive.

Diverges since we compared with a divergent series.

EX#2: $\sum_{n=0}^{\infty}\frac{7}{n^3+5} \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{7}{n^3+5}}{\frac{1}{n^3}}\right| = \lim_{n\to\infty}\left|\frac{7}{n^3+5}\cdot\frac{n^3}{1}\right| = 7$ Limit is finite and positive.

Converges since we compared with a convergent series.

Geometric Series Test

A geometric series $\sum_{n=m}^{\infty} a\cdot r^n$ converges iff $|r| < 1$. A geometric series $\sum_{n=m}^{\infty} a\cdot r^n$ diverges iff $|r| \ge 1$.

If a geometric series converges, then $\sum_{n=m}^{\infty} a\cdot r^n = \frac{ar^m}{1-r}$ **Sum:** $S = \frac{a}{1-r}$

EX#1: $\sum_{n=0}^{\infty} 2\cdot\left(\frac{3}{5}\right)^n$ Converges because $r = \frac{3}{5}$. The sum of the series is $S = \frac{2}{1-3/5} = 5$

EX#2: $\sum_{n=0}^{\infty} 6\cdot\left(\frac{7}{2}\right)^n$ Diverges because $r = \frac{7}{2}$. No sum since series diverges.

Ratio Test (for nonnegative series) (Used for really ugly problems)

The series $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then this test is inconclusive.

EX#1: $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n}$ $\quad \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{e}\right| = \infty$ so series Diverges

EX#2: $\sum_{n=1}^{\infty} \frac{(-1)^n n^5}{2^n}$ $\quad \lim_{n\to\infty}\left|\frac{\frac{(n+1)^5}{2^{n+1}}}{\frac{n^5}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^5}{2^{n+1}}\cdot\frac{2^n}{n^5}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^5}{2n^5}\right| = \frac{1}{2}$ so series Converges

Root Test for Convergence (for nonnegative series)

The sequence $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$

The sequence $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$

If $\lim_{n\to\infty}\sqrt[n]{a_n} = 1$, then this test is inconclusive.

EX#1: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{e^n}$ $\quad \lim_{n\to\infty}\left|\sqrt[n]{\frac{2^n}{e^n}}\right| = \lim_{n\to\infty}\left|\frac{2}{e}\right| = \frac{2}{e}$ so series Converges

EX#2: $\sum_{n=1}^{\infty} \frac{(-1)^n 7^n}{5^n}$ $\quad \lim_{n\to\infty}\left|\sqrt[n]{\frac{7^n}{5^n}}\right| = \lim_{n\to\infty}\left|\frac{7}{5}\right| = \frac{7}{5}$ so series Diverges

Telescoping Series (Limit = 0 and the terms get smaller as we approach ∞)

If a series is a telescoping series then it is convergent.

EX#1: $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ is convergent. The *sum* $= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+ \ldots\ldots = \quad 1$

EX#1: $\sum_{n=1}^{\infty} \frac{3}{n} - \frac{3}{n+2}$ is convergent. The *sum* $= (3-1)+\left(\frac{3}{2}-\frac{3}{4}\right)+\left(1-\frac{3}{5}\right)+\left(\frac{3}{4}-\frac{3}{6}\right)+ \ldots\ldots = \quad 3+\frac{3}{2} = \quad \frac{9}{2}$

SERIES AND SEQUENCES

Power Series : A power series is defined as $\sum_{n=0}^{\infty} c_a x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

Taylor's Theorem and Related Series (If the series is centered at c = 0 then it is a MacLaurin Series)

Taylor's theorem for approximating $f(x)$ to the nth term:

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots\ldots + \frac{f^n(c)}{n!}(x-c)^n$$

Taylor series of $f(x)$ $\rightarrow$ this is a power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$

EX#1 : Find the fourth degree MacLaurin Series (centered at $c = 0$) for $f(x) = \cos x$.

$f(x) = \cos x \qquad f(0) = 1 \qquad f(x) = 1 + 0\frac{(x-0)^1}{1!} - 1\frac{(x-0)^2}{2!} + 0\frac{(x-0)^3}{3!} + 1\frac{(x-0)^4}{4!}$

$f'(x) = -\sin x \qquad f'(0) = 0 \qquad f(x) = 1 - 1\frac{(x-0)^2}{2!} + 1\frac{(x-0)^4}{4!}$

$f''(x) = -\cos x \qquad f''(0) = -1 \qquad f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^4(x) = \cos x \qquad f^4(0) = 1$

EX#2 : Find the fourth degree Taylor Series for $f(x) = \ln x$ centered at $c = 1$.

$f(x) = \ln x \qquad f(1) = 0 \qquad f(x) = 0 + 1\frac{(x-1)^1}{1!} - 1\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!}$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1 \qquad f(x) = (x-1) - 1\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!}$

$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1 \qquad f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$

$f^4(x) = \frac{-6}{x^4} \qquad f^4(1) = -6$

EX #3 : $f(2) = 5 \qquad f'(2) = 4 \qquad f''(2) = 2 \qquad f'''(2) = 17$

3rd Degree Taylor Polynomial : $P_3(x) = 5 + 4(x-2) + 2\frac{(x-2)^2}{2!} + 17\frac{(x-2)^3}{3!}$

Using 3rd Degree Taylor Polynomial to approximate $x = 2.1$:

$$P_3(2.1) = 5 + 4(2.1-2) + 2\frac{(2.1-2)^2}{2!} + 17\frac{(2.1-2)^3}{3!} = 5.41283$$

Radius of convergence / Interval of convergence

EX#1: Find the radius and interval of convergence

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n \quad \text{Geometric Series}$$

$$\left|\frac{x+1}{2}\right| < 1 \qquad \text{Converges iff } |r| < 1$$

$$-1 < \frac{x+1}{2} < 1$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$

Don't have to check endpoints because endpoints of geometric series never work.
I'll check anyway.

Check endpoints

Check -3, $\displaystyle\sum_{n=0}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}}$ Diverges

Check 1, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}}$ Diverges

Center of convergence: -1

Radius of convergence: 2

Interval of Convergence: $-3 < x < 1$

EX#2: Find the radius and interval of convergence

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{n \cdot 3^n} \qquad \text{Use Ratio Test}$$

$$\lim_{n\to\infty} \left|\frac{(x-4)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-4)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{(x-4)}{3} \cdot \frac{n}{n+1}\right|$$

$$-1 < \frac{x-4}{3} < 1$$

$$-3 < x-4 < 3$$

$$1 < x < 7$$

Check endpoints

Check 1, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{2n} \cancel{3^n}}{n \cdot \cancel{3^n}}$ Diverges

Check 7, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{3^n}}{n \cdot \cancel{3^n}}$ Converges

Since 7 works include 7 in interval.

Center of convergence: 4

Radius of convergence: 3

Interval of Convergence: $1 < x \le 7$

When finding the interval of convergence using ratio test:

If $\displaystyle\lim_{n\to\infty} \left| \quad \right| = 0$ then the series converges from $-\infty$ to ∞.

If $\displaystyle\lim_{n\to\infty} \left| \quad \right| = \infty$ then the series converges at the center c only.

EX#1: $\displaystyle\sum_{n=0}^{\infty} \frac{(x-4)^n}{3^n \cdot n!}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(x-4)^{n+1}}{3^{n+1}\cdot(n+1)!}}{\dfrac{(x-4)^n}{3^n \cdot n!}} \right| = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{3^{n+1}\cdot(n+1)!} \cdot \frac{3^n \cdot n!}{(x-4)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-4)}{3(n+1)} \right| = 0$$

so the series converges from $-\infty$ to ∞.

EX#2: $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n n!}{2^n}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(x+1)^{n+1}(n+1)!}{2^{n+1}}}{\dfrac{(x+1)^n n!}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}(n+1)!}{2^{n+1}} \cdot \frac{2^n}{(x+1)^n n!} \right| = \lim_{n\to\infty} \left| \frac{(x+1)(n+1)}{2} \right| = \infty$$

so the series converges at the center -1 only.

Memorize these known series

Series for e^x : $= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+... = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

Series for $\frac{1}{1-x}$: $= 1+x+x^2+x^3... = \sum_{n=0}^{\infty}x^n$

Series for $\sin x$: $= x-\frac{x^3}{3!}+\frac{x^5}{5!}-... = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Series for $\cos x$: $= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-... = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

EX#1 : Use the series from above to write each of the following series

$f(x)=\cos x^2 \Rightarrow f(x)=1-\frac{x^4}{2!}+\frac{x^8}{4!}-... =\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$ (Replace x with x^2 in the known cosine series)

$f(x)=x\cos x \Rightarrow f(x)=x-\frac{x^3}{2!}+\frac{x^5}{4!}-... =\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$ (Multiply the known cosine series by x)

$f(x)=\sin x \Rightarrow f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-... =\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ (Take the integral of the known cosine series)

$f(x)=e^{2x} \Rightarrow f(x)=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+... =\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$ (Replace x with $2x$ in the known e^x series)

$f(x)=\int_0^1 e^{2x}dx \Rightarrow f(x)=x+x^2+\frac{(2x)^3}{2\cdot 3!}+\frac{(2x)^4}{2\cdot 4!}+...\Big|_0^1 = \left(1+1+\frac{2}{3}+\frac{1}{3}\right)-(0) \doteq 3$ (Take the definite integral of e^{2x})

$f(x)=\frac{1}{1+x}$: $= 1-x+x^2-x^3... =\sum_{n=0}^{\infty}(-1)^n x^n$ (Replace x with $-x$ in the known series $\frac{1}{1-x}$)

$f(x)=\frac{x^2}{1+x}$: $= x^2-x^3+x^4-x^6... =\sum_{n=0}^{\infty}(-1)^n x^{n+2}$ (Multiply the found series $\frac{1}{1+x}$ by x^2)

DIFFERENTIAL EQUATIONS (Separating Variables)

Separable Differential Equations

$$\text{If } P(x)+Q(y)\frac{dy}{dx}=0 \text{ then } Q(y)dy=-P(x)dx \text{ then } \int Q(y)dy=-\int P(x)dx$$

EX#1 : Find the general solution given $\frac{dy}{dx}=\frac{x^2}{y}$

$$\frac{dy}{dx}=\frac{x^2}{y} \Rightarrow y\,dy=x^2\,dx \Rightarrow \int y\,dy=\int x^2\,dx \Rightarrow \frac{y^2}{2}=\frac{x^3}{3}+C \Rightarrow \underline{y^2=\frac{2x^3}{3}+C_1}$$

EX#2 : Find the particular solution $y=f(x)$ for **EX#1** given $(3,-5)$

$$y^2=\frac{2x^3}{3}+C_1 \Rightarrow 25=18+C_1 \Rightarrow 7=C_1 \Rightarrow y^2=\frac{2x^3}{3}+7 \Rightarrow \underline{y=-\sqrt{\frac{2x^3}{3}+7}}$$

EX#3 : Find the particular solution $y=f(x)$ given $\frac{dy}{dx}=6xy$ and $(0,5)$

$$\frac{dy}{y}=6x\,dx \Rightarrow \int\frac{dy}{y}=\int 6x\,dx \Rightarrow \ln y=3x^2+C \Rightarrow y=e^{3x^2+C} \Rightarrow y=C_1e^{3x^2} \Rightarrow 5=C_1(1) \Rightarrow \underline{y=5e^{3x^2}}$$

L' Hôpital's Rule : If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ Reminder: In limits $\dfrac{1}{\infty}=0$ and $\dfrac{1}{0}=\infty$

EX #1 : $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}=\dfrac{\infty}{\infty}$ so $\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}=\dfrac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{e^x}{6x}=\dfrac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

EX #2 : $\lim\limits_{x\to 2}\dfrac{x^2+5x-14}{x-2}=\dfrac{0}{0}$ so $\lim\limits_{x\to 2}\dfrac{2x+5}{1}=9$

Indeterminate forms : $0\cdot\infty,\ \infty-\infty,\ 1^\infty,\infty^0,\ 0^0$ $\left(\text{Indeterminate forms must be changed to } \dfrac{0}{0} \text{ or } \dfrac{\infty}{\infty}\right)$

EX #3 : $\lim\limits_{x\to 0^+} x\cdot\ln x = 0\cdot\infty$ so we must convert to $\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x}=\dfrac{0}{0}$ so $\lim\limits_{x\to 0^+}\dfrac{1/x}{-1/x^2}=\lim\limits_{x\to 0^+}-x=0$

EX #4 : $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=1^\infty$ so we must bring down the x so let $y=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ and take ln of both sides.

$\ln y=\lim\limits_{x\to\infty}\ln\left(1+\dfrac{1}{x}\right)^x$ $\ln y=\lim\limits_{x\to\infty}x\ln\left(1+\dfrac{1}{x}\right)=0\cdot\infty$ $\ln y=\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{x}}=\dfrac{0}{0}$ $\ln y=\lim\limits_{x\to\infty}\dfrac{\left(\dfrac{1}{1+\dfrac{1}{x}}\right)\cdot\dfrac{-1}{x^2}}{\dfrac{-1}{x^2}}$

$\ln y=\lim\limits_{x\to\infty}\dfrac{1}{1+\dfrac{1}{x}}$ $\ln y=1$ $\underline{y=e}$

Work

$W=F\cdot d\cos\theta$

or the dot product of the Force vector and the distance vector.

$W=\vec{F}\cdot\vec{d}$

$W=\int_a^b F(x)\,dx$; where $F(x)$ is a continuously varying force.

Hooke's Law : The force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is,

$F=kd$

$W=\int_a^b kx\,dx$

where the constant of proportionality k (the spring constant) depends on the specific nature of the spring.

EX : Compressing a Spring

A force of 750 pounds compresses a spring 3 inches from its natural length of 15 inches. Find the work done in compressing the spring an additional 3 inches.

$750=k(3)$ $k=250$ $F(x)=kx$ so $F(x)=250x$

$W=\int_3^6 250x\,dx=3375$ inch – pounds

Logistical Growth

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad ; \quad y = \frac{L}{1 + be^{-kt}} \quad ; \quad b = \frac{L - Y_0}{Y_0}$$

$k =$ constant of proportionality

$L =$ carrying capacity

$Y_0 =$ initial amount

Exponential Growth

$$\frac{dy}{dt} = ky \quad ; \quad y = Ce^{kt}$$

$k =$ growth constant

$C =$ initial amount

Logistic Growth

$$\frac{dy}{dt} = ky(1 - \frac{y}{L})$$

where L is the carrying capacity (upper bound) and k is the constant of proportionality.

If you solve the differential equation using very tricky separation of variables, you get the general solution:

$$y = \frac{L}{1 + be^{-kt}} \quad ; \quad b = \frac{L - Y_0}{Y_0}$$

EX#1:

A highly contagious "pinkeye" (scientific name: Conjunctivitus itchlikecrazius) is ravaging the local elementary school. The population of the school is 900 (including students and staff), and the rate of infection is proportional both to the number infected and the number of students whose eyes are pus-free. If seventy-five people were infected on December 15 and 250 have contracted pinkeye by December 20, how many people will have gotten the gift of crusty eyes by Christmas Day?

Solution

Because of the proportionality statements in the problem, logistic growth is the approach we should take. The upper limit for the disease will be L = 900; it is impossible for more than 900 people to be infected since the school only contains 900 people. This gives us the equation

$$L = 900 \qquad b = \frac{900 - 75}{75} = 11 \qquad \Rightarrow \qquad y = \frac{900}{1 + 11e^{-kt}}$$

Five days later, 250 people have contracted pinkeye, so plug that information to find k:

$$250 = \frac{900}{1 + 11e^{-k \cdot 5}} \quad \Rightarrow \quad 250(1 + 11e^{-5k}) = 900$$

$$11e^{-5k} = \frac{13}{5} \quad \Rightarrow \quad e^{-5k} = \frac{13}{55}$$

$$\ln e^{-5k} = \ln\left(\frac{13}{55}\right) \quad \Rightarrow \quad -5k = -1.442383838$$

$$k = 0.2884767656$$

Finally, we have the equation $y = \dfrac{900}{1 + 11e^{-0.2884767656t}}$. We want to find the number of infections on December 25, so $t = 10$.

$$y = \frac{900}{1 + 11e^{-(0.2884767656)(10)}}$$

$$y \approx 557.432$$

So almost 558 students have contracted pinkeye in time to open presents.

Euler's Method

Uses tangent lines to approximate points on the curve.

$New\ y \ = \ Old\ y \ + \ dx \cdot \dfrac{dy}{dx}$ $\qquad dx:\ change\ in\ x$ $\qquad \dfrac{dy}{dx}$ = Derivative (slope) at the point.

EX#1:

Given: $f(0) = 3$ $\qquad \dfrac{dy}{dx} = \dfrac{xy}{2}$ $\qquad$ Use step of $h = 0.1$ to find $f(0.3)$

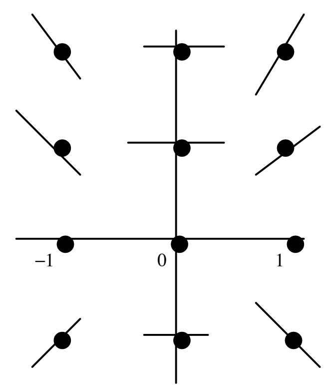
Original $(0,3)$

$New\ y = 3 + (0.1)(0) = 3$ $\qquad f(0.1) \doteq 3$ $\qquad$ At $(0,3)$ $\quad \dfrac{dy}{dx} = \dfrac{0 \cdot 3}{2} = 0$

$New\ y = 3 + (0.1)(0.15) = 3.015$ $\qquad f(0.2) \doteq 3.015$ $\qquad$ At $(0.1,3)$ $\quad \dfrac{dy}{dx} = \dfrac{0.1 \cdot 3}{2} = 0.15$

$New\ y = 3.015 + (0.1)(0.3015) = 3.04515$ $\qquad \underline{f(0.3) \doteq 3.04515}$ $\qquad$ At $(0.2, 3.015)$ $\quad \dfrac{dy}{dx} = \dfrac{0.2 \cdot 3.015}{2} = 0.3015$

Slope Fields



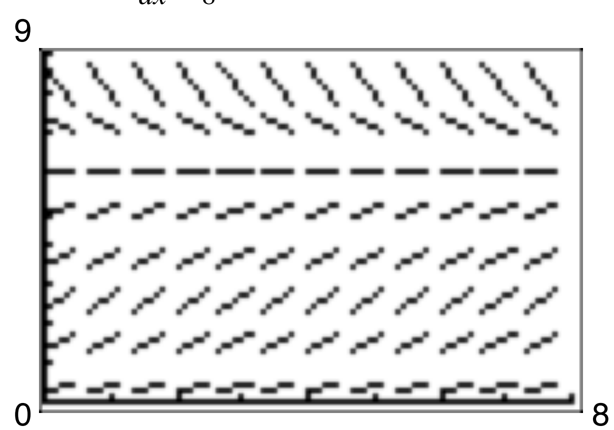
Draw the slope field for $\dfrac{dy}{dx} = xy$

Plug each point into $\dfrac{dy}{dx}$ and graph the tangent line at the point

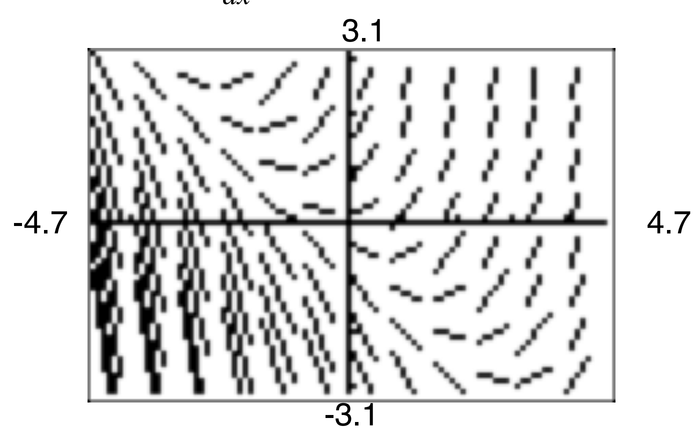
At $(1,\ 1)$ $\dfrac{dy}{dx} = 1.$ $\quad$ At $(-1, 1)$ $\dfrac{dy}{dx} = -1.$ $\quad$ At $(0, 1)$ $\dfrac{dy}{dx} = 0$

EX: Here are the slope fields for the given differential equations

$\dfrac{dy}{dx} = \dfrac{y}{8}(6 - y)$



$\dfrac{dy}{dx} = x + y$



The slope fields give **ALL** the solutions to the differential equation **.**

Lagrange Error Bound

$Error = |f(x) - P_n(x)| \le |R_n(x)|$ where $|R_n(x)| \le \dfrac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}$

$f^{n+1}(z) =$ the **maximum** of the $(n+1)$th derivative of the function

EX #1: $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots.$ $\cos(0.1) \doteq 0.99500416667$

$R_4(x) = \dfrac{f^5(z)(x-c)^5}{5!}$ $f^5(z) = -\sin z$ We need $-\sin z$ to be as large as possible (which is 1 because $-1 \le \sin z \le 1$)

$R_4(0.1) < \dfrac{1 \cdot (0.1)^5}{5!} = 0.0000000833$

EX #2: $f(1) = 2$ $f'(1) = 5$ $f''(1) = 7$ $f'''(1) = 12$

2nd Degree Taylor Polynomial: $P_2(x) = 2 + 5(x-1) + 7(x-1)^2$

Using 2nd Degree Taylor Polynomial to approximate $x = 1.1$: $P_2(1.1) = 2 + 5\dfrac{(1.1-1)^1}{1!} + 7\dfrac{(1.1-1)^2}{2!} = 2.535$

Lagrange Error Bound: $R_3(x) = \dfrac{f^3(z)(x-c)^3}{3!} = \dfrac{12(1.1-1)^3}{3!} = 0.002 = \text{Error}$

Area as a limit

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{b-a}{n} i\right)}_{\text{height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{width}}$$

i = interval

Summation formulas and properties

1) $\sum_{i}^{n} c = cn$

2) $\sum_{i}^{n} i = \dfrac{n(n+1)}{2}$

3) $\sum_{i}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

4) $\sum_{i}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

5) $\sum_{i}^{n} (a_i \pm b_i) = \sum_{i}^{n} a_i \pm \sum_{i}^{n} b_i$

6) $\sum_{i}^{n} k \cdot a_i = k \sum_{i=1}^{n} a_i$, k is a constant

EX#1: $f(x) = x^3$ $[0, 1]$ n subdivisions Find Area under curve.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(0 + \frac{1-0}{n} i\right)\left(\frac{1-0}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \frac{1}{4}$$

EX#2: $\lim\limits_{n\to\infty} \dfrac{1}{n}\left[\sqrt{\dfrac{1}{n}} + \sqrt{\dfrac{2}{n}} + \ldots\ldots\ldots + \sqrt{\dfrac{n}{n}}\right] =$

Answer: $\int_0^1 \sqrt{x}\, dx$ (You have to recognize that the problem is asking for Area from $[0,1]$ with n subdivisions)

Summary of tests for Series

Test	Series	Converges	Diverges	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert \mathrm{R_N} \rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$.
Direct comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Power Series for Elementary Functions

Function	**Interval of Convergence**
$\dfrac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\dfrac{1}{1+x} = 1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\ln x = (x-1)-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}-\dfrac{(x-1)^4}{4}+\cdots+\dfrac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\cdots+\dfrac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dfrac{x^9}{9!}-\cdots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dfrac{x^8}{8!}-\cdots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\dfrac{x^9}{9}-\cdots+\dfrac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x = x+\dfrac{x^3}{2\cdot 3}+\dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\dfrac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$
$(1+x)^k = 1+kx+\dfrac{k(k-1)x^2}{2!}+\dfrac{k(k-1)(k-2)x^3}{3!}+\dfrac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$	$-1<x<1$*

*The convergence at $x=\pm 1$ depends on the value of k.

Procedures for fitting integrands to Basic Rules

Technique	**Example**
Expand (numerator)	$\int \left(1+e^x\right)^2 dx = \int \left(1+2e^x+e^{2x}\right) dx \quad = x+2e^x+\frac{e^{2x}}{2}+C$
Separate numerator	$\int \frac{1+x}{x^2+1} dx = \int \left(\frac{1}{x^2+1}+\frac{x}{x^2+1}\right) dx \quad = \arctan x+\frac{1}{2}\ln\left(x^2+1\right)+C$
Complete the square	$\int \frac{1}{\sqrt{2x-x^2}} dx = \int \frac{1}{\sqrt{1-(x-1)^2}} dx \quad = \arcsin(x-1)+C$
Divide improper rational function	$\int \frac{x^2}{x^2+1} dx = \int \left(1-\frac{1}{x^2+1}\right) dx \quad = x-\arctan x+C$
Add and subtract terms in the numerator	$\int \frac{2x}{x^2+2x+1} dx = \int \left(\frac{2x+2}{x^2+2x+1}-\frac{2}{x^2+2x+1}\right) dx =$ $\int \left(\frac{2x+2}{x^2+2x+1}-\frac{2}{(x+1)^2}\right) dx \quad = \ln\left\lvert x^2+2x+1\right\rvert+\frac{2}{x+1}+C$
Use trigonometric Identities	$\tan^2 x = \sec^2 x-1 \qquad \sin^2 x = 1-\cos^2 x \qquad \cot^2 x = \csc^2 x-1$
Multiply and Divide by Pythagorean conjugate	$\int \frac{1}{1+\sin x} dx = \int \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) dx = \int \frac{1-\sin x}{1-\sin^2 x} dx$ $= \int \frac{1-\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right) dx$ $= \int \left(\sec^2 x-\sec x\tan x\right) dx \quad = \tan x+\sec x+C$

Happy studying and good luck!

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VECTOR CALCULUS

A vector has two components: magnitude and direction, expressed as an ordered pair of real numbers: (a_1, a_2) where $a_1 = x_1 - x_0$ and $a_2 = y_1 - y_0$: (x_0, y_0) and (x_1, y_1) represent the initial and terminal point of the vector's directed line segment respectively. θ is the angle between two vectors.

Length (magnitude) of a vector : $\| \vec{a} \| = \sqrt{a_1^2 + a_2^2}$

Unit vectors : $i = (1,0)$ $\quad j = (0,1)$

$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$ $\quad \vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2)$

$c\vec{a} = (ca_1, ca_2)$ $\quad \vec{a} = a_1 i + a_2 j$

$\| c\vec{a} \| = |c| \| \vec{a} \|$ $\quad \vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2$

$\vec{a} \bullet \vec{b} = \| \vec{a} \| \, \| \vec{b} \| \cos\theta$ $\quad \vec{a} \times \vec{a} = \|\vec{a}\|^2$

If $\vec{a} \bullet \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are perpendicular $\quad$ If $\vec{b} = c\vec{a}$, then $\vec{a}$ and $\vec{b}$ are parallel

Angle between two vectors : $\cos\theta = \dfrac{\vec{a} \bullet \vec{b}}{\|\vec{a}\| \times \|\vec{b}\|}$

Projection (of vector $\vec{b}$ onto $\vec{a}$)

$$pr_a\vec{b} = \frac{\vec{a} \bullet \vec{b}}{(\|a\|)^2} \qquad \vec{b} = pr_a\vec{b} + pr_{a'}\vec{b} \qquad \text{where } \vec{a}\,' \text{ is perpendicular to } \vec{a}$$

Vector - Valued Functions

A vector - valued function has a domain (a set of real numbers) and a rule (which assigns to each number in the domain a vector)

e.g. $\mathbf{F}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$

If $\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}$

$\lim \mathbf{F}(t)$ exists only if $\lim f_1(t)$ and $f_2(t)$ exist

$$\lim_{t \to c} \mathbf{F}(t) = \lim_{t \to c} f_1(t))\mathbf{i} + \lim_{t \to c} f_2(t))\mathbf{j}$$

A vector valued function is continuous at t_0 if its component functions are continuous at t_0

$$\mathbf{F}'(t) = f_1'(t) + f_2'(t)$$

$$\int \mathbf{F}(t)dt = \int f_1(t)\,dt + \int f_2(t)\,dt$$

Motion (defined as a vector valued function of time)

Position:	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$
Velocity:	$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$
Speed:	$\|\mathbf{v}(t)\|$
Acceleration:	$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$
Tangent vector:	$\mathbf{T}(t) = \mathbf{r}'(t) / \|\mathbf{r}'(t)\|$
Curvature:	$\mathbf{k} = \|\mathbf{T}'(t)\| / \|\mathbf{r}'(t)\|$
Normal vector:	$\mathbf{N}(t) = \mathbf{T}'(t) / \|\mathbf{T}'(t)\|$

OTHER DIFFERENTIAL EQUATIONS (Calculus III)

Linear First Order Differential Equations

A linear first order differential equation has this form: $\frac{dy}{dx}+P(x)y=Q(x)$

Solve using this equation: $y=e^{-S(x)}\int e^{S(x)}Q(x)\,dx$ where $S(x)=\int P(x)\,dx$

Second Order Linear Differential Equations

A second order differential equation has this general form: $\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=g(x)$

Homogeneous Equations

A second order linear equation is homogeneous if $g(x)=0$; thus

$$\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

To solve, make use of these equations:

$$s_1=\frac{-b+\sqrt{b^2-4c}}{2} \text{ and } s_1=\frac{-b-\sqrt{b^2-4c}}{2}$$

There are three cases depending on the solution to b^2-4c

If $b^2-4c>0$	$y=C_1e^{s_1x}+C_2e^{s_2x}$	where C_1 and C_2 are constants
If $b^2-4c=0$	$y=C_1e^{s_1x}+C_2e^{s_2x}$	where C_1 and C_2 are constants
If $b^2-4c<0$	$y=C_1e^{ux}\sin vx+C_2e^{ux}\cos vx$	where C_1 and C_2 are constants

$u=-\frac{b}{2}$ and $v=\frac{1}{2}\sqrt{4c-b^2}$

Nonhomogeneous Equations

A second order differential equation is nonhomogeneous if $g(x)$ does not equal zero

To solve:

1. Find the solution for $\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

   See "Homogeneous Equations"

2. Find y_p

   $y_p=u_1y_1+u_2y_2$

   where:

   $$u_1'=\frac{-(y_2g)}{y_1y_2'-y_1'y_2}$$

   $$u_2'=\frac{y_1g}{y_1y_2'-y_1'y_2}$$

   and y_1 and y_2 are solutions in Step 1

3. Express the solution as

   $y=y_p+$ the solution given by Step 1